Diophantine Gaussian excursions and random walks

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GT Combinatoire et Transcendance

Gaussian fields

- **Random field** : Random function $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$
- **Gaussian random field** : For all $x_1, \ldots, x_q \in \mathbb{R}^d, (\mathbf{X}(x_1), \ldots, \mathbf{X}(x_q))$ is a Gaussian vector, characterised by

$$\mathbb{E}(\mathbf{X}(x_i)) = 0, \quad C_{\mathbf{X}}(x_i, x_j) := \mathbf{Cov}(\mathbf{X}(x_i), \mathbf{X}(x_j)) = \mathbb{E}(\mathbf{X}(x_i)\mathbf{X}(x_j))$$

- $C_{\mathbf{X}}$ uniquely characterises the law of $\mathbf{X}$, and has to be SDP : for all $\alpha_1, \ldots, \alpha_q \in \mathbb{R}, x_1, \ldots, x_q \in \mathbb{R}^d$

$$\sum_{i,j} \alpha_i \alpha_j C_{\mathbf{X}}(x_i, x_j) = \sum_{i,j} \alpha_i \alpha_j \mathbb{E}(\mathbf{X}(x_i)\mathbf{X}(x_j)) = \mathbb{E}((\sum_i \alpha_i \mathbf{X}(x_i))^2) \geqslant 0$$

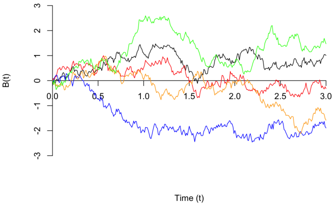


FIGURE – Brownian motion : $C_{\mathbf{X}}(t,s) = \mathbb{E}(\mathbf{X}(t)\mathbf{X}(s)) = \min(s,t), s,t \in \mathbb{R}_+$

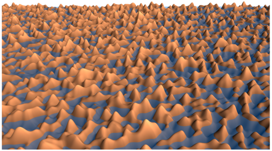


FIGURE – Random planar wave, $C_{\mathbf{X}}(x,y) = Bessel_0(\|x-y\|), x,y \in \mathbb{R}^d$.*Crédit : V. Beffara, D. Gayet*

- $\mathbf{X}$ stationary :

$$\forall y \in \mathbb{R}^d, \{\mathbf{X}(x); x \in \mathbb{R}^d\} \overset{(d)}{=} \{\mathbf{X}(x+y) : x \in \mathbb{R}^d\}$$

- Equivalent to $C_{\mathbf{X}}(x,y) = \mathbf{C}_{\mathbf{X}}(x-y)$
- Bochner Theorem : $\mathbf{C} : \mathbb{R}^d \to \mathbb{R}$ is SDP iff there is $\mu = \hat{\mathbf{C}}$ finite measure on $\mathbb{R}^d$ such that

$$\mathbf{C}(x) = \int_{\mathbb{R}^d} e^{\imath\langle t,x\rangle}\mu(dx)$$

Example :

$$Bessel_0(x) = \int_{\mathbb{S}^1} e^{\imath\langle t,x\rangle}\mathcal{H}^1(dx) \quad \Leftrightarrow \quad \Delta\mathbf{X} = -\mathbf{X}$$

- For a stationary process $\mathbf{X}$, $\mu_{\mathbf{X}} = \hat{\mathbf{C}}_{\mathbf{X}}$ is the *spectral measure* of $\mathbf{X}$

Example

$$\mu_{\mathbf{X}}(dx) = \frac{1}{2}(\delta_{-1} + \delta_1)(dx), x \in \mathbb{R}$$

$$\mathbf{C}_{\mathbf{X}}(x) = \cos(x)$$

$$\text{Claim : } \mathbf{X}(x) \overset{(d)}{=} A\cos(x) + B\sin(x),\ A, B\ iid \sim \mathcal{N}(0,1)$$

$$\begin{aligned}
\text{Proof : } \mathbb{E}(\mathbf{X}(x)\mathbf{X}(y)) &= \mathbb{E}[(A\cos(x) + B\sin(x))(A\cos(y) + B\sin(y))] \\
&= \underbrace{\mathbb{E}(A^2)}_{=1}\cos(x)\cos(y) + \underbrace{\mathbb{E}(AB)(... + ...)}_{=0} + \underbrace{\mathbb{E}(B^2)}_{=1}\sin(x)\sin(y) \\
&= \cos(x - y)
\end{aligned}$$

for $\omega_1, \ldots, \omega_m > 0$

$$\mu = \frac{1}{2}\sum_i (\delta_{\omega_i} + \delta_{-\omega_i})$$

$$\mathbf{X}(x) = \sum_i [A_i\cos(\omega_i x) + B_i\sin(\omega_i x)]$$

Excursions

$$\mathbf{E}_\ell = \{x : \mathbf{X}(x) > \ell\}$$

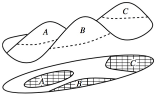


FIGURE – Credit : R. Adler & J. Taylor

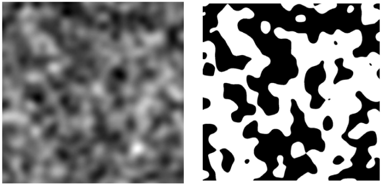
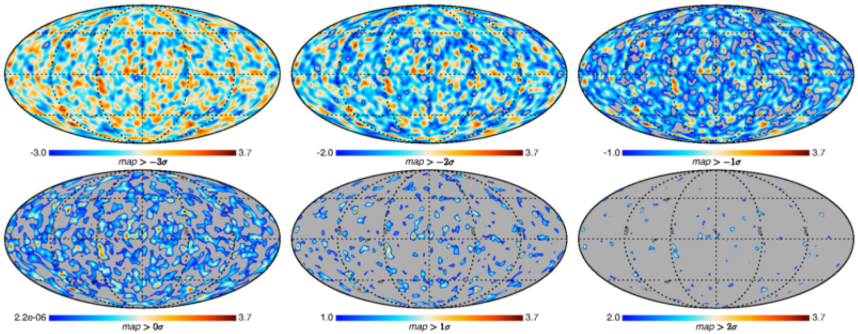


FIGURE – Gaussian excursion. Credit : **Biermé & Desolneux 2020, Biermé et al. 2021**

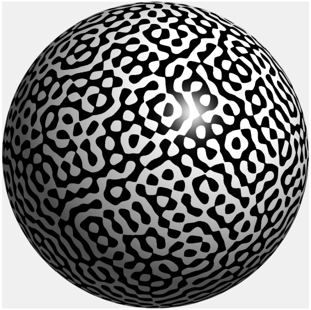
Cosmological Microwave Background (CMB)

FIGURE – Credit : **Y. Fantaye, F. Hansen, D. Maino, D. Marinucci**, *Cosmological Applications of the Gaussian Kinematic Formula*, Physical Review D 91(6), 2014



Harmoniques gaussiennes

FIGURE – Credit : Dimitri Beliaev



- $\mathbf{X}_n$: Combinaison linéaire isotrope gaussienne de fonctions propres du Laplacien : $\Delta \mathbf{X}_n = -n\mathbf{X}_n$ avec $n \to \infty$
- $\mathbf{X}_n \to$ Random Planar Wave "locally" (after rescaling)
- **Variance des lignes nodales**

$$\mathbf{Var}(\mathcal{H}^1(\mathbf{E}_0(\mathbf{X}_n)))$$

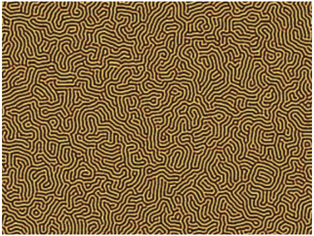
  Comportement inattendu pour $\ell \neq 0$
- **Berry's conjecture** : Variance "cancellation" phenomenon (**Wigman, Marinucci, Peccati, Rossi, Cammarota, Maffuci**)

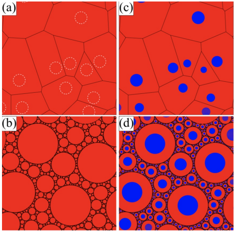
Hyperuniform random sets

- A stationary random set $\mathbf{E}$ is hyperuniform if

$$\frac{\mathbf{Var}(\mathbf{Leb}^d(\mathbf{E} \cap \mathbf{B}(0, R)))}{R^d} \to 0$$

- **Torquato, Stillinger** : Labyrinth-like Turing pattern (Left), hard sphere packings (Right)





Does there exist a stationary Gaussian field **X** such that

$$\mathbf{V}_{\mathbf{X}}^{\ell}(R) := \mathbf{Var}(\mathbf{Leb}^d(\{\mathbf{X} > \ell\} \cap B(0,R))) = o_{R\to\infty}(|B(0,R)|)?$$

For "normal" fields the answer is **no** :

$$\mathbf{Leb}^d(\{\mathbf{X} > \ell\} \cap B(0,R)) = \int_{B(0,R)} \mathbf{1}_{\{\mathbf{X}(x)>\ell\}} dx$$

$$\mathbf{V}_{\mathbf{X}}^{\ell}(R) = \int_{B(0,R)^2} \underbrace{\mathbf{Cov}(\mathbf{1}_{\{\mathbf{X}(x)>\ell\}}, \mathbf{1}_{\{\mathbf{X}(y)>\ell\}})}_{\text{Small if } \|x-y\| \text{ large}} dxdy$$

$$\approx \text{Magnitude of diagonal terms}$$

$$\sim |B(0,R)|.$$

It's the same for a sum of iid variables Y_i (non-diagonal terms vanish) :

$$\mathbf{Var}(\sum_{i=1}^n Y_i) = \sum_{i,j=1}^n \underbrace{\mathbf{Cov}(Y_i, Y_j)}_{=0 \text{ if } i\neq j} = \sum_{i=1}^n \mathbf{Var}(Y_i) = n\mathbf{Var}(Y_1)$$

General case with $\ell = 0$

$$\varphi(x-y) = \mathbf{Cov}(\mathbf{1}_{\{\mathbf{X}(x)>0,\mathbf{X}(y)>0\}})$$

recall that $(\mathbf{X}(x), \mathbf{X}(y))$ is a Gaussian vector characterised by $\mathbf{Var}(\mathbf{X}(x)) = \mathbf{Var}(\mathbf{X}(y)) = 1$ and

$$\mathrm{cov}(\mathbf{X}(x), \mathbf{X}(y)) = \mathbf{C}_{\mathbf{X}}(x-y)$$

hence (computations...)

$$\varphi(x-y) = \arcsin(\mathbf{C}_{\mathbf{X}}(x-y)) = \sum_{n \text{ odd}} (n^{-3/2} + o(1))\mathbf{C}_{\mathbf{X}}(x-y)^n$$

$$\mathbf{V}^0_{\mathbf{X}}(R) =: \mathbf{V}_{\mathbf{X}}(R) \sim \sum_{n \text{ odd}} n^{-3/2} \int_{B(0,R)^2} \mathbf{C}_{\mathbf{X}}(x-y)^n dxdy$$

$$(\text{ spectral analysis } ...) = c_d R^{2d} \sum_{n \text{ odd}} n^{-3/2} \int \widehat{\mathbf{1}_{\{B(0,1)\}}}(Rx)^2 \mu^{\otimes n}(dx)$$

Gaussian excursions volume variance

Crux : $|x| \leqslant \eta R^{-1}$ contains the main contribution (for $\eta > 0$ well chosen)

$$\underbrace{\widehat{\mathbf{1}_{\{B(0,1)\}}}(Rx)^2}_{4arcsin(Rx)^2 \text{ in dimension 1}} \sim \underbrace{\widehat{\mathbf{1}_{\{B(0,1)\}}}^2(\eta)}_{\xrightarrow[\eta\to 0]{} c_d > 0} \mathbf{1}_{\{|x|<\eta R^{-1}\}}$$

is a good estimate for $x \sim \mu^{\otimes n}$.

Theorem (Lr 21)

- $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ *stationary*
- $\mu_{\mathbf{X}}$: *Spectral measure (normalised)*
- $\mathbf{U}_n = \sum_{k=1}^n Y_k \text{ where } Y_k \sim \mu_{\mathbf{X}} \text{ i.i.d.}$
- $\mathbf{K}(\varepsilon) := \sum_n n^{-3/2}\mathbb{P}(\|\mathbf{U}_{2n+1}\| < \varepsilon)$.

Then if $\mathbf{K}(\varepsilon) \sim \varepsilon^\alpha$,

$$\mathbf{V}_{\mathbf{X}}(R) \sim R^{2d-\alpha}$$

(and $\alpha \leqslant d+1$*).*

Example : Gaussian planar wave and isotropic models

$$\mu_{\mathbf{X}}(dx) = \mathbf{1}_{\{\mathbb{S}^{d-1}\}}(x)\mathcal{H}^{d-1}(dx) \quad \Leftrightarrow \quad \Delta\mathbf{X} = -\mathbf{X} a.s.$$

We can prove for ε small

$$\mathbb{P}(\|\mathbf{U}_1\| < \varepsilon) = 0$$
$$\mathbb{P}(\|\mathbf{U}_2\| < \varepsilon) \sim \varepsilon^{d-1}$$
$$\mathbb{P}(\|\mathbf{U}_n\| < \varepsilon) \sim \varepsilon^{d}, n \geqslant 3$$

hence

$$\underbrace{\mathbf{V}_{\mathbf{X}}^{\ell}(R) \geqslant c' R^{d+1}}_{\text{Similar spectral analysis}} \quad \text{and} \quad \mathbf{K}(\varepsilon) \sim \varepsilon^{d} \text{ and } \mathbf{V}_{\mathbf{X}}^{0}(R) \sim R^{d}$$

- Variance cancellation phenomenon at the level $\ell = 0$ (cf. Marinucci-Wigman '11, Rossi '19, ...)
- Every isotropic model has a higher variance $\Rightarrow$ **No model with sublinear variance !**

Gaussian excursions

We consider spectral measures with finite support, for instance

$$\mathbf{C}(x) = \cos(x) + \cos(\omega x) \text{ where } \omega \in \mathbb{R} \setminus \mathbb{Q}$$
$$\mathbf{X}(x) = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(\omega x) + A_4 \sin(\omega x)$$

where the A_i are i.i.d. centered standard Gaussian. Let

$$\mathbf{V}(R) = \mathbf{Var}(\mathbf{Leb}^1(\mathbf{E}_0 \cap [0, R])).$$

Theorem

Let $\beta \in [0, 2)$, L a slowly varying function in some sense. Then there are uncountably many $\omega \in \mathbb{R}$ such that

$$0 < c_- R^\beta L(R) \overset{inf.often}{\leqslant} \mathbf{V}(R) \leqslant c_+ R^\beta L(R) < \infty$$

Generalisations

Several frequencies :

$$\mathbf{C}(x) = \sum_{i=0}^{m} \cos(\omega_i x)$$

Dimension d **:**

$$\mathbf{C}(x_1, \ldots, x_d) = \sum_{k=1}^{d} \sum_{i=0}^{m} \cos(\omega_{k,i} x_k)$$

$$\mathbf{X} : \text{Stationary Gaussian Field with spectral measure } \mu$$

$$\mathbf{U_n} : \text{Random walk with i.i.d. increments distributed as } \mu$$

$$\mathbf{K}(\varepsilon) := \sum_{n} n^{-3/2} \mathbb{P}(\|\mathbf{U}_{2n+1}\| < \varepsilon)$$

Recall that $\mathbf{K}(\varepsilon) \sim \varepsilon^{\alpha} \Rightarrow \mathbf{V_X}(R) \sim R^{2d-\alpha}$

Irrational random walk

- Spectral measure

$$\mu = \frac{1}{2} \sum_{k,i} (\delta_{\omega_{k,i}} + \delta_{-\omega_{k,i}}) \mathbf{e}_k$$

- Assume μ is $\mathbb{Z}$-free : $\forall k$,

$$p_i \in \mathbb{Z}, \sum_i p_i \omega_{k,i} = 0 \Rightarrow \forall i, p_i = 0.$$

- Y_j i.i.d. with law μ and

$$\mathbf{U}_n = \sum_{j=1}^{n} Y_j$$

$$\bar{\mathbf{U}}_n = \mathbf{U}_n - [\mathbf{U}_n] \in \mathbb{T}^d$$

- What are

$$\mathbb{P}(0 < \|\mathbf{U}_n\| < \varepsilon)? \quad \mathbb{P}(0 < \|\bar{\mathbf{U}}_n\| < \varepsilon)?$$

Random walk (Cont'd) Dimension $d = 1$

- Known results (Diaconis, Saloff-Coste, Rosenthal, Porod and Su 1998)

$$\sup_{I \text{ interval of } [0,1]} |\mathbb{P}(\bar{\mathbf{U}}_n \in I) - \mathbf{Leb}^1(I)| \xrightarrow[n\to\infty]{} 0$$

$$\text{Hence } \sup_{0<\varepsilon<1} |\mathbb{P}(|\bar{\mathbf{U}}_n| < \varepsilon) - 2\varepsilon| \xrightarrow[n\to\infty]{} 0$$

- $\mathbb{P}(|\bar{\mathbf{U}}_n| < \varepsilon) \sim \varepsilon$ does not lead to accurate estimates
- **Need :** Uniform bound over n and ε
- Lower bounds for $\mathbb{P}(|\mathbf{U}_n| < \varepsilon)$ are actually unstable, it is easier to control

$$\mathbf{K}(\varepsilon) = \sum_n n^{-3/2} \mathbb{P}(|\mathbf{U}_{2n+1}| < \varepsilon).$$

Approximability of ω

Definition

Let $\psi : \mathbb{N} \to (0, \infty)$ such that $\psi(q) \to 0$ as $q \to \infty$.
$\omega = (\omega_1, \dots, \omega_m) \in \mathbb{R}^m$ is ψ-BA (Badly Approximable) iff

$$\inf_{p \in \mathbb{Z}} |p - \omega \cdot q| > \psi(|q|), q \in \mathbb{Z}^m$$

and ψ-WA (Well Approximable) iff for some $c > 0$

$$|p - \omega \cdot q| \overset{inf.often}{\leqslant} c\psi(|q|)$$

Subtlety : We actually need the following concept ; ω is ψ-WA* iff for infinitely many $q = (q_i) \in \mathbb{Z}^m$ such that $\sum_{i=1}^m q_i$ is odd

$$|p - \omega \cdot q| \leqslant c\psi(|q|)$$

Theorem (Khintchine-Groshev (Hussain, Yusupova '17))

If

$$\sum_{q \in \mathbb{Z}^m} \psi(|q|) < \infty$$

the set of ψ-WA ω is $\mathbf{Leb}^m$*-negligible. If the sum diverges,* $\mathbf{Leb}^m$*-a.a. ω is ψ-WA* (If $m = 1$, ψ has to decrease also).*

Theorem (Jarnik '31)

The set of $\omega \in \mathbb{R}$ that are ψ-BA and ψ-WA is uncountable if $q\psi(q)$ decreases to 0.*

Unfortunately, ψ-WA probably does not imply ψ-WA*, but some ω' obtained from ω by adding $\pm\mathbf{e}_i$ is ψ-WA*

Power functions

- For $\eta > 0$, $\mathbf{Leb}^m$-a.a. $\omega \in \mathbb{R}^m$ are q^{-m}-WA* and $q^{-m-\eta}$-BA because

$$\sum_{q \in \mathbb{Z}^m \setminus \{0\}} |q|^{-m} = \infty, \qquad \sum_{q \in \mathbb{Z}^m \setminus \{0\}} |q|^{-m-\eta} < \infty$$

- For all $\eta \in [0, \infty]$, uncountably many ω are $q^{-m-\eta}$-BA and WA

Theorem

If the ω_i are $\mathbb{Z}-$free and η-approximable,

$$c_- \varepsilon^{\frac{1+d(m+1)}{m+\eta}} \overset{\textbf{inf.often}}{\leqslant} \mathbf{K}(\varepsilon) \leqslant c_+ \varepsilon^{\frac{1+d(m+1)}{m+\eta}}$$

If $d = 1$,

$$\mathbb{P}(\|\bar{\mathbf{U}}_n\| < \varepsilon) \leqslant c\varepsilon^{\frac{m}{m+\eta}}, \quad \mathbb{P}(\|\mathbf{U}_n\| < \varepsilon) \leqslant cn^{-1/2}\varepsilon^{\frac{m}{m+\eta}}$$

Case $m = d = 1, \eta = 0$ **:** *linear order ε^1*

Theorem (In dimension $d = 1$)

Let $\beta \in [0, 2)$, L a slowly varying function in some sense. Then there are uncountably many $\omega \in \mathbb{R}$ such that

$$0 < c_- R^\beta L(R) \overset{inf.often}{\leqslant} \mathbf{V}(R) \leqslant c_+ R^\beta L(R) < \infty$$

The variance is bounded for $\mathbf{Leb}^d$*-a.a.* ω

- In dimension d, if ω is $q^{-m-\eta}$-(WA) and -(BA)

$$\mathbf{C}(x_1, \ldots, x_d) = \cos(x_1) + \cos(x_1\omega) + \cdots + \cos(x_d) + \cos(x_d\omega)$$

the variance on $\mathbf{B}(0, R)$ is in

$$\underbrace{R^{d-1}}_{\sim|\partial B(0,R)|\text{ (minimal)}} \quad << \quad R^{\max(d-1, 2d - \frac{1+2d}{1+\eta})} \quad << \quad \underbrace{R^{2d}}_{\sim|B(0,R)|^2\text{ (maximal)}}$$

Other functions

Let $\psi(q) = q^{-\tau} L(q)$ where $L : \mathbb{N} \to \mathbb{R}$ is "slowly varying"

$$\frac{L(q+1) - L(q)}{q^{-1} L(q)} \to 0.$$

There are uncountably many $\omega \in \mathbb{R}^m$ that are ψ-WA and ψ-BA, and

$$\mathbf{V}_{\mathbf{X}}(R) \begin{cases} \sim \frac{R^{2d}}{\psi^{-1}(R^{-1})^{1+d(m+1)}} \text{ if } \tau > \tau^* = \frac{1+d(m+1)}{1+d} \\ \leqslant R^{d-1} \ln(R) \text{ if } \tau := \tau^* \\ \leqslant R^{d-1} \text{ if } \tau < \tau^* \end{cases}$$

Different frequencies along dimensions

$$\mathbf{C}(x) = \sum_{k,i} \cos(\omega_{k,i} x_k) \text{ (with } \omega_0 = 1)$$

the variance upper bound depends on the worst approximable ω_i, and the lower bound on the properties of **simultaneous approximation** :

$\omega = (\omega_1, \ldots, \omega_d) \in (\mathbb{R}^m)^d$ is ψ-SWA if for infinitely many $q \in \mathbb{R}^m$

$$\inf_{p \in \mathbb{Z}} |p - \omega_k \cdot q| < \psi(q) \text{ for all } k$$

It typically does not match if $\omega_k \neq \omega_{k'}$ for $k \neq k'$.

Lower bound proof in the case $m = d = 1$

- $\mu = \frac{1}{4}(\delta_1 + \delta_{-1} + \delta_\omega + \delta_{-\omega})$ where $\omega \in \mathbb{R} \setminus \mathbb{Q}$.
- Consider the 2D random walk $\mathbf{V}_0 = 0$ with independent increments

$$\mathbf{V}_{n+1} = \mathbf{V}_n + \begin{cases} +\mathbf{e}_1 \\ -\mathbf{e}_1 \\ +\omega\mathbf{e}_2 \\ -\omega\mathbf{e}_2 \end{cases} \quad \text{all with probability } \frac{1}{4}$$

$$\mathbf{U}_n \overset{(d)}{=} \mathbf{V}_{n,1} - \mathbf{V}_{n,2}$$

$$\mathbb{P}(|\mathbf{U}_n| < \varepsilon) = \mathbb{P}((\mathbf{V}_{n,1}, \mathbf{V}_{n,2}) \in \{(p,q) : |p - \omega q| < \varepsilon\})$$

- Assume ω is η-Well Approximable : $\exists$ integers $q_j \to \infty, p_j$ such that

$$\varepsilon_j := c q_j^{-1-\eta} \leqslant |p_j - \omega q_j|$$

Proof(C'td)

$$\mathbb{P}(|\mathbf{U}_n| < \varepsilon_j) \geqslant \mathbb{P}(\mathbf{V}_{n,1} = p_j, \mathbf{V}_{n,2} = q_j)$$

- **First approximation :** $\mathbf{V}_{n,1}$ and $\mathbf{V}_{n,2}$ are independent random walks
- **Second approximation :** $\mathbf{V}_{n,i} \sim G_i \overset{(d)}{=} \mathcal{N}(0, c_i n)$ (Central Limit Theorem)

$$\mathbb{P}(|\mathbf{U}_n| < q_j^{-1-\eta}) \geqslant \mathbb{P}(G_1 = p_j)\mathbb{P}(G_2 = q_j)$$

$$
\begin{aligned}
\mathbb{P}(|\mathbf{U}_n| < q_j^{-1-\eta}) \geqslant & \mathbb{P}(G_1 = p_j)\mathbb{P}(G_2 = q_j) \\
\geqslant & c' \frac{1}{\sqrt{n}} \exp\left(-\frac{p_j^2}{c_1' n^2}\right) \frac{1}{\sqrt{n}} \exp\left(-\frac{q_j^2}{c_2' n^2}\right) \\
\mathbf{K}(\varepsilon_j) = \sum_{n \text{ odd}} n^{-3/2}\mathbb{P}(|\mathbf{U}_n| < \varepsilon_j) \geqslant & \sum n^{-3/2} n^{-1} \exp\left(-\frac{p_j^2 + q_j^2}{n}\right) \\
\sim & \int_1^\infty y^{-5/2} \exp\left(-\frac{p_j^2 + q_j^2}{y}\right) dy \\
\sim & (p_j^2 + q_j^2)^{-3/2} \\
\sim & (\varepsilon_j^{2(1+\eta)})^{-3/2} \text{ because } |p_j| \sim |q_j \omega|
\end{aligned}
$$