Nodal sets variance for Gaussian stationary processes

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New trends in Point processes
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Gaussian fields

- **Centered Gaussian fields** : $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ such that :
  - $(X(x_1), \dots, X(x_q))$ is a Gaussian vector for $x_1, \dots, x_q \in E$
  - $\mathbb{E}(\mathbf{X}(x)) = 0, x \in E$.
- Stationarity : $\mathbf{X}(x + \cdot) \stackrel{(d)}{=} \mathbf{X}$ for $x \in \mathbb{R}^d$

Excursion and level sets are privileged observables : for $\ell \in \mathbb{R}$

$$\mathbf{E}_\ell = \{x : \mathbf{X}(x) \geqslant \ell\}$$
$$\mathbf{L}_\ell = \{x : \mathbf{X}(x) = \ell\}.$$

Variance linearity :

- in $\mathbb{R}$:

$$\mathbf{Var}(\mathbf{L}_0 \cap [0, R]) = o(R)? \sim R?$$

- in $\mathbb{R}^d$:

$$\mathbf{Var}(\mathbf{E}_\ell \cap \mathbf{B}(0, R)) = o(R^d)? \sim R^d?$$

A non-stationary field with stationary zeros

Let $\mathbf{X} : \mathbb{C} \to \mathbb{C}$ be a GAF, i.e. a Gaussian field such that

- $\mathbf{X}$ is a.s. holomorphic
- For all $x_1, \dots, x_q, (\mathbf{X}(x_1), \dots, \mathbf{X}(x_q)) \in \mathbb{C}^q$ has a centered standard Complex distribution ($\neq$ standard distribution that is complex)

Then $\mathbf{X}$ is not stationary. Still it is possible that the point process

$$\mathbf{Z} = \mathbf{L}_0 = \mathbf{X}^{-1}(\{0_{\mathbb{C}}\})$$

is stationary, in this case the law of $\mathbf{Z}$ is uniquely determined up to a scaling factor :

$$\exists \mathbf{Z_1} \text{ stationary point process such that } \mathbf{Z} \overset{(d)}{=} \alpha \mathbf{Z_1} \text{ for some } \alpha > 0.$$

and $\mathbf{Z_1}$ is hyperuniform :

$$\mathbf{Var}(\#\mathbf{Z}_1 \cap \mathbf{B}(0, R)) = o(R^d)$$

Hyperuniformity (S. Torquato, J. Lebowitz, S. Ghosh, ...)

- A point process **Z** is **hyperuniform** if

$$\lim_{T\to\infty} \frac{\mathbf{Var}(\#\mathbf{Z} \cap \mathbf{B}(0,R))}{R^d} = 0.$$

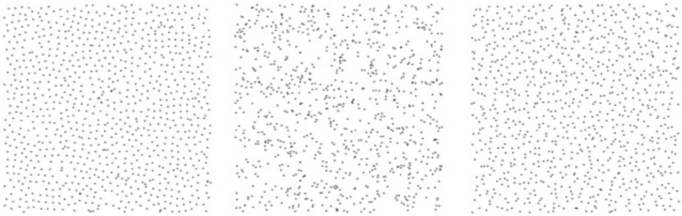


FIGURE – (*left :* critical points of the RPW, *middle* : Poisson, *Right* : DPP. Credit : Torquato et al.)

Stationary zeros

- A problem going back to the 50's is the study of zeros of a smooth Stationary Gaussian Process (SGP) $\mathbf{X}$ in dimension 1 :

$$\mathbf{Z} := \mathbf{L}_0 = \{x \in \mathbb{R} : \mathbf{X}(x) = 0\}$$

- **"Nodal"** : the properties of $\mathbf{Z}$ might differ from those of the $\mathbf{L}_\ell, \ell \neq 0$.
- **First order** : $\mathbb{E}(\mathbf{Leb}^1([0,T] \cap \mathbf{Z}))$ is proportionnal to T ("linear")

Tools

A SGP $\mathbf{X}$ is characterised by :

- Its reduced covariance function $\mathbf{C_X} : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\mathbb{E}(\mathbf{X}(x)\mathbf{X}(y)) = \mathbf{C_X}(x-y), x, y \in E$$

- Its spectral measure $\mu_{\mathbf{X}}$, defined by

$$\mathbf{C_X}(x) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} \mu_{\mathbf{X}}(dt)$$

- Example :

$$C_{\mathbf{X}}(x) = \cos(x),\ \mu_{\mathbf{X}} = \frac{\delta_1 + \delta_{-1}}{2},\ \mathbf{X}(x) = A\cos(x) + B\sin(x),$$

$$A, B i.i.d. \sim \mathcal{N}(0,1)$$

Zeros number variance

- Define

$$\mathbf{V_X}(T) = \mathbf{Var}(\mathbf{Z} \cap [0, T])$$

- If $\mathbf{X}$ is τ-periodical, $\mathbf{V_X}(T) \sim T^2 \mathbf{Var}(\mathbf{Z} \cap [0, \tau])$, hence **quadratic** ($\sim T^2$), except if

$$\mathbf{X}(t) = A\cos(\frac{2\pi x}{\tau}) + B\sin(\frac{2\pi x}{\tau}) \Leftrightarrow \mathbf{C_X}(x) = \mathbf{C_X}(0)\cos(\frac{2\pi x}{\tau}), x \in \mathbb{R}$$

  for A, B i.i.d. Gaussian variables.

- **Kac-Rice (1950')** : Expression of $\mathbf{V_X}$ in fonction of $\mathbf{C_X}$.
- **Cramer & Leadbetter (1967)** : $\mathbf{V_X}(T) < \infty$ if $\mathbf{C_X}$ is twice differentiable and a little bit mode : for some $\delta > 0$

$$\int_0^\delta \frac{1}{t^2}(\mathbf{C_X'}(t) - \mathbf{C_X''}(0)t)dt < \infty. \tag{1}$$

Bibliography

- **Geman (1972)** : Sufficient condition ("Geman's condition")
- **Cuzick (1976)** : If **furthermore** $\mathbf{C_X} \in \mathbf{L^2}, \mathbf{C''_X} \in \mathbf{L^2}$, the variance is at most linear

$$\limsup_{T\to\infty} T^{-1}\mathbf{V_X}(T) < \infty$$

Central Limit Theorem under the additional assumption that the variance is at least linear :

$$\lim_{T\to\infty} T^{-1}\mathbf{V_X}(T) = \sigma > 0$$

- **Slud (1991) :** gets rid of the "at least linear" assumption (chaotic decomposition of $\mathbf{X}$)
- **Kratz & Léon (2001)** : Chaotic decomposition in $(\mathbf{X}, \mathbf{X}')$: generalisations, levels $\ell \neq 0$, ...

Variance linearity

Can we have hyperuniform zeros ?

Theorem (Lr 20)

- *The variance is sub-linear only if* $\mathbf{C_X}(x) = \cos(2\pi x/\tau), \tau \geqslant 0$
- *If the variance is linear,* $\mathbf{C_X''} - \mathbf{C_X} \in \mathbf{L^2}$
- *It is a NSC equivalent to* $[\mathbf{C_X}, \mathbf{C_X''} \in \mathbf{L^2}]$ *iff* $\mathbf{C_X}$ *has a density* $\mathbf{L^2}$ *in the neighbourhood of* $\pm\sqrt{-\mathbf{C}''(0)}$.
- *Extension to* **linear statistics** *of zeros*

Proof : Based on the decomposition of Kratz & Léon

Legendre, Ancona '20 : Linear statistics in the linear regime

Assaf, Buckley, Feldheim '21 : Similar results + upper bounds

Rigidity

- Zeros of a GSP are not hyperuniform :(In dimension 1 !
- A stationary Point process $\mathbf{Z}$ is rigid if $\#(\mathbf{Z} \cap \mathbf{B}(0, R))$ is measurable wrt $\mathbf{Z} \cap \mathbf{B}(0, R)^c$
- Most HU examples are rigid :
  - Some DPP
  - Zeros of the planar Gaussian Analytic Function
  - Coulomb systems
- Link hyperuniformity / rigidity ?

An exemple rigid and hyper-fluctuating

Exemple

Let $\mathbf{X}$ *with covariance*

$$\mathbf{C}(x) = \prod_{k=1}^{\infty} \cos(x/k!)$$

The zeros $\mathbf{Z}$ *of* $\mathbf{X}$ *are hyper-fluctuating and super rigid*
($\mathbf{X}$ *is not too much dependent : it is weakly mixing, as is the PP* $\mathbf{Z}$*, and* $\mathbf{X}$ *is a.s. unbounded.)*

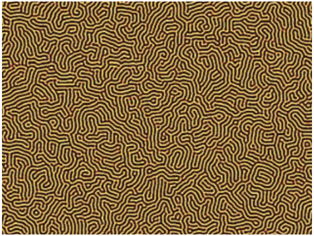
- **Klatt & Last '20** : Other (hyperfluctuating rigid) example in dimension $d \geqslant 2$ with "random grids"

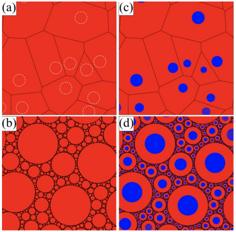
Hyperuniform random sets

- A stationary random set **E** is HU if

$$\frac{\mathbf{Var}(\mathbf{Leb}^d(\mathbf{E} \cap \mathbf{B}(0, R)))}{R^d} \to 0$$

- **Torquato, Stillinger** : Labyrinth-like Turing pattern (Left), hard sphere packings (Right)





Gaussian excursions volume variance

Theorem (Lr 21)

- $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}$ *stationary*
- $\mathbf{V}_{\mathbf{X}}^{\ell}(R) := \mathbf{Var}(\mathrm{Leb}(\{\mathbf{X} > \ell\} \cap \mathbf{B}(0, R)))$
- μ_X : *Spectral measure*
- $\mathbf{U}_n$: *Random walk with i.i.d. increments with law* μ
- $\mathbf{K}(\varepsilon) := \sum_n n^{-3/2}\mathbb{P}(\|\mathbf{U}_{2n+1}\| < \varepsilon)$.

Then

$$c_- R^{2d}\mathbf{K}(R^{-1}) \leqslant \mathbf{V}_{\mathbf{X}}^{0}(R) \leqslant c_+ R^{2d}\mathbf{K}(R^{-1}) + I(R)$$

$$cR^{2d}\mathbb{P}(\|\mathbf{U}_2\| < R^{-1}) \leqslant \mathbf{V}_{\mathbf{X}}^{\ell}(R), \ell \neq 0$$

If $\mathbf{K}(\varepsilon) \sim \varepsilon^{\alpha}$, *then* $I(R) \sim \mathbf{V}_X^0(R) \sim T^{2d-\alpha}$ *(and* $\alpha \leqslant d+1$*).*

Example : Gaussian planar wave and isotropic models

$$\mu_X(dx) = \mathbf{1}_{\{\mathbb{S}^{d-1}\}}(x)\mathcal{H}^{d-1}(dx) \Leftrightarrow \Delta \mathbf{X} = -\mathbf{X} a.s.$$

We can prove for ε small

$$\mathbb{P}(\|\mathbf{U}_1\| < \varepsilon) = 0$$
$$\mathbb{P}(\|\mathbf{U}_2\| < \varepsilon) \sim \varepsilon^{d-1}$$
$$\mathbb{P}(\|\mathbf{U}_n\| < \varepsilon) \sim \varepsilon^{d}, n \geqslant 3$$

hence

$$\mathbf{V}^{\ell}_{\mathbf{X}}(R) \geqslant c' R^{d+1} > 0 \quad \text{and} \quad \mathbf{K}(\varepsilon) \sim \varepsilon^{d} \text{ and } \mathbf{V}^{0}_{\mathbf{X}}(R) \sim R^{d}$$

- Variance cancellation phenomenon (cf. Marinucci-Wigman '11, Rossi '19, ...)
- Every isotropic model has a higher variance ⇒ No isotropic hyperuniform model !

Gaussian excursions

We consider spectral measures with finite support, for instance

$$\mathbf{C}(x) = \cos(x) + \cos(\omega x) \text{ where } \omega \in \mathbb{R} \setminus \mathbb{Q}$$
$$\mathbf{X}(x) = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(\omega x) + A_4 \sin(\omega x)$$

where the A_i are i.i.d. centered standard Gaussian. Let

$$\mathbf{V}(R) = \mathbf{Var}(\mathbf{Leb}^1(\mathbf{E}_0 \cap [0, R])).$$

Theorem

Let $\beta \in [0, 2)$, L a slowly varying function in some sense. Then there are uncountably many $\omega \in \mathbb{R}$ such that

$$0 < c_- R^\beta L(R) \overset{inf.often}{\leqslant} \mathbf{V}(R) \leqslant c_+ R^\beta L(R) < \infty$$

Variance exponent and approximability of ω

- ω is η-approximable :

$$c_- q^{-1-\eta} \leqslant \min_{p\in\mathbb{Z}} |p - \omega q| \overset{inf.often}{\leqslant} c_+ q^{-1-\eta}, q \in \mathbb{N}^*,$$

- If $\omega = \sqrt{2}$ (badly approximable, $\eta = 0$), $\beta = 0$, the variance is bounded (true for $\mathbf{Leb}^1$-a.e. ω)
- If $\omega = \sum_{k=1}^{\infty} 10^{-k!}$ (Liouville number ; well approximated, η-approx $\forall\eta$), for all $\varepsilon > 0$, $R^{2-\varepsilon} << \mathbf{V}(R) << R^2$
- In dimension d, if

$$\mathbf{C}(x_1, \dots, x_d) = \cos(x_1) + \cos(x_1\omega) + \cdots + \cos(x_d) + \cos(x_d\omega)$$

the variance on $\mathbf{B}(0, R)$ is in

$$R^{\max(d-1, 2d - \frac{1+2d}{1+\eta})},$$

Several frequencies

- **Dimension** 1 :

$$\mathbf{C}(x) = \sum_{i=0}^{m} \cos(\omega_i x) \text{ (with } \omega_0 = 1)$$

the variance depends on the diophantine properties of the vector $(\omega_1, \dots, \omega_m)$, i.e. on the number $\eta \geqslant 0$ such that

$$c_+ \|q\|^{-m-\eta} \overset{\textbf{inf.often}}{\geqslant} \operatorname{dist}(q_1\omega_1 + \cdots + q_m\omega_m, \mathbb{Z}) \geqslant c_- \|q\|^{-m-\eta}$$

- For $\mathbf{Leb}^m$−a.a. $(\omega_1, \dots, \omega_m)$, the variance is in $R^{1-\frac{2}{m+\varepsilon}}, \varepsilon$ arb. small
- **Dimension** d : Several vectors $\omega_k = (\omega_k, i)_{1 \leqslant i \leqslant m}$, for $1 \leqslant k \leqslant d$,

$$\mathbf{C}(x_1, \dots, x_d) = \sum_{k=1}^{d} \sum_{i=1}^{m} \cos(\omega_{k,i} x_k)$$

The lower bound depends on the properties of **simultaneous** diophantine approximations of the ω_k

Variance - random walk

$$\mathbf{X} : \text{Stationary Gaussian Field with spectral measure } \mu$$
$$\mathbf{U_n} : \text{Random walk with i.i.d. increments distributed as } \mu$$
$$\mu \text{ is } \mathbb{Z}\text{-free} : \mathbb{P}(\mathbf{U}_{2n+1} = 0) = 0$$
$$\mathbf{K}(\varepsilon) := \sum_n n^{-3/2}\mathbb{P}(\|\mathbf{U}_{2n+1}\| < \varepsilon)$$

Recall that

$$\mathbf{K}(\varepsilon) \sim \varepsilon^{\alpha} \Rightarrow \mathbf{V_X}(R) \sim R^{2d-\alpha}$$

Irrational random walk

- Spectral measure

$$\mu = \sum_{k,i} (\delta_{\omega_{k,i}} + \delta_{-\omega_{k,i}}) \mathbf{e}_k$$

- $\mathbf{X}_j$ i.i.d. with law μ and

$$\mathbf{U}_n = \sum_{j=1}^{n} \mathbf{X}_j$$
$$\bar{\mathbf{U}}_n = \mathbf{U}_n - [\mathbf{U}_n] \in \mathbb{T}^d$$

- What are

$$\mathbb{P}(0 < \|\mathbf{U}_n\| < \varepsilon)?$$
$$\mathbb{P}(0 < \| \bar{\mathbf{U}}_n\| < \varepsilon)?$$

Random walk (Cont'd)

- Known results (Su 1998)

$$\sup_{I \text{ interval of } [0,1]} |\mathbb{P}(\bar{\mathbf{U}}_n \in I) - \mathbf{Leb}^1(I)| \xrightarrow[n\to\infty]{} 0$$

$$\text{Hence } \sup_{0<\varepsilon<1} |\mathbb{P}(|\bar{\mathbf{U}}_n| < \varepsilon) - 2\varepsilon| \xrightarrow[n\to\infty]{} 0$$

- Need : Uniform bound over n and ε of the form

$$\mathbb{P}(|\bar{\mathbf{U}}_n| \in (0,\varepsilon)) < cn^{-\frac{1}{2}}\varepsilon^{\gamma}.$$

- Let

$$\mathbf{J}(\varepsilon) = \sum_n n^{-3/2}\mathbb{P}(\overline{\mathbf{U}}_{2n+1} \in (0,\varepsilon))$$

Random walk bounds

Theorem

If the $\omega_{k,i}$ are $\mathbb{Z}-$free and η-approximable, there are finite $c, c', c'' > 0$ such that

$$\mathbb{P}(|\bar{\mathbf{U}}_n| \in (0,\varepsilon)) \leqslant cn^{-d/2}\varepsilon^{\frac{md}{m+\eta}}$$

$$c''\varepsilon^{-\frac{1+d(m+1)}{m/d+\eta}} \overset{\textbf{inf.often}}{\leqslant} \bar{\mathbf{J}}(\varepsilon) \leqslant c'\varepsilon^{-\frac{1+d(m+1)}{m+\eta}}$$

- **Case m=d=1** : If $\eta = 0$ (badly approximable numbers, e.g. $\sqrt{2}$), we retrieve the linear order ε^1, otherwise the optimal bound is larger.