Ergodicity of STIT tessellations

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1 Random tessellations

2 STIT tessellations

3 Direct construction of the model

Tessellations

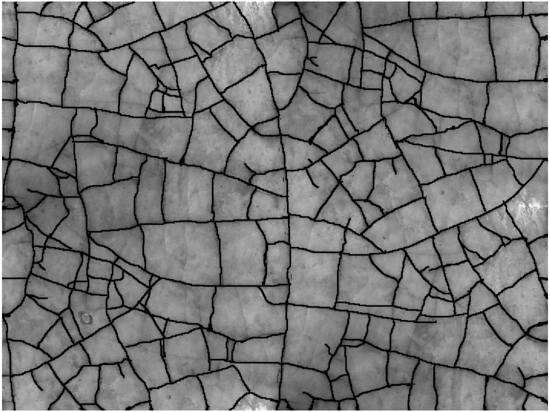
Definition

Tessellation: Set $\mathcal{C} = \{C_1, C_2, ...\}$ of convex compact cells, locally finite (for all compact K, $\{i;\ C_i \cap K \neq \emptyset\}$ is finite), such that

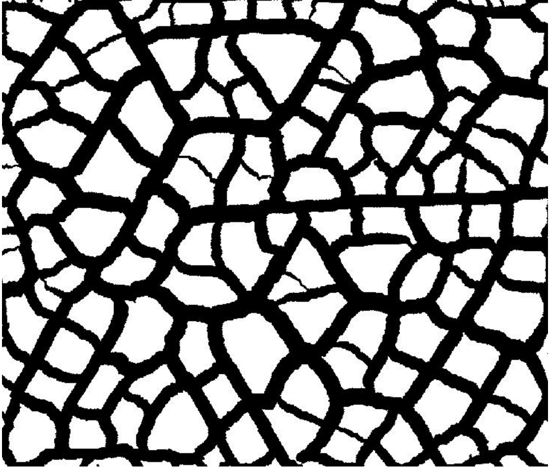
$$\mathbb{R}^d = \cup_i C_i,$$
$$\mathrm{int}(C_i) \cap \mathrm{int}(C_j) = \emptyset,\ i \neq j.$$

The corresponding closed set is $M = \cup_i \partial C_i$.

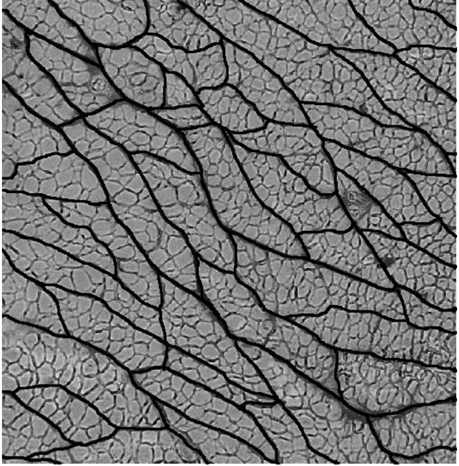
Some examples of real structures – Potential applications.



Craquelée on a ceramic (Photo: G. Weil)



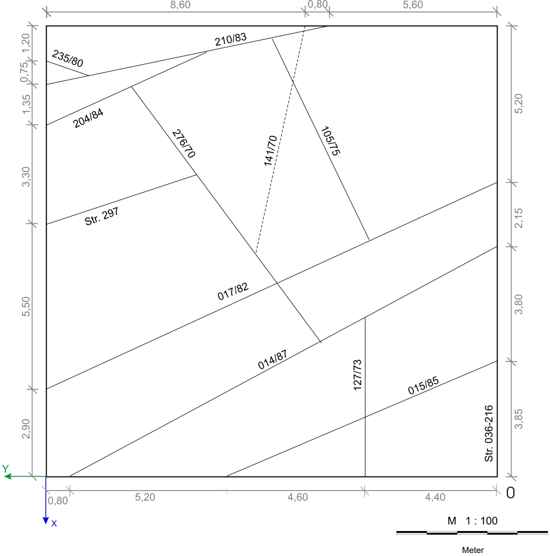
cracking simulation (H.-J. Vogel)



Rat muscle tissue (I. Erzen)

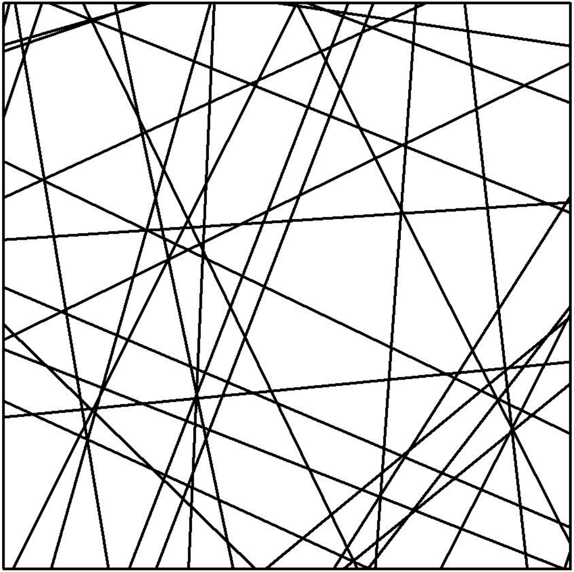
Perla 2.jpg 

Granit joints (D. Nikolayev, S. Siegesmund, S. Mosch, A. Hoffmann)



Gris Perla.jpg

Poisson tessellations

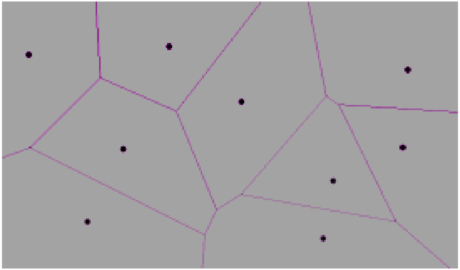


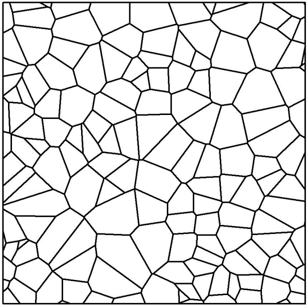
Union of random lines.

Voronoi and Delaunay tessellation

- Π: Point process on $\mathbb{R}^d$.
- $x \in \Pi$, V_x : Set of points of $\mathbb{R}^d$ for which x is the closest element of Π,

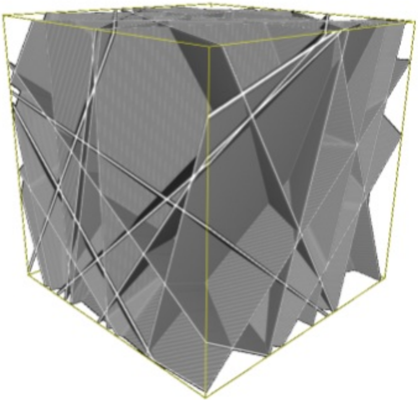
$$V_x = \{y \in \mathbb{R}^2;\ \|x - y\| = \inf_{x' \in \Pi} \|x' - y\|\}$$





3D tessellations

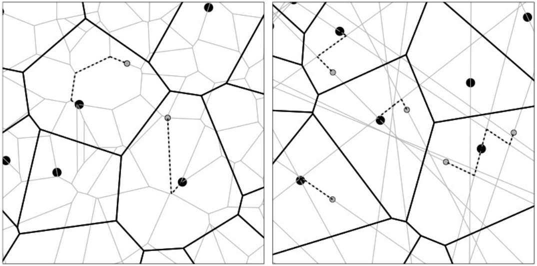
3D Poisson hyperplanes tesselation:



Mixed tessellations

(Schmidt, Voss 2010)

- **Grey tessellation**: Low-level servers.
- **Black spots**: High-level servers.
- **Black tessellation** (High-level network): Voronoi tessellation corresponding to black points.



Left: Low-level servers= Voronoi tessellation.
Right: Low-level servers=Poisson line tessellation.

1 Random tessellations

2 STIT tessellations

3 Direct construction of the model

Parameters of the construction

Let $\mathcal{H}$ be the class of hyperplanes of $\mathbb{R}^d$.

- **Intensity:** $a > 0$.
- **Stationary measure** ν **on** $\mathcal{H}$, i.e. invariant under the action of translations, and locally finite.
- W: Compact window of $\mathbb{R}^d$.

$$[W] = \{H \in \mathcal{H} : H \cap W \neq \emptyset\}.$$

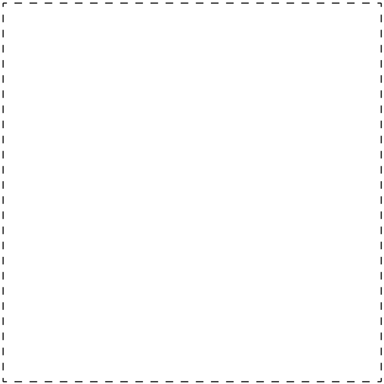
ν locally finite:

$$\nu([W]) < +\infty.$$

Renormalised restriction of ν to W:

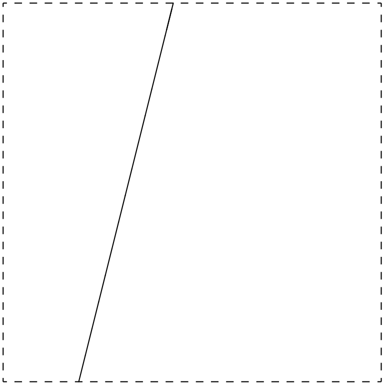
$$\nu_W(\cdot) = \frac{1}{\nu([W])}\nu([W] \cap \cdot).$$

Modelisation of cracking on compact window



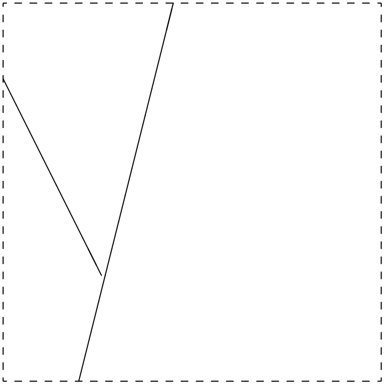
Start from a bounded window W, and after a random exponential time with rate $\nu([W])$, cut the window W by a random line drawn according to ν_W.

Modelisation of cracking on compact window



Each sub-cell C created behaves independantly: it is divided after a random time $\sim \mathcal{E}(\nu([C]))$ by a random line drawn according to ν_C.

Modelisation of cracking on compact window



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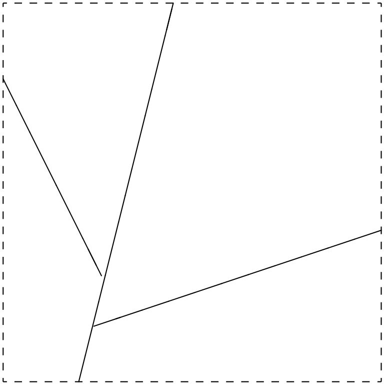

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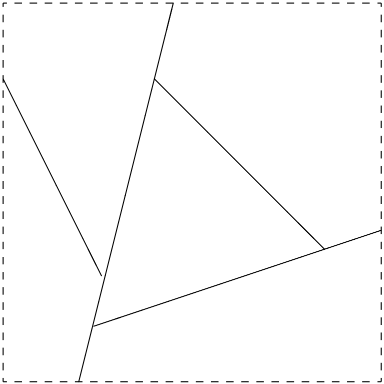

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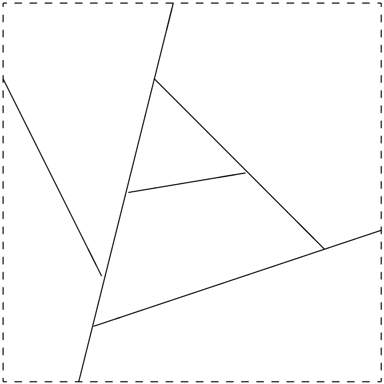

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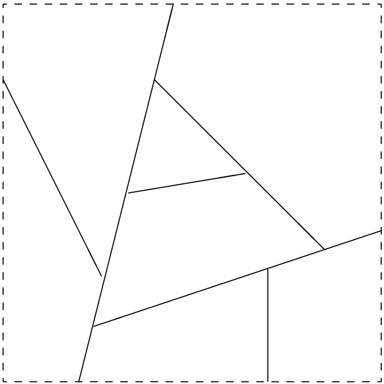

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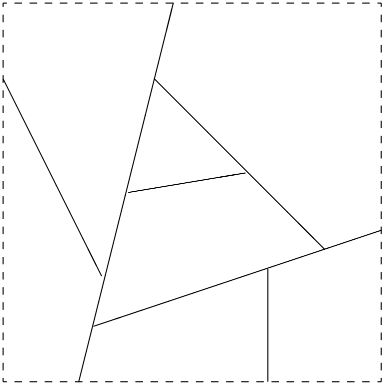
Modelisation of cracking on compact window



Stop the process when time a is reached.

Nagel, Weiss, Mecke (Jena)

- Model of cell division.
- Time process without memory (birth and death process).

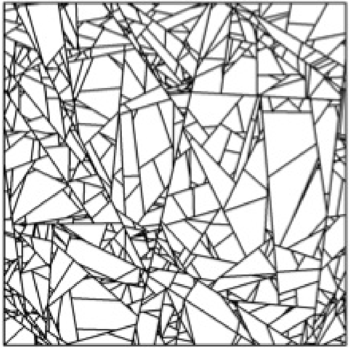
Call $M_{W,a,\nu}$ the obtained "tessellation",under the form of the union of the cells boundaries.

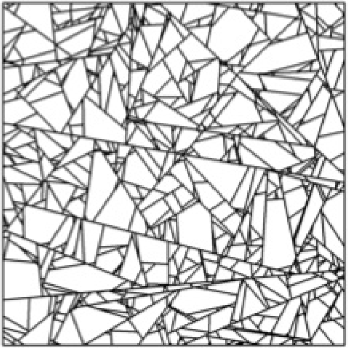
$$M_{W,a,\nu} = \bigcup_{\text{C cell existing at time } a} \partial C.$$

It is a random element with values in the class $\mathcal{F}(W)$ of closed sets of W. $\mathcal{F}(W)$ is endowed with the **Fell topology**, and the corresponding Borel σ-algebra $\mathcal{B}$.

Examples(Simulations: J. Ohser)

Simulations of isotropic STIT tessellations. (ν is stationary and isotropic, i.e. invariant under the action of rotations).

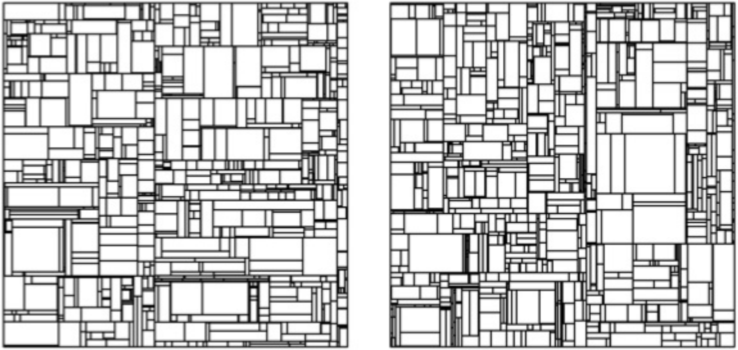




In the isotropic case, the death rate of a cell $\nu([C])$ is proportionnal to its perimeter.

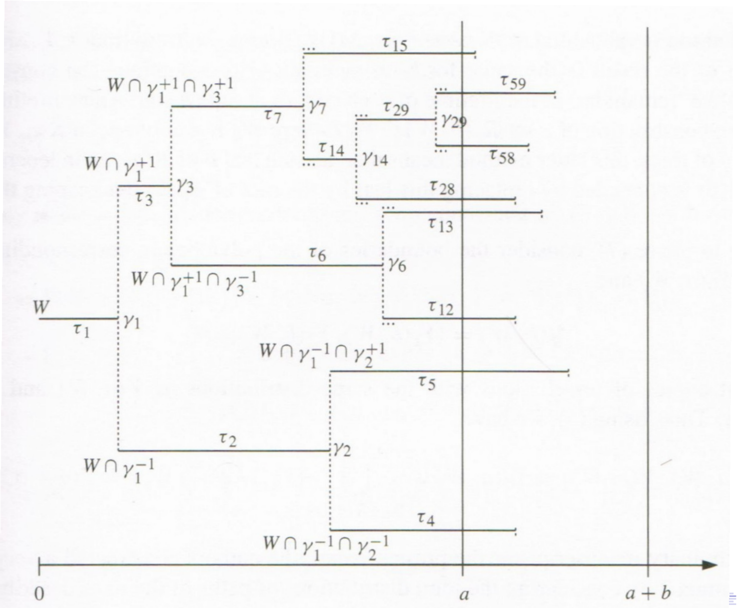
Non-isotropic example

Examples where ν is stationary (but not isotropic).



Here, $\nu([C])$ is proportionnal to the perimeter of the smallest rectangle with sides parallels to the axes containing C.

Binary tree



τ_{15}
$W \cap \gamma_1^{+1} \cap \gamma_3^{+1}$
τ_{59}
τ_7
γ_7
τ_{29}
γ_{29}
τ_{14}
γ_{14}
τ_{58}
$W \cap \gamma_1^{+1}$
γ_3
τ_3
τ_{28}
τ_{13}
τ_6
γ_6
$W \cap \gamma_1^{+1} \cap \gamma_3^{-1}$
W
τ_1
γ_1
τ_{12}
$W \cap \gamma_1^{-1} \cap \gamma_2^{+1}$
τ_5
τ_2
γ_2
$W \cap \gamma_1^{-1}$
τ_4
$W \cap \gamma_1^{-1} \cap \gamma_2^{-1}$
0
a
$a + b$

Tessellation on $\mathbb{R}^d$.

Consistency properties

Nagel and Weiss 2005:

If $W \subseteq W'$, then

$$M_{W,a,\nu} \cap \mathrm{int}(W) \stackrel{(d)}{=} M_{W',a,\nu} \cap \mathrm{int}(W).$$

Theorem

Let $\{W_i;\ i \in \mathbb{N}\}$ be a family of compact windows such that

(i) $W_i \uparrow \mathbb{R}^d$,

(ii) $W_i \subset \mathrm{int}(W_{i+1})$.

If a family of random closed sets $\{F_{W_i} \subseteq W_i\}$ satisfy

$$F_{W_i} \cap \mathrm{int}(W_i) \stackrel{(d)}{=} F_{W_j} \cap \mathrm{int}(W_i),\ j > i,$$

then there exists a random closed set F of $\mathbb{R}^d$ such that
$F \cap \mathrm{int}(W_i) \stackrel{(d)}{=} F_{W_i},\ i \in \mathbb{N}.$

There exists a random tessellation $M_{a,\nu} \in \mathcal{F}(\mathbb{R}^d)$ such that

$$(M_{a,\nu} \cap W) \cup \partial W \overset{(d)}{=} M_{a,\nu,W}$$

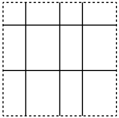
for all compact W. It is the **STIT tessellation with parameters a and ν.** We have

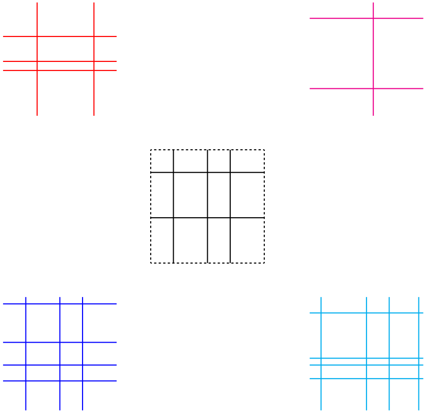
$$a = \mathbb{E}\mathcal{H}^{d-1}(M \cap [0,1]^d)$$

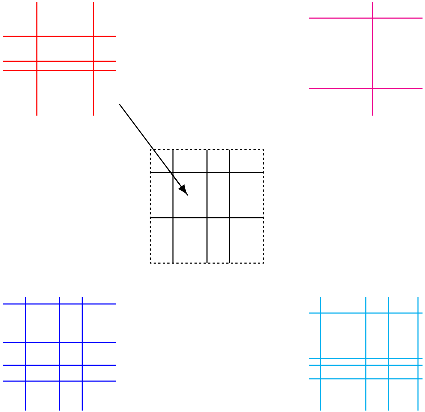
and a is the intensity.

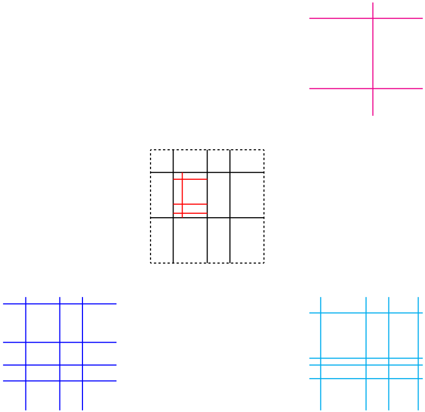
There exists a direct construction of the tessellation with the help of a point process on $\mathcal{H} \times \mathbb{R}_+$ (marked point process of hyperplanes).

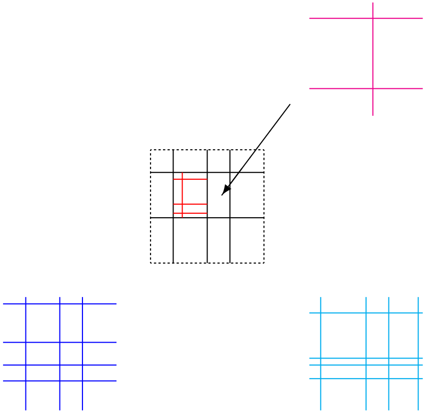
Iteration

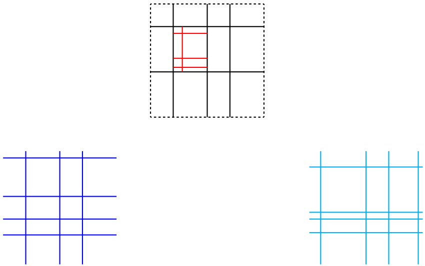


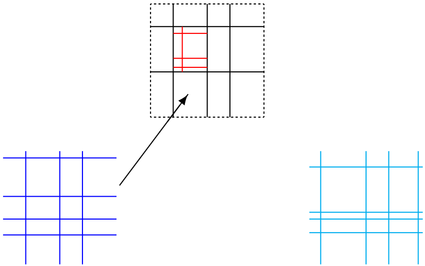


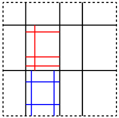


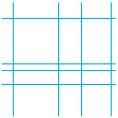


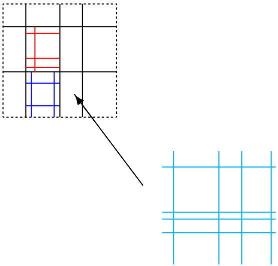


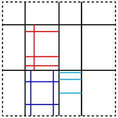




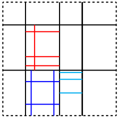








Rescaling



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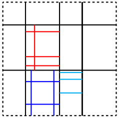
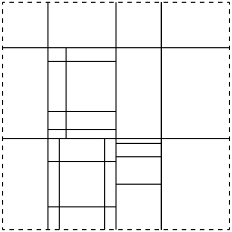


$\times\ 2$
$\Rightarrow$



Iteration

Let M, M' be two random tessellations.

- $C_1, C_2, \dots$ cells of M.
- $M'_1, M'_2, \dots$ independent copies of M', independent of the C_i.

Define the iterate of M and $M' by$

$$M \boxplus M' = 2 \cup_i \cup_{C_j \text{cell of } M'_i} \partial(C_j \cap C_i).$$

- It is a definition in distribution.
- The operation is not commutative.

STable under ITeration (Mecke, Nagel, Weiss)

Every STIT tessellation $M_{a,\nu}$ satisfies

$$M_{a,\nu} \boxplus M_{a,\nu} \overset{(d)}{=} M_{a,\nu}.$$

Furthermore, every random tessellation M that satisfies this property is a STIT.

Attraction pool

Let M be a stationary tessellation. Define by induction

$$\begin{cases} M_1 = M, \\ M_{n+1} = M_n \boxplus M_n. \end{cases}$$

Then

$$M_n \Rightarrow M_{a,\nu},$$

for a certain STIT tessellation $M_{a,\nu}$.

Mixing properties

A stationary tessellation M is mixing if

$$\mathbb{P}(M \cap K = \emptyset, M \cap (K' + h) = \emptyset) \to_{\|h\| \to \infty} \mathbb{P}(M \cap K = \emptyset)\mathbb{P}(M \cap K' = \emptyset),$$

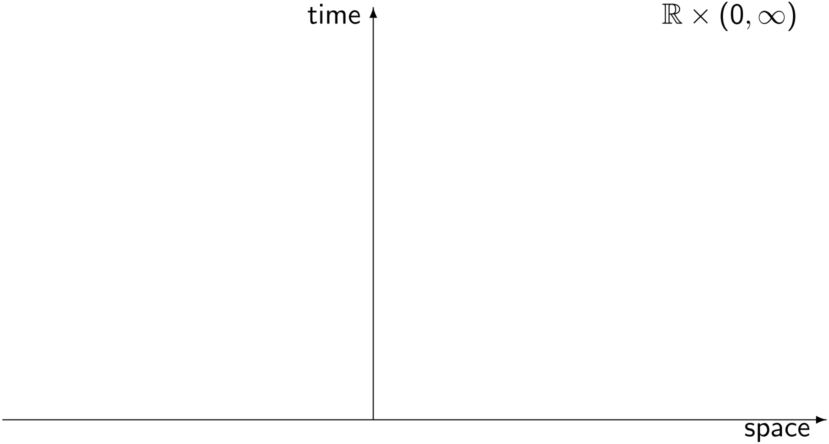
for all compacts K, K'.

Theorem (L.,2009)

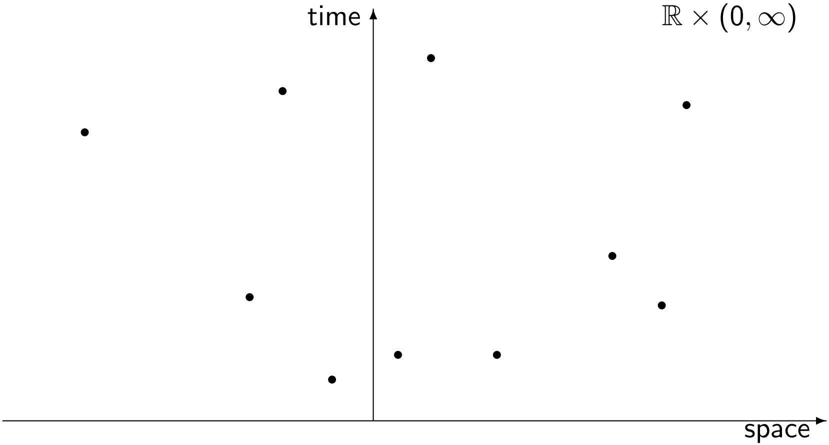
Let M be a STIT tessellation. Then for all compacts K and K',

$$\mathbb{P}(K \cap M = \emptyset, (K' + h) \cap M = \emptyset) - \mathbb{P}(K \cap M = \emptyset)\mathbb{P}((K' + h) \cap M = \emptyset) = O(1/\|h\|)$$

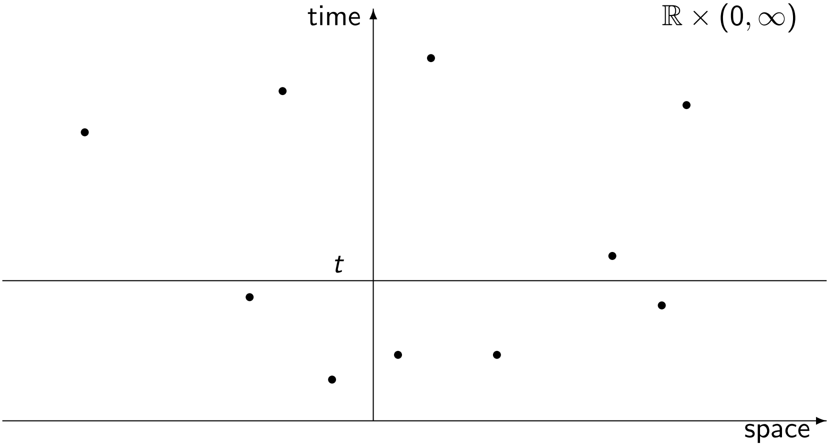
Poisson point processes, marked with birth times



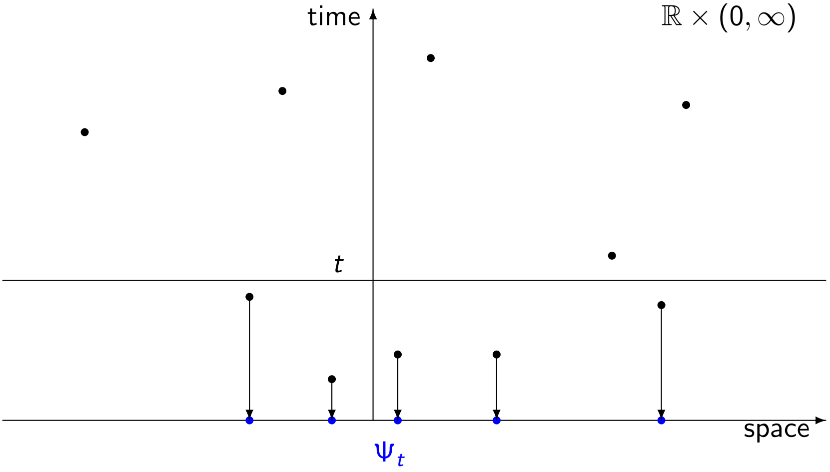
Poisson point processes, marked with birth times



2. Poisson point processes, marked with birth times



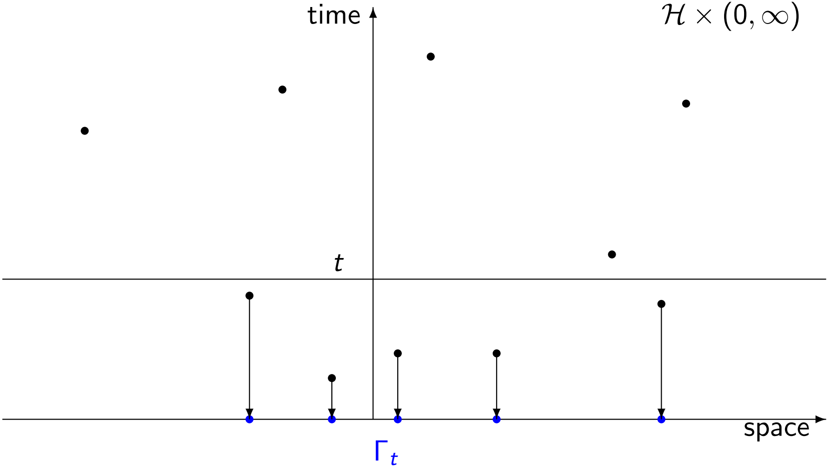
Poisson point processes, marked with birth times

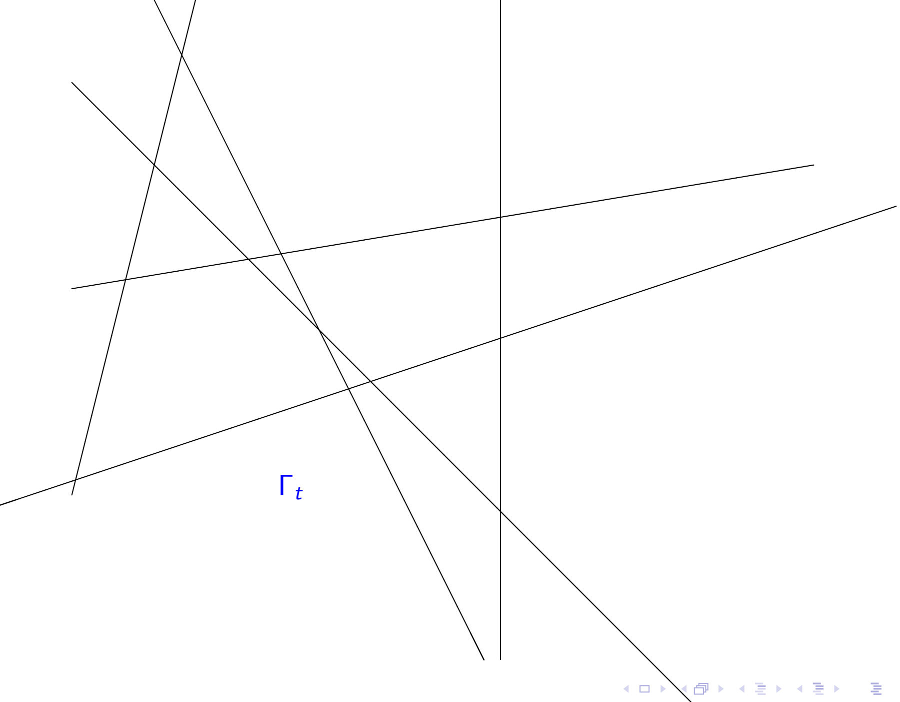


Now replace $\mathbb{R}$ by

$\mathcal{H}$ ... the set of all lines in $\mathbb{R}^2$

Poisson line processes Γ_t, marked with birth times




Γ_t

Poisson point process Γ on $\mathcal{H} \times (0, \infty)$
with intensity measure $\nu \times \ell_+$
ν translation invariant on $\mathcal{H}$
ℓ_+ ... Lebesgue measure on $(0, \infty)$
For all $t > 0$

$$\Gamma_t = \{h : \ (h, s) \in \Gamma : \ s < t\}$$

is a spatially homogeneous Poisson line process in $\mathbb{R}^2$.

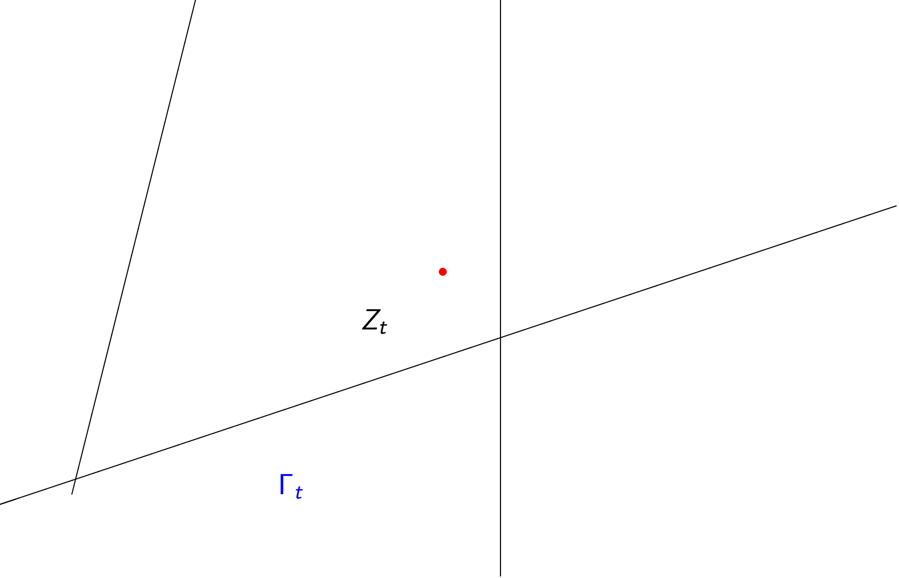
A preliminary construction

How to start a division of the whole plane when all segments have to have a finite length?

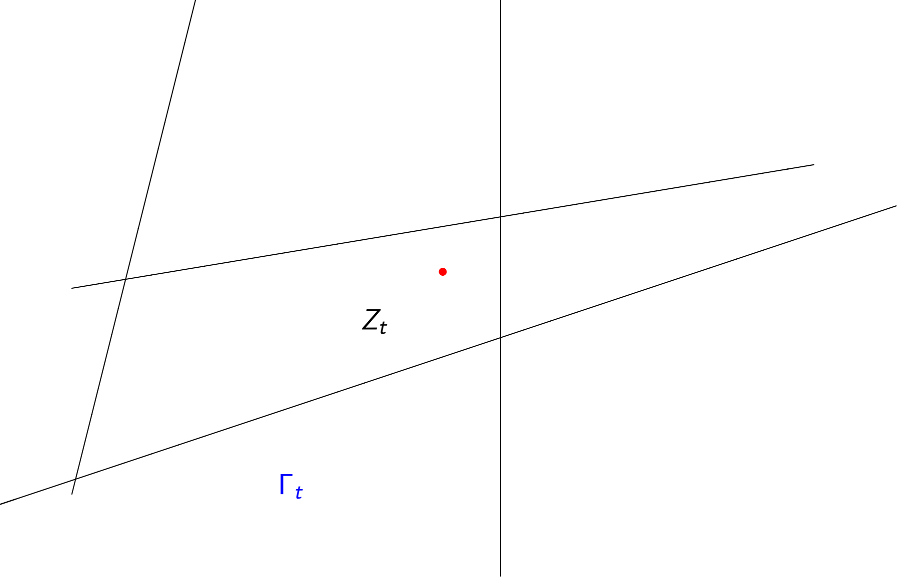
And all nodes are of T-type.

The crucial idea (by Joseph Mecke):

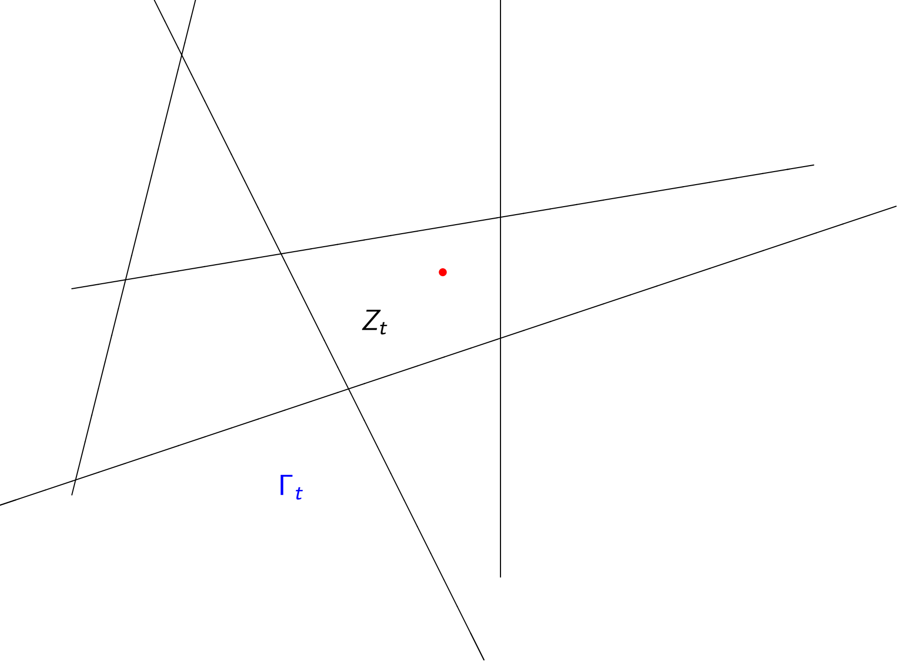
Consider the process $(Z_t)_{t>0}$ of the o-cells of $(\Gamma_t)_{t>0}$.



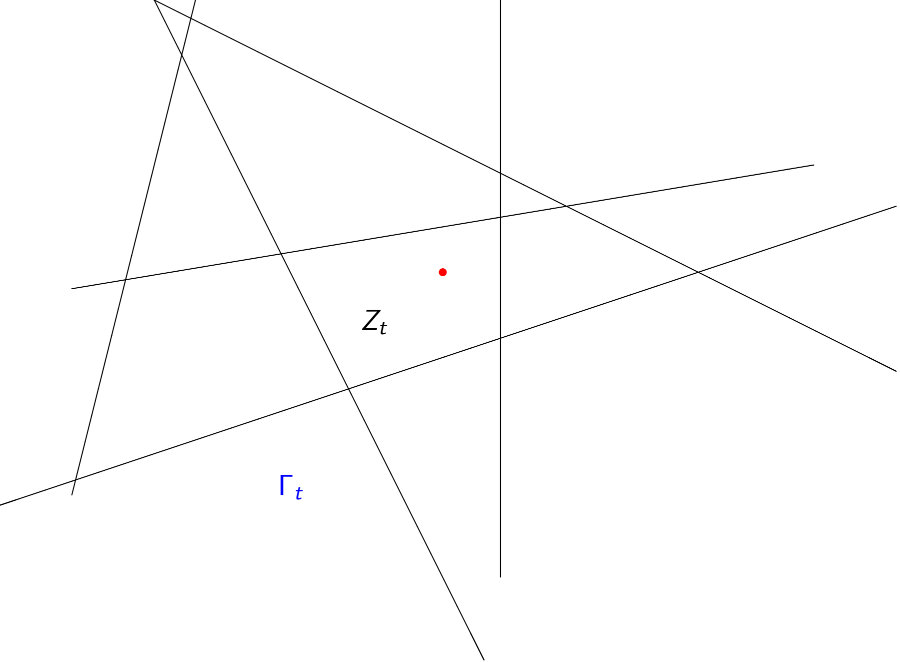
Z_t
Γ_t

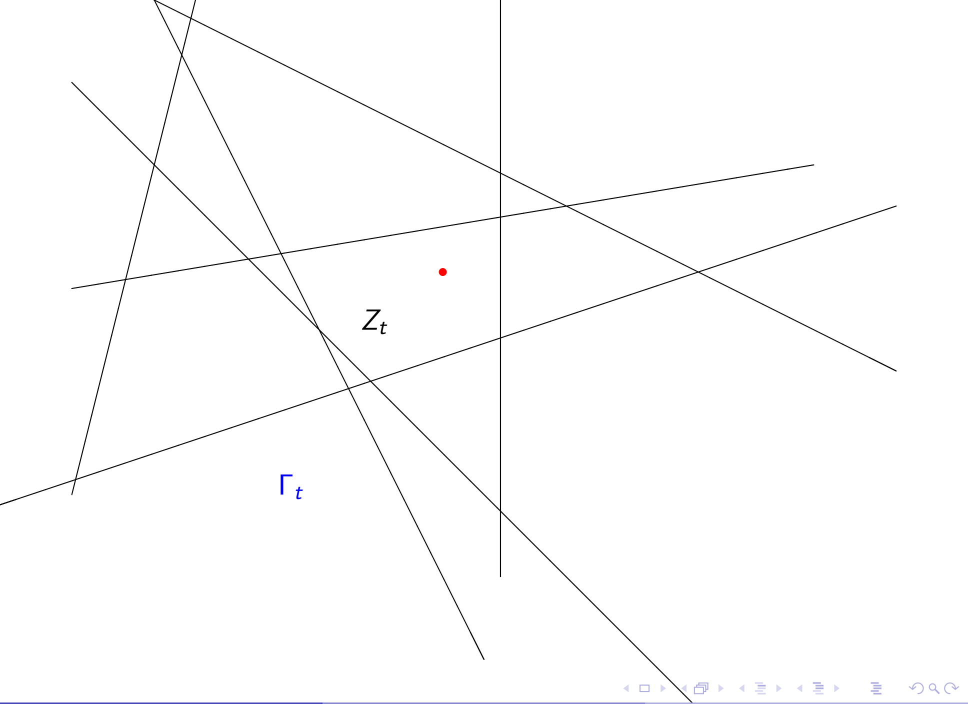


Z_t
Γ_t



Z_t
Γ_t


Z_t
Γ_t


Z_t
Γ_t

For all $t > 0$ the Poisson line process Γ_t is a.s. not empty.
Assume that the directional distribution $\mathcal{R}$ of the lines is not concentrated in a single point.
Then Γ_t generates a tessellation with a compact convex polygon Z_t that a.s. contains the origin o in its interior.
$\Longrightarrow$ Stochastic process $(Z_t)_{t>0}$ of o-cells of $(\Gamma_t)_{t>0}$.

The isotony

$$\Gamma_{t_1} \subseteq \Gamma_{t_2} \quad \text{for} \quad t_1 < t_2$$

implies

$$Z_{t_1} \supseteq Z_{t_2} \quad \text{for} \quad t_1 < t_2.$$

The process $(Z_t)_{t>0}$ is piecewise constant.
$\mathbb{Z} \ldots$ the set of all integers.
Monotonic sequence $(\sigma_k)_{k\in\mathbb{Z}}$ of times where $(Z_t)_{t>0}$ changes its state.
σ_k is the time when the interior *int* $Z_{\sigma_{k-1}}$ is hit by a line from Γ.

$$\ldots < \sigma_{-2} < \sigma_{-1} < \sigma_0 < \sigma_1 < 1 < \sigma_2 < \ldots$$

We obtain

$$\lim_{k \to -\infty} \sigma_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \sigma_k = \infty$$

Crucial for the construction

$$Z_{\sigma_k} \uparrow \mathbb{R}^2 \ \ a.s. \quad \text{if} \quad k \downarrow -\infty$$

Also

$$Z_{\sigma_k} \downarrow \{o\} \ \ a.s. \quad \text{if} \quad k \uparrow \infty$$

A preliminary tessellation of $\mathbb{R}^2$

For $t > 0$ we define a tessellation Ψ_t with the cells

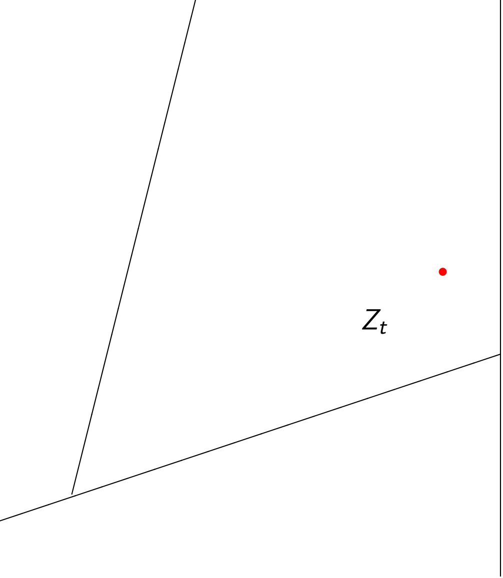
$$Z_t \quad \text{and} \quad \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}, \quad \sigma_k < t.$$

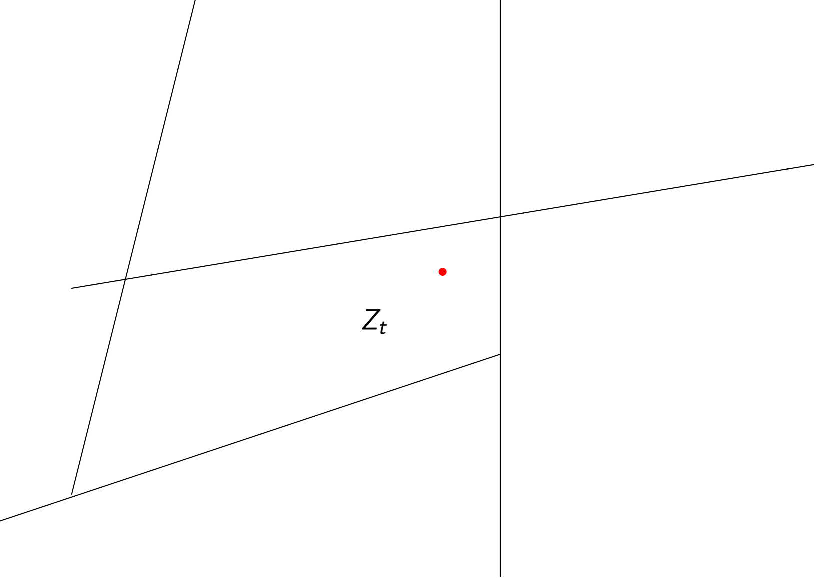
All these cells are compact, convex and have a pairwise disjoint interior.
Due to

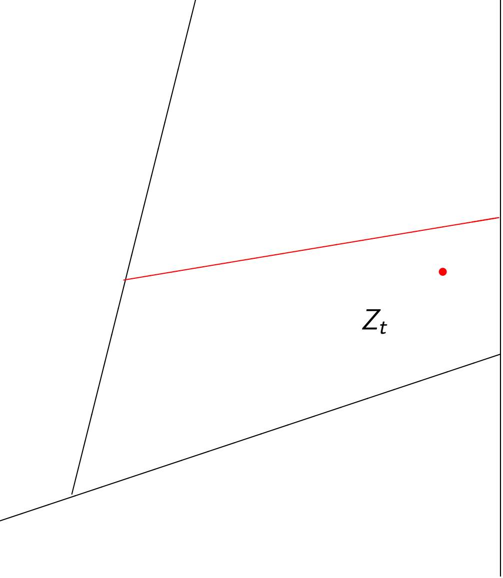
$$Z_{\sigma_k} \uparrow \mathbb{R}^2 \;\; a.s. \quad \text{if} \quad k \downarrow -\infty$$

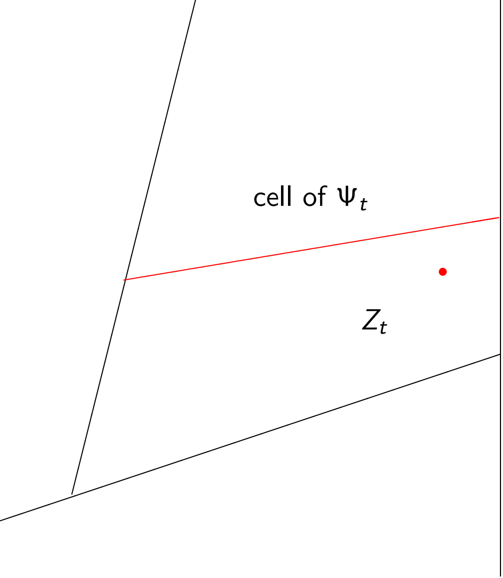
the cells fill the plane, i.e. for all $t > 0$

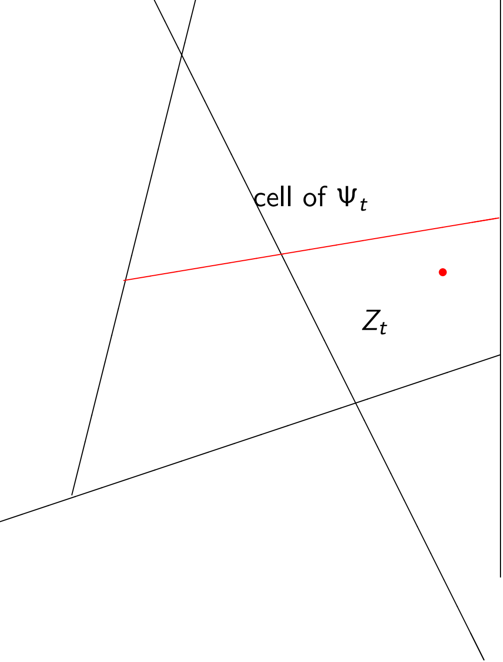
$$Z_t \cup \bigcup_{\sigma_k < t} \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} = \mathbb{R}^2$$

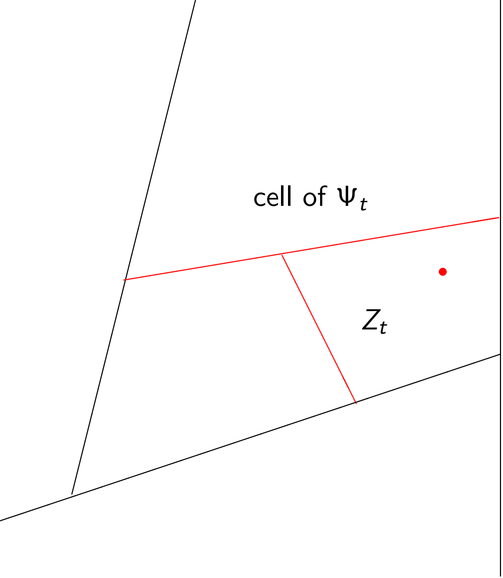

Z_t

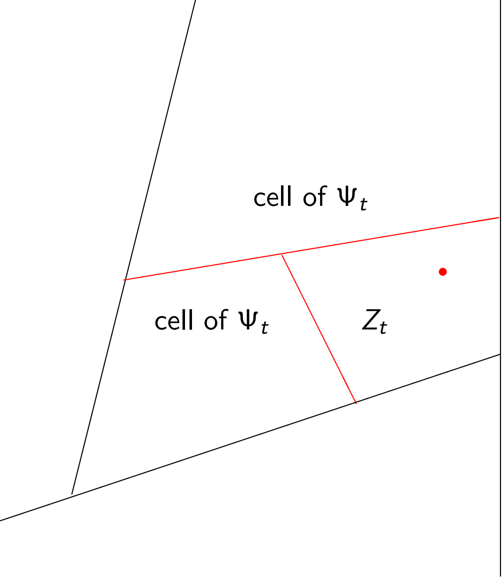

Z_t


Z_t

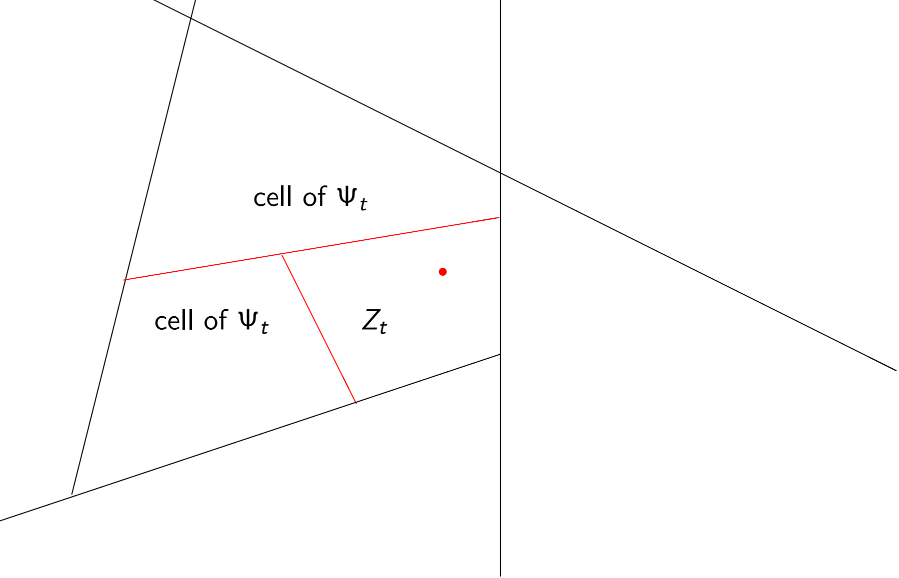

cell of Ψ_t
Z_t

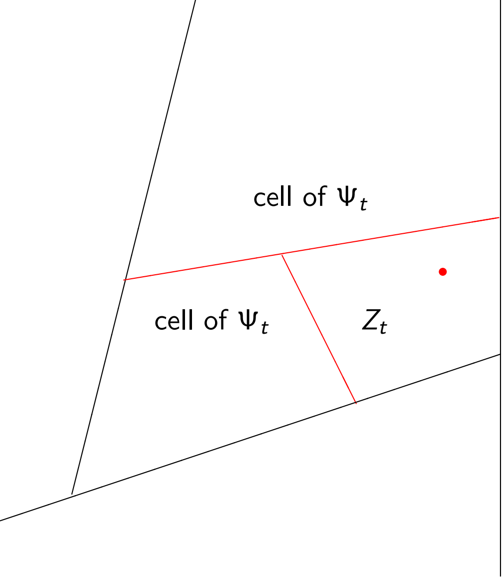

cell of Ψ_t
Z_t


cell of Ψ_t
Z_t



cell of Ψ_t
cell of Ψ_t
Z_t


cell of Ψ_t
cell of Ψ_t
Z_t



cell of Ψ_t
cell of Ψ_t
Z_t

The random tessellation Ψ_t is non-homogeneous (spatially non-stationary).

Intuitively, the older cells of Ψ_t,

$\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ with σ_k close to the time 0 (the moment of the 'Big Bang')

are very far from the origin $o \in \mathbb{R}^2$

and they tend to be larger than the younger ones.

The final steps of the construction – generating a spatially homogeneous random tessellation

For $t > 0$: non-homogeneous tessellation Ψ_t with the cells

$$Z_t \quad \text{and} \quad \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}, \quad \sigma_k < t.$$

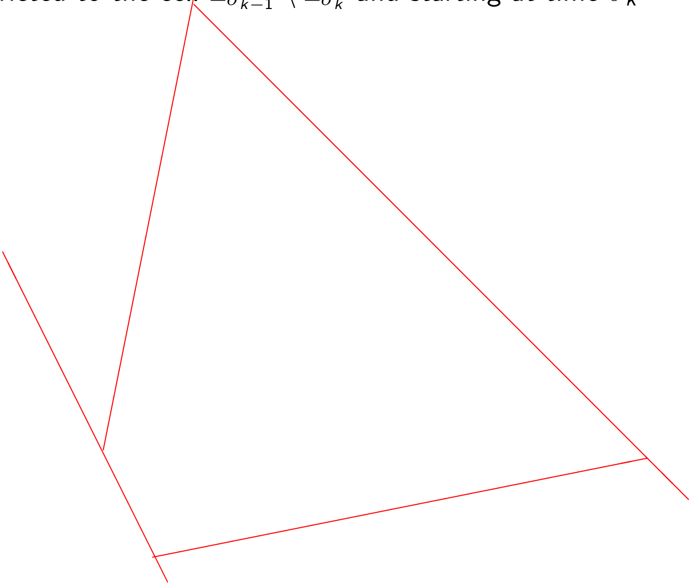
A cell

$$\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$$

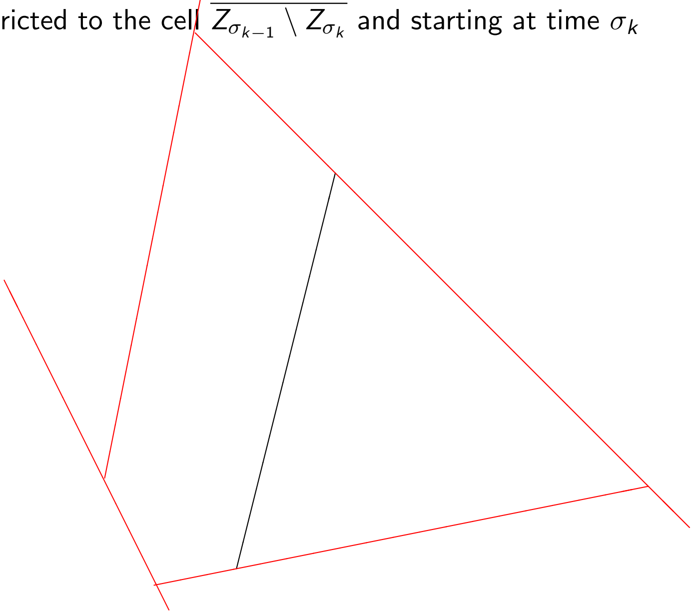
is born at the time $\sigma_k < t$.

During the time interval (σ_k, t) this bounded cell is divided by random chords as described in the beginning.

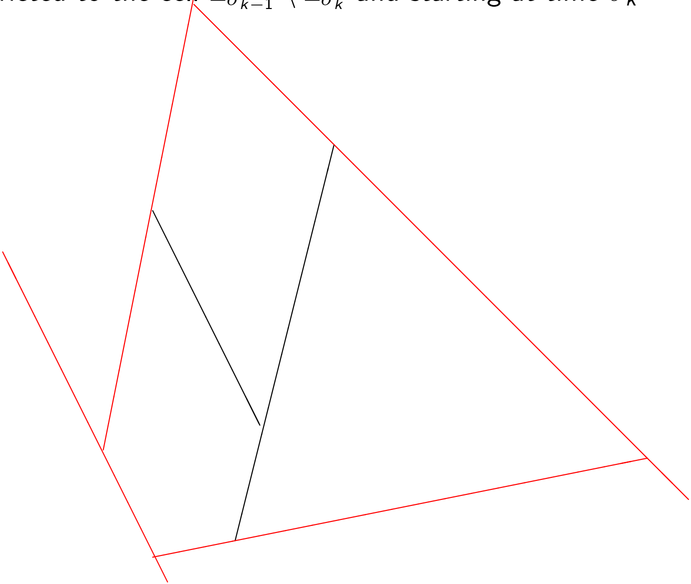
Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



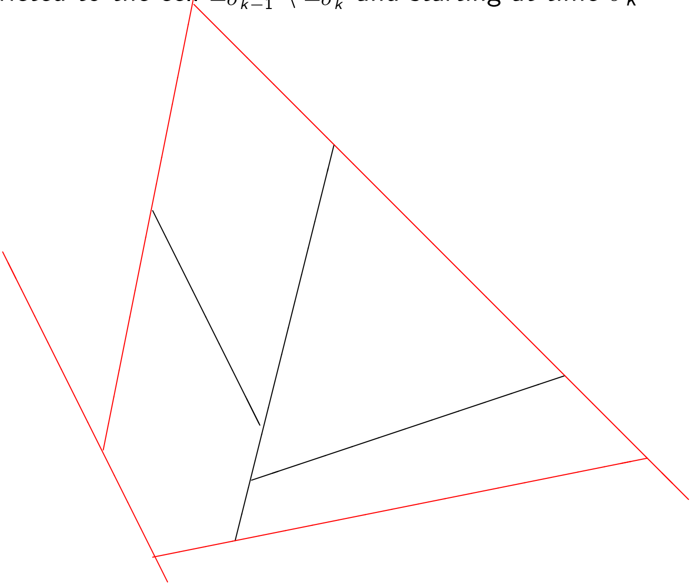
Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



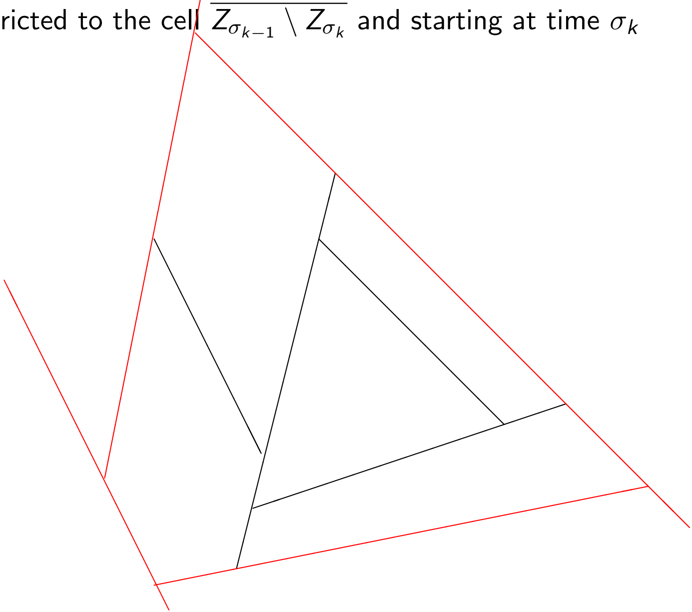
Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



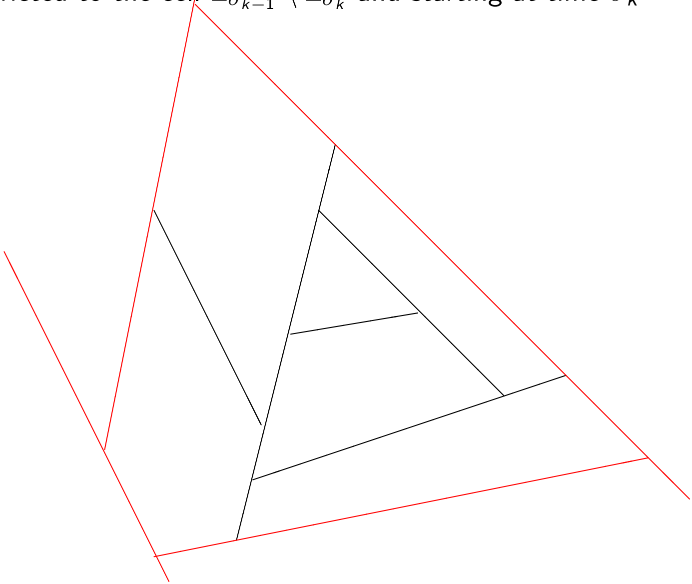
Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



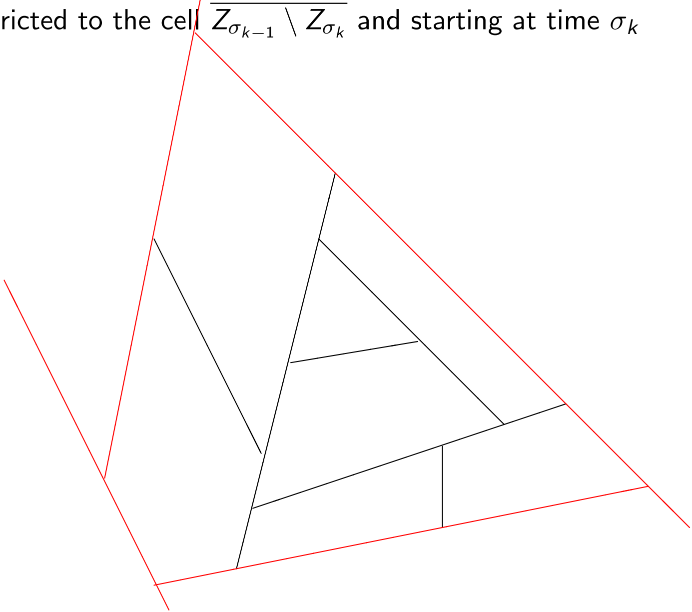
Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



... and finishing at time t.

Thus the cells $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ are filled during the time interval (σ_k, t) such that the resulting tessellation Φ_t

- is spatially homogeneous,
- STIT, i.e. stable under iteration/nesting of tessellations.

Mecke, J., Nagel, W., Weiß, V. (2008). A global construction of homogeneous random planar tessellations that are stable under iteration. *Stochastics: An Int. J. of Prob. and Stoch. Proc.* **80**, 51-67.

Mecke, J., Nagel, W., Weiß, V. (2008). The iteration of random tessellations and a construction of a homogeneous process of cell divisions. *Advances in Applied Probability* **40**, 49-59.

Nagel, W., Weiß, V. (2005). Crack STIT tessellations – characterization of the stationary random tessellations which are stable with respect to iteration.
Advances in Applied Probability **37**, 859-883.

Lachièze-Rey R., Mixing properties of STIT tessellations.
*Advances in Applied Probability***43.1** (march 2011)