

Ergodicité des mosaïques STIT.

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- 1 Mosaiques aléatoires
- 2 Mosaiques STIT
- 3 Construction directe du modèle

1 Mosaiques aléatoires

2 Mosaiques STIT

3 Construction directe du modèle

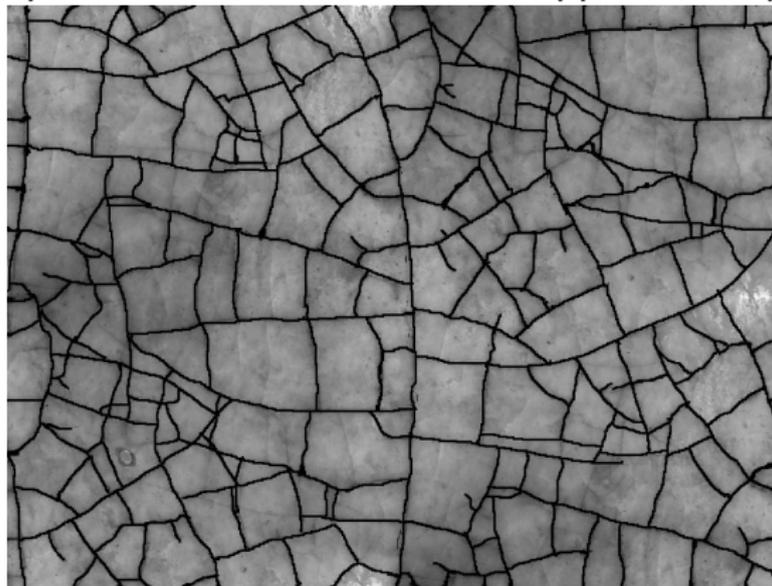
Definition

Mosaïque : Ensemble $\mathcal{C} = \{C_1, C_2, \dots\}$ de cellules convexes compactes, localement fini (pour tout compact K , $\{i; C_i \cap K \neq \emptyset\}$ est fini), tel que

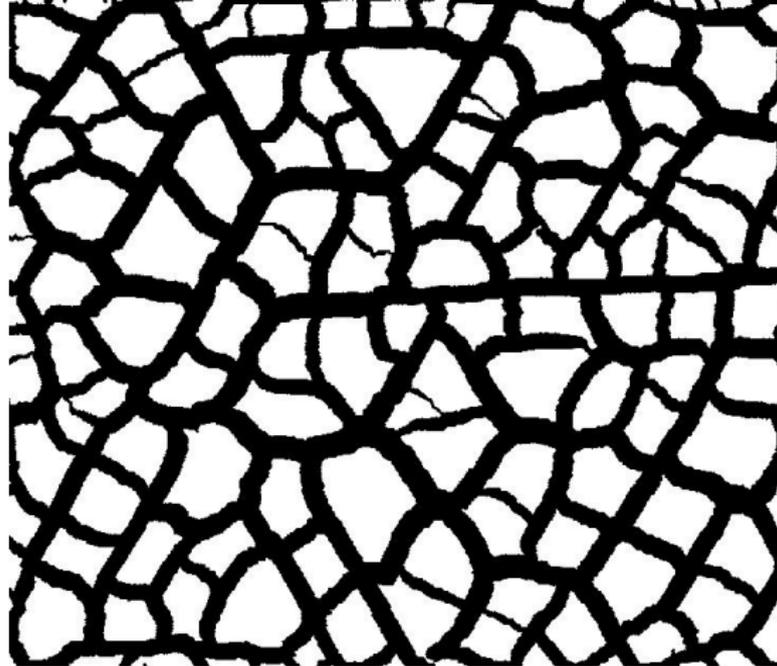
$$\begin{aligned}\mathbb{R}^d &= \cup_i C_i, \\ \partial C_i \cap \partial C_j &= \emptyset, \quad i \neq j.\end{aligned}$$

Le fermé correspondant est $M = \cup_i \partial C_i$.

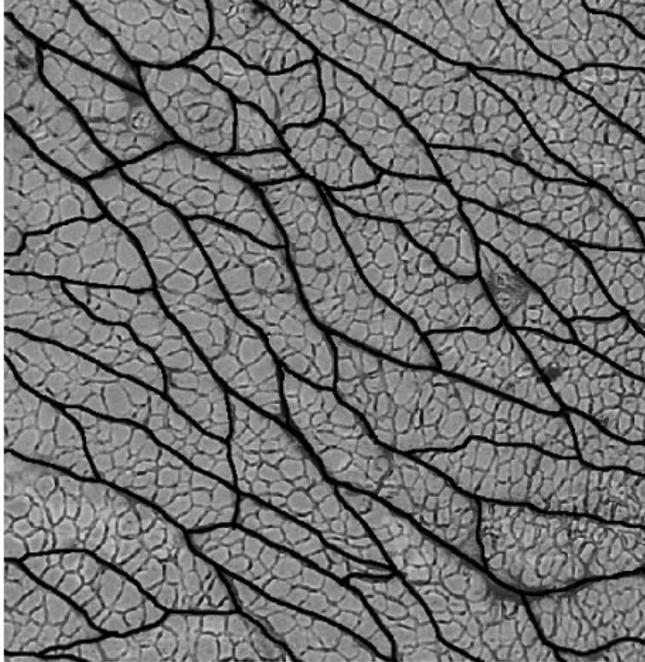
Quelques exemples de structures réelles – Applications potentielles.



Craquelée sur une surface de céramique (Photo : G. Weil)



Simulation de craquement d'un sol séché (H.-J. Vogel)

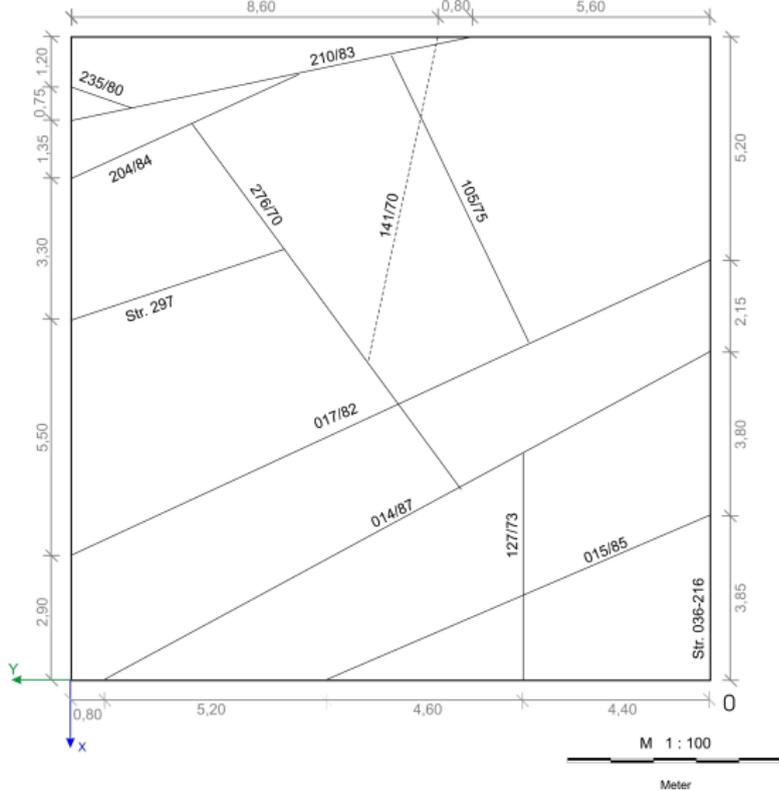


Tissu dans un muscle de rat (I. Erzen)



Perla 2.jpg

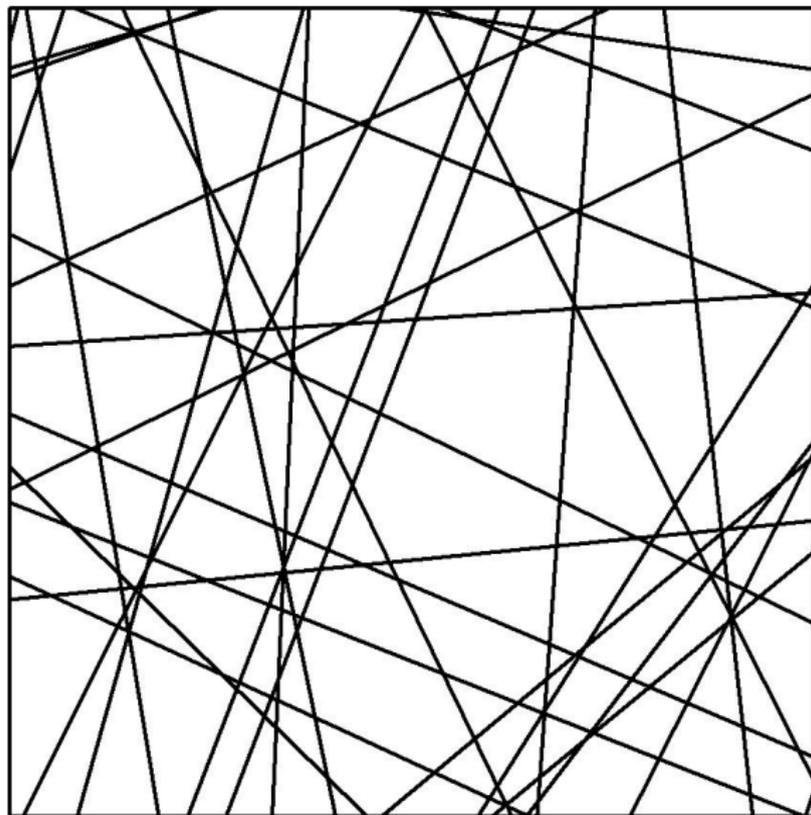
Joints dans le granit (D. Nikolayev, S. Siegesmund, S. Mosch, A. Hoffmann)



Gris Perla.jpg
Schéma des joints dans le granite.

Mosaïque poissonnienne

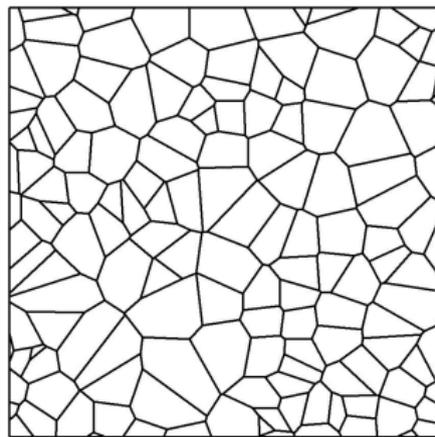
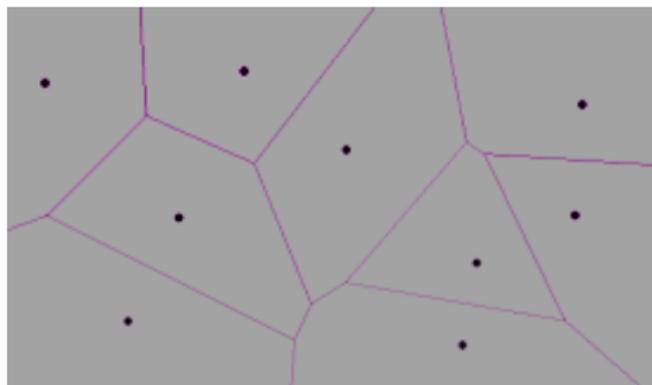
On tire des droites "aléatoirement".



Mosaïques de Voronoi, Delaunay

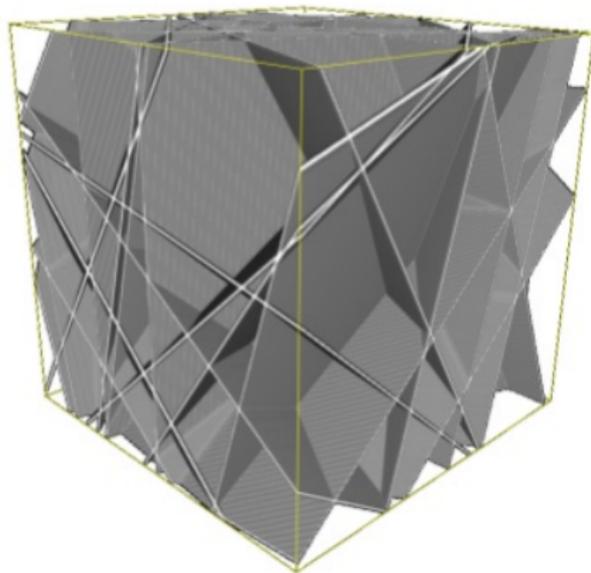
- Π : Processus ponctuel sur \mathbb{R}^d .
- $x \in \Pi$, V_x : Ensemble des points de \mathbb{R}^d dont x est le plus proche élément de Π ,

$$V_x = \{y \in \mathbb{R}^2; \|x - y\| = \inf_{x' \in \Pi} \|x' - y\|\}$$



Mosaïques en 3D

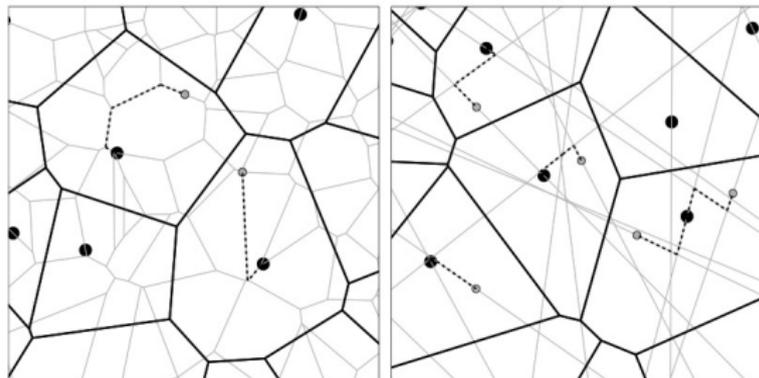
Mosaïque d'hyperplans Poissonniens en dimension 3 :



Mosaïques mixtes

(Schmidt, Voss 2010)

- **Mosaïque grise** : Réseau de bas niveau.
- **Points noirs** : Serveurs de haut niveau.
- **Mosaïque noire** (Réseau de haut niveau) : Mosaïque de Voronoi correspondant aux points noirs.



gauche : Serveur de bas niveau= mosaïque de Voronoi.

droite : Seveurs de bas niveau=mosaïque poissonnienne de droites.

1 Mosaiques aléatoires

2 Mosaiques STIT

3 Construction directe du modèle

Paramètres de la construction

Soit \mathcal{H} l'ensemble des hyperplans de \mathbb{R}^d .

- **Intensité** : $a > 0$.
- **Mesure stationnaire ν sur \mathcal{H}** , i.e. invariante sous l'action des translations, et localement finie.
- W : Fenêtre compacte de \mathbb{R}^d .

$$[W] = \{H \in \mathcal{H} : H \cap W \neq \emptyset\}.$$

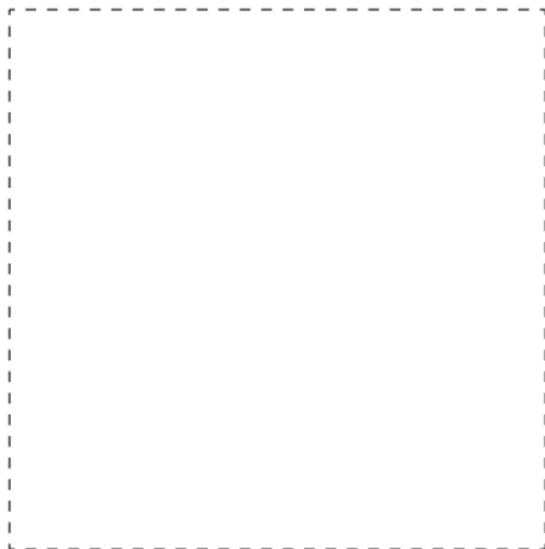
ν localement finie :

$$\nu([W]) < +\infty.$$

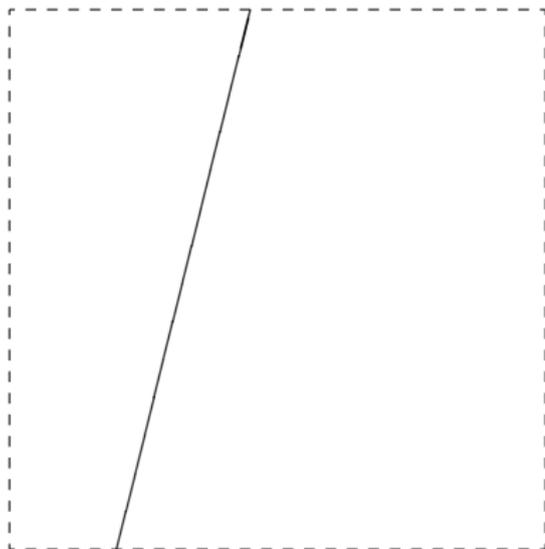
Restriction renormalisée de ν à W :

$$\nu_W(\cdot) = \frac{1}{\nu([W])} \nu([W] \cap \cdot).$$

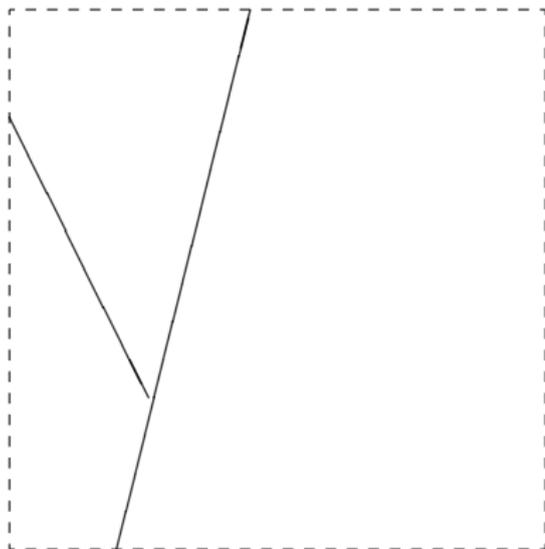
Modélisation de craquèlement sur une fenêtre bornée



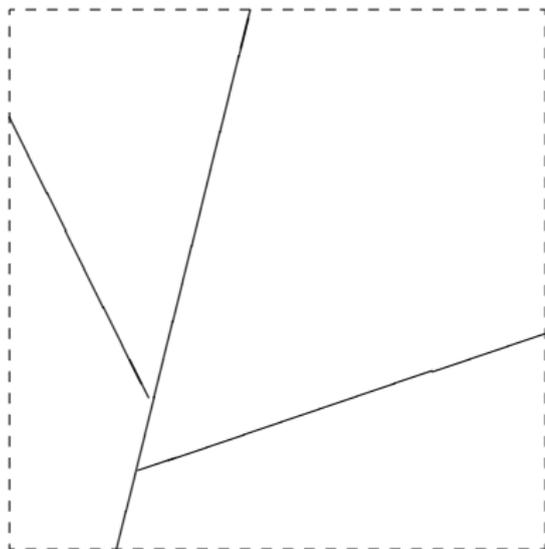
On part d'une fenêtre bornée W , et au bout d'un temps exponentiel de taux $\nu([W])$, on coupe la fenêtre W par une droite tirée aléatoirement selon ν_W .



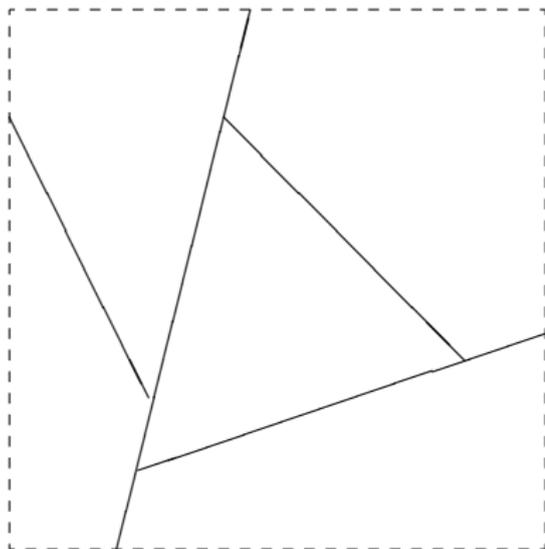
Chaque sous-cellule C ainsi créée se comporte indépendamment : Elle est coupée au bout d'un temps $\sim \mathcal{E}(\nu([C]))$ par une droite tirée selon ν_C .



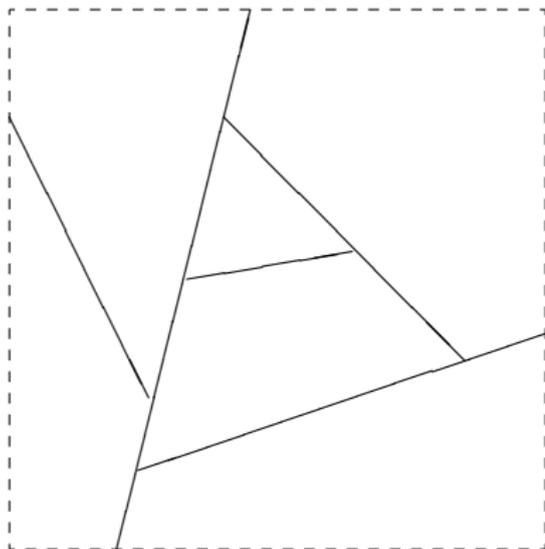
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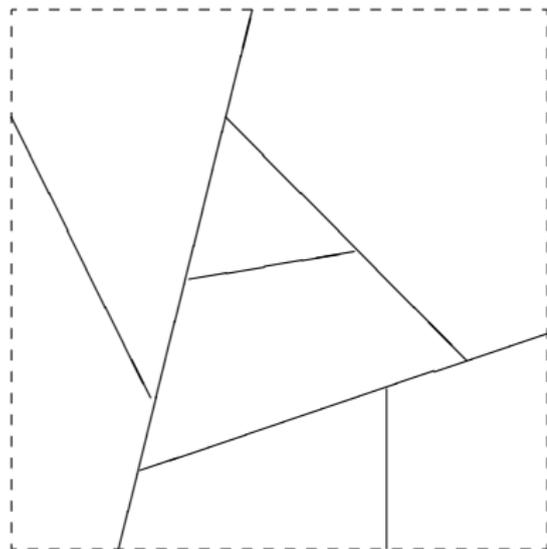
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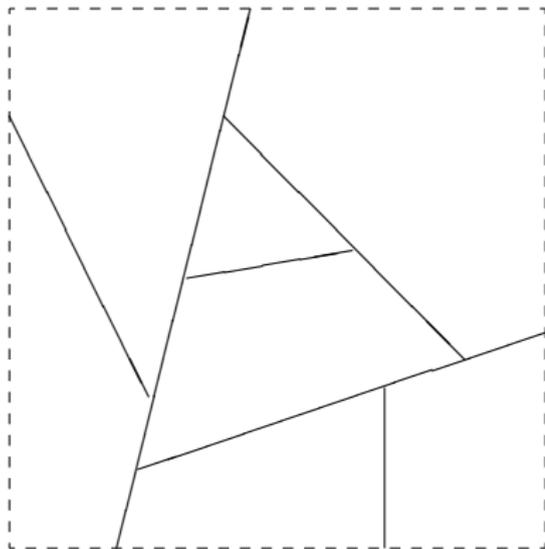


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modélisation de craquèlement sur une fenêtre bornée



Stop the process when time a is reached.

Nagel, Weiss, Mecke (Jena)

- Modèle de division cellulaire.
- Processus sans mémoire (processus de vie et de mort).

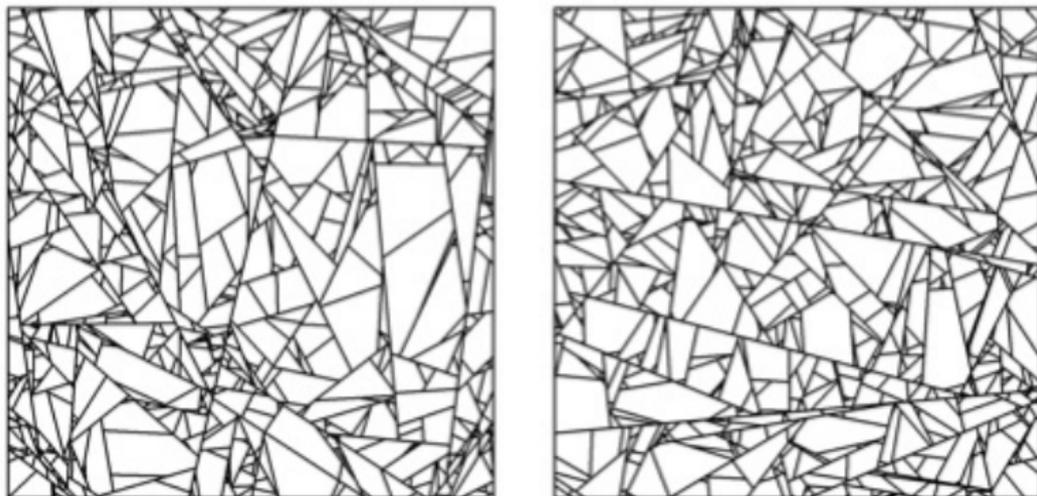
On appelle $M_{W,a,\nu}$ la “mosaïque” obtenue, sous forme de l’union des frontières des cellules

$$M_{W,a,\nu} = \bigcup_{C \text{ cellule au temps } a} \partial C.$$

C’est un élément aléatoire à valeurs dans l’ensemble $\mathcal{F}(W)$ des fermés de W . $\mathcal{F}(W)$ est muni de la **topologie de Fell**, et de la tribu Borélienne associée \mathcal{B} .

Exemples(Simulations : J. Ohser)

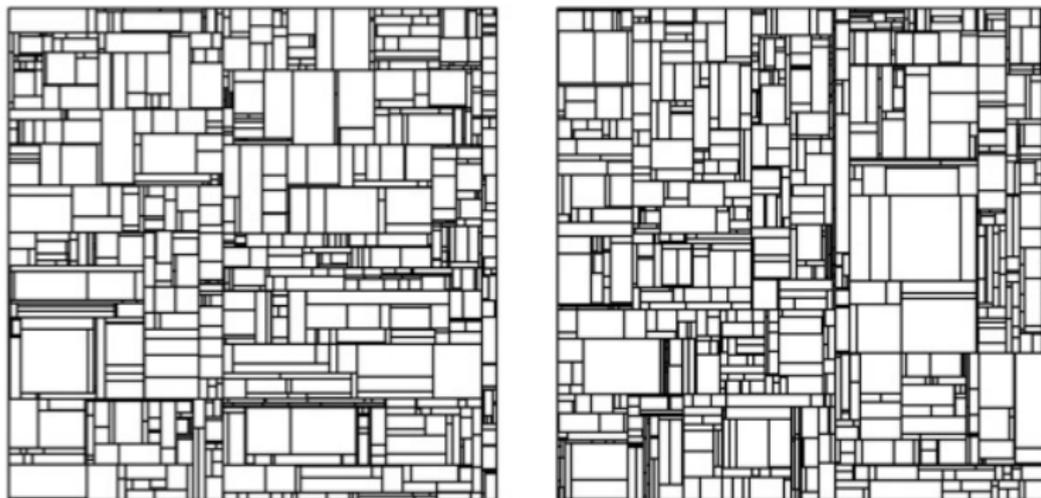
Simulations de mosaïques STIT isotropes. (ν est stationnaire et isotrope).



Dans le cas isotrope, le taux de mort d'une cellule $\nu([C])$ est proportionnel à son périmètre.

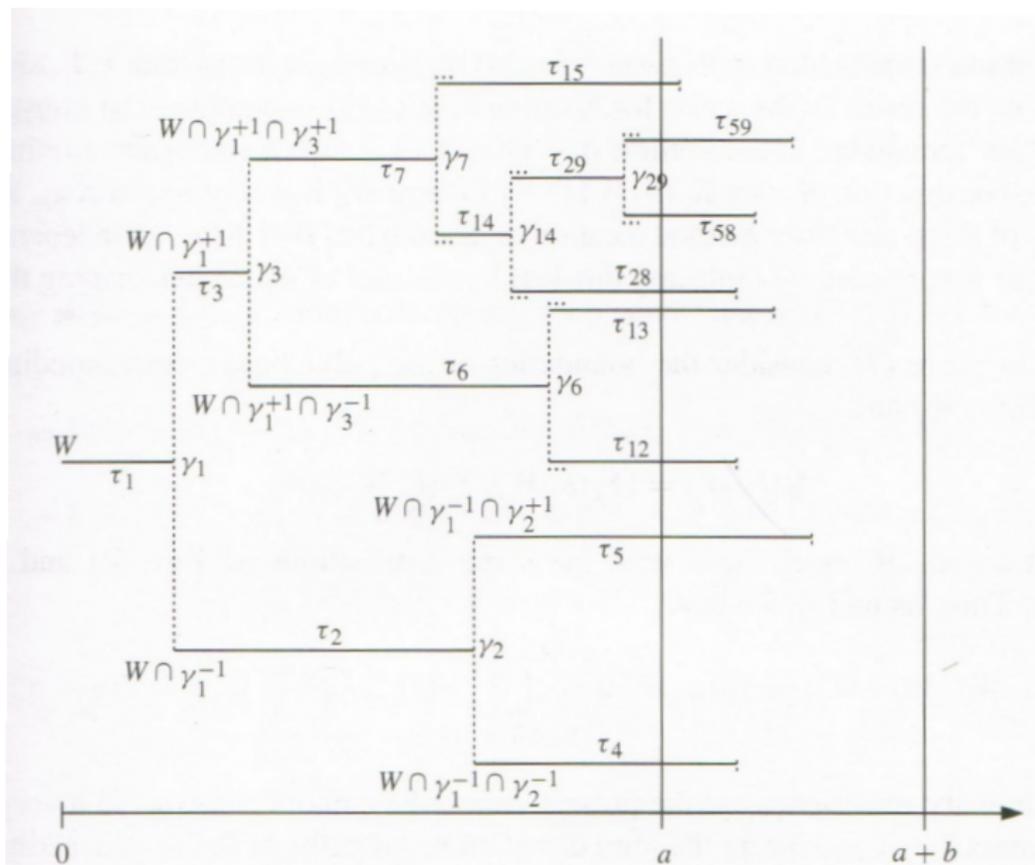
Exemple non-isotrope

Exemples où ν est stationnaire (mais pas isotrope).



Ici, $\nu([C])$ est proportionnel au périmètre du plus petit rectangle de côtés parallèle aux axes qui contient la cellule C .

Arbre binaire



Mosaïque sur \mathbb{R}^d .

Propriété de consistance

Nagel and Weiss 2005 :

Si $W \subseteq W'$, alors

$$M_{W,a,\nu} \cap \text{int}(W) \stackrel{(d)}{=} M_{W',a,\nu} \cap \text{int}(W).$$

Théorème

Soit une famille de compacts $\{W_i; i \in \mathbb{N}\}$ telle que

- (i) $W_i \uparrow \mathbb{R}^d$,
- (ii) $W_i \subset \text{int}(W_{i+1})$.

Si une famille de fermé aléatoires $\{F_{W_i}\}$ définis resp. sur les W_i vérifient

$$F_{W_i} \cap \text{int}(W_i) \stackrel{(d)}{=} F_{W_j} \cap \text{int}(W_i), j > i,$$

alors il existe un fermé aléatoire F de \mathbb{R}^d tel que

$$F \cap \text{int}(W_i) \stackrel{(d)}{=} F_{W_i}, i \in \mathbb{N}.$$

Il existe une mosaïque aléatoire $M_{a,\nu} \in \mathcal{F}(\mathbb{R}^d)$ telle que

$$(M_{a,\nu} \cap W) \cup \partial W \stackrel{(d)}{=} M_{a,\nu,W}$$

pour tout compact W . C'est la **mosaïque STIT de paramètres a et ν** .

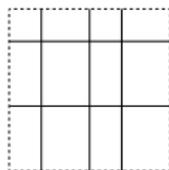
On a

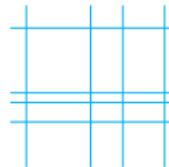
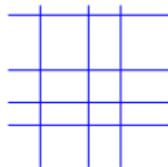
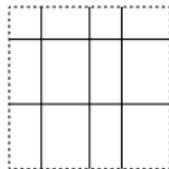
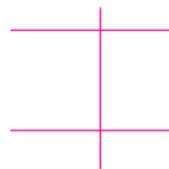
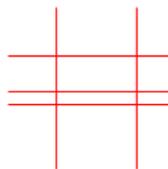
$$a = \mathbb{E} \mathcal{H}^{d-1}(M \cap [0, 1]^d)$$

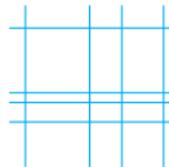
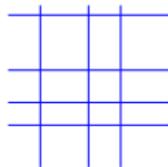
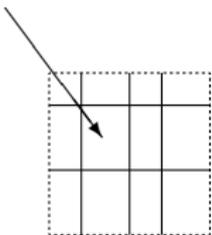
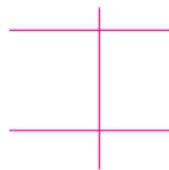
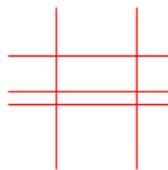
et a est l'intensité.

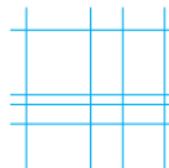
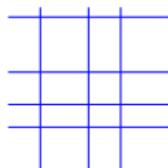
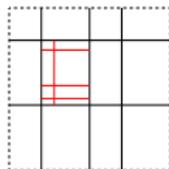
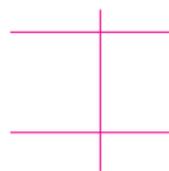
Il existe une construction directe de la mosaïque à l'aide d'un processus ponctuel sur $\mathcal{H} \times \mathbb{R}_+$.

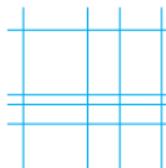
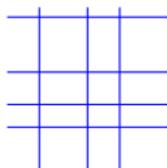
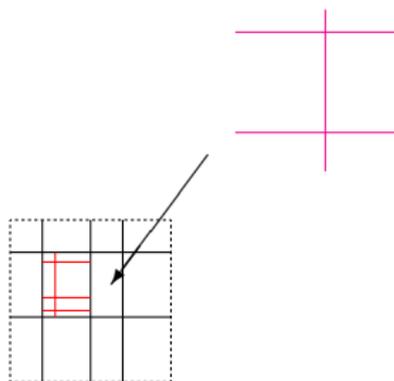
Itération

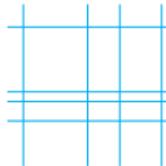
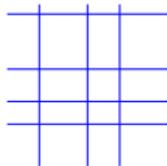
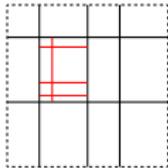


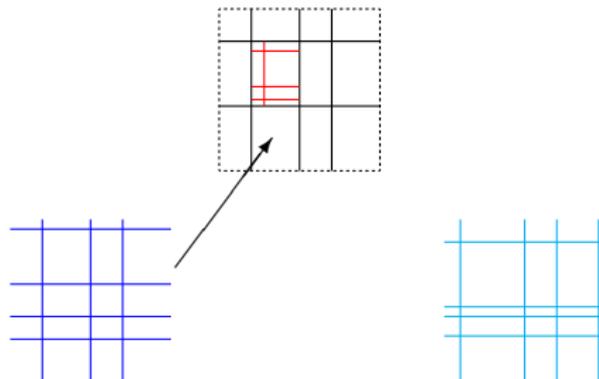


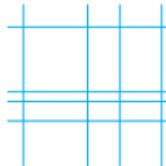
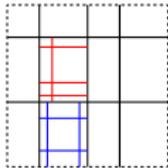


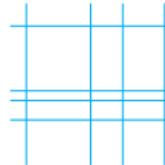
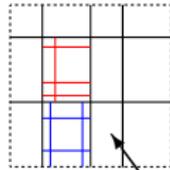


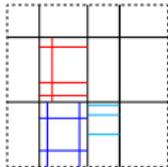




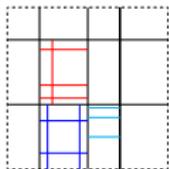




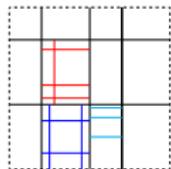




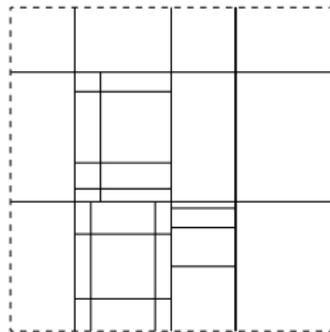
Rescaling



Rescaling



$\times 2$
 \Rightarrow



Iteration

Soit M, M' deux mosaïques aléatoires.

- C_1, C_2, \dots cellules de M .
- M'_1, M'_2, \dots des copies indépendantes de M' , et indépendantes des C_i .

Soit l'itérée de M et M'

$$M \boxplus M' = 2 \cup_i \cup_{C \text{ cellule de } M'_i} \partial C_i.$$

- C'est une définition en loi.
- L'opération n'est pas commutative.

STable to IIteration (Mecke, Nagel, Weiss)

Toute mosaïque STIT $M_{a,\nu}$ vérifie

$$M_{a,\nu} \boxplus M_{a,\nu} \stackrel{(d)}{=} M_{a,\nu}.$$

De plus, toute mosaïque M qui vérifie cette propriété est une mosaïque STIT.

Bassin d'attraction

Soit M une mosaïque stationnaire. On définit par récurrence

$$\begin{cases} M_1 = M, \\ M_{n+1} = M_n \boxplus M_n. \end{cases}$$

Alors

$$M_n \Rightarrow M_{a,\nu},$$

pour une certaine mosaïque STIT $M_{a,\nu}$.

Propriété de mélange

Une mosaïque stationnaire M vérifie la propriété de mélange si

$$\mathbb{P}(M \cap K = \emptyset, M \cap (K' + h) = \emptyset) \xrightarrow{\|h\| \rightarrow \infty} \mathbb{P}(M \cap K = \emptyset) \mathbb{P}(M \cap K' = \emptyset),$$

pour tous compacts K, K' .

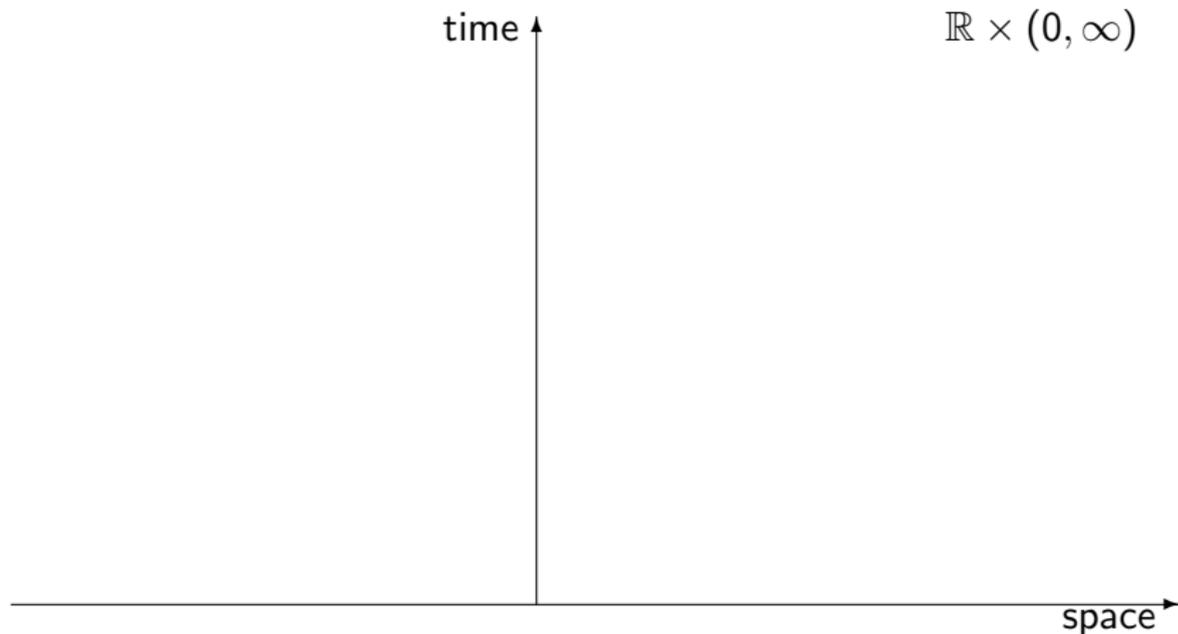
Théorème

Soit M une mosaïque STIT. Alors pour tous compacts K et K' ,

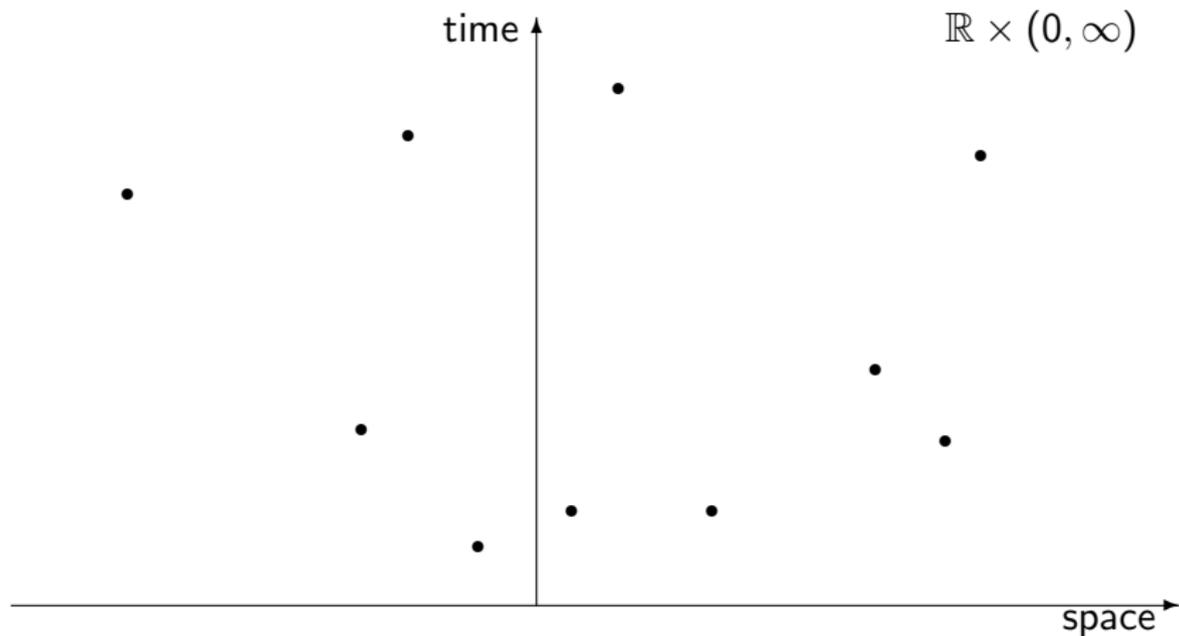
$$\begin{aligned} \mathbb{P}(K \cap M = \emptyset, (K' + h) \cap M = \emptyset) - \mathbb{P}(K \cap M = \emptyset) \mathbb{P}((K' + h) \cap M = \emptyset) \\ = O(1/\|h\|) \end{aligned}$$

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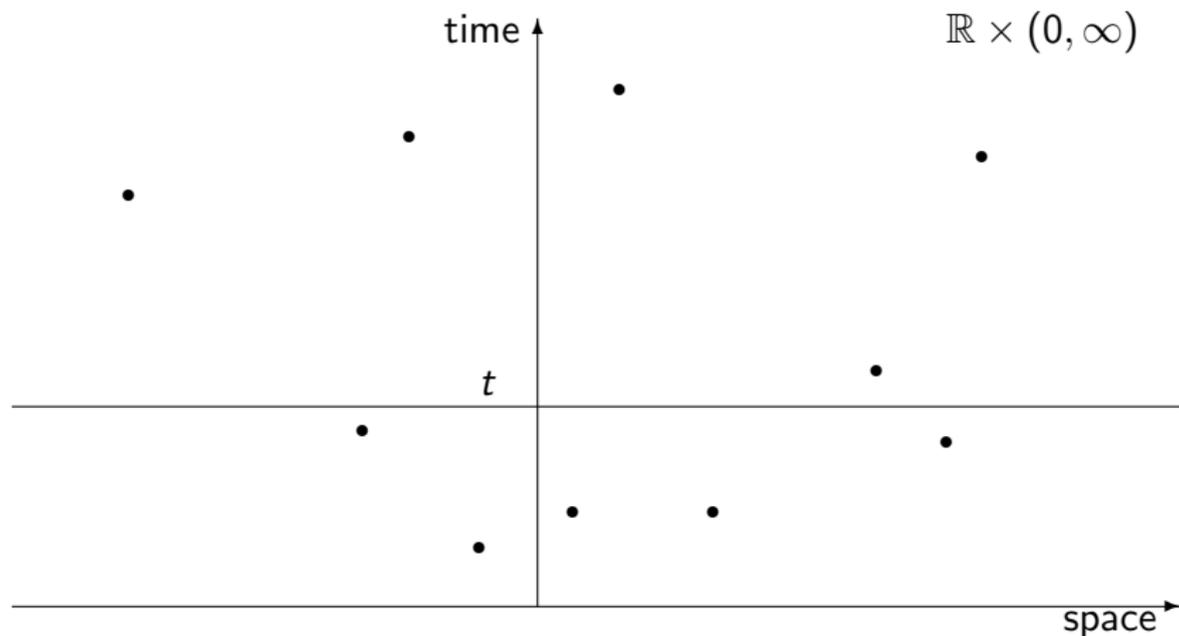
Poisson point processes, marked with birth times



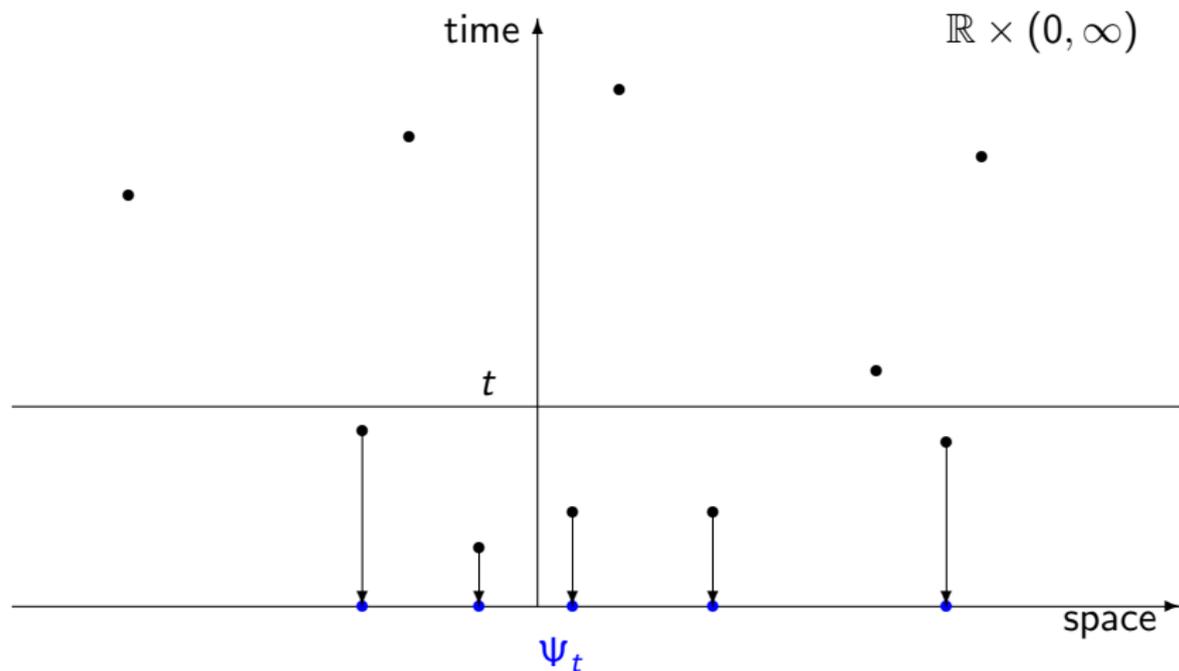
Poisson point processes, marked with birth times



2. Poisson point processes, marked with birth times



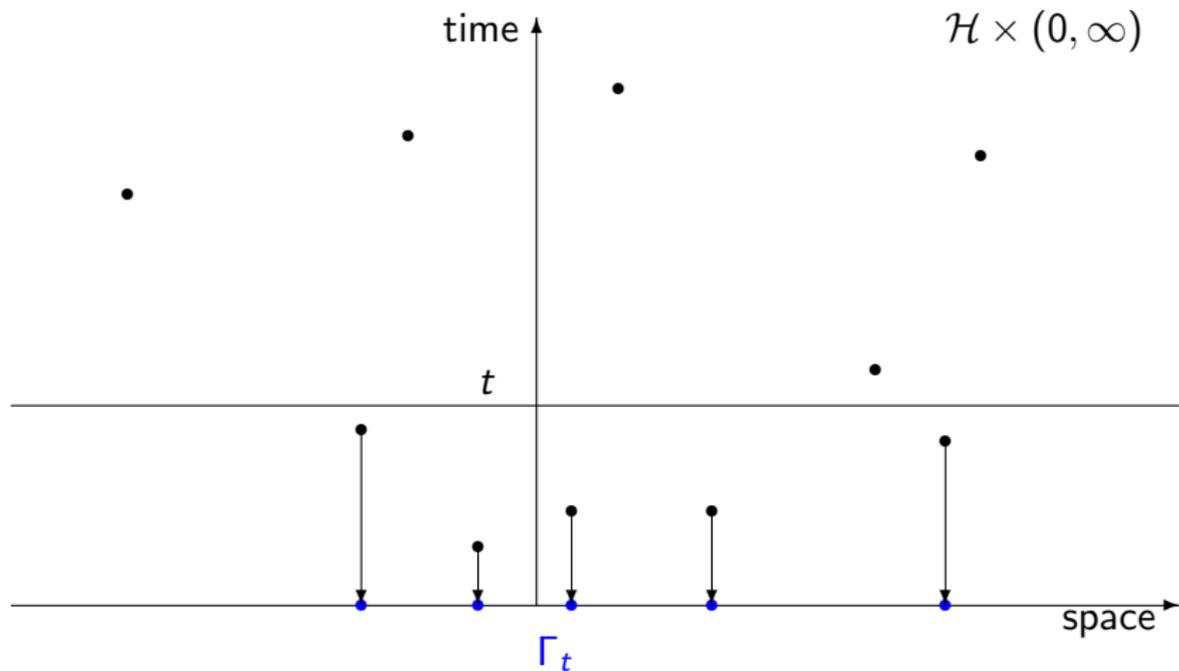
Poisson point processes, marked with birth times

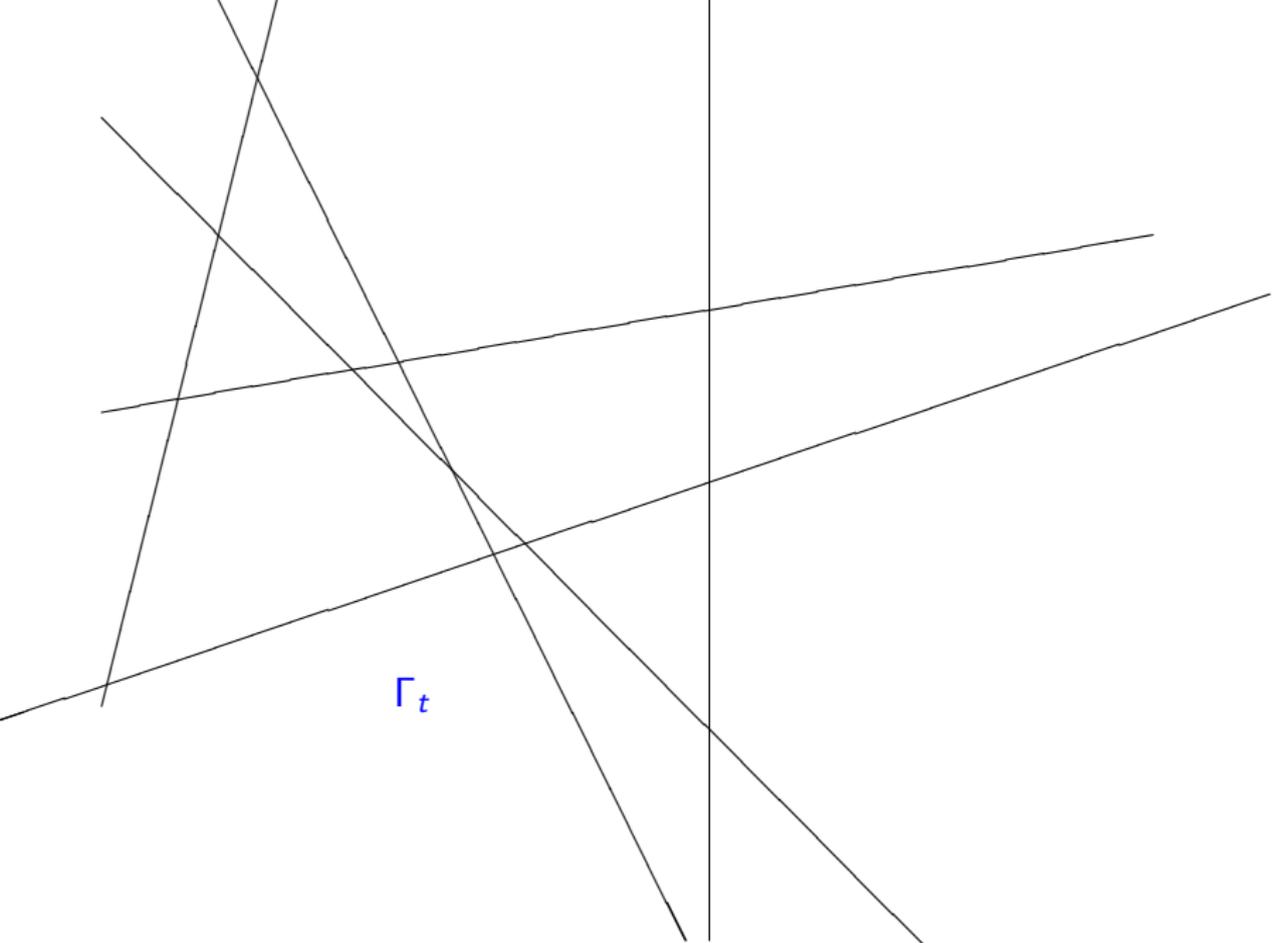


Now replace \mathbb{R} by

\mathcal{H} ... the set of all lines in \mathbb{R}^2

Poisson line processes Γ_t , marked with birth times





Poisson point process Γ on $\mathcal{H} \times (0, \infty)$
with intensity measure $\nu \times \ell_+$

ν translation invariant on \mathcal{H}

ℓ_+ ... Lebesgue measure on $(0, \infty)$

For all $t > 0$

$$\Gamma_t = \{h : (h, s) \in \Gamma : s < t\}$$

is a spatially homogeneous Poisson line process in \mathbb{R}^2 .

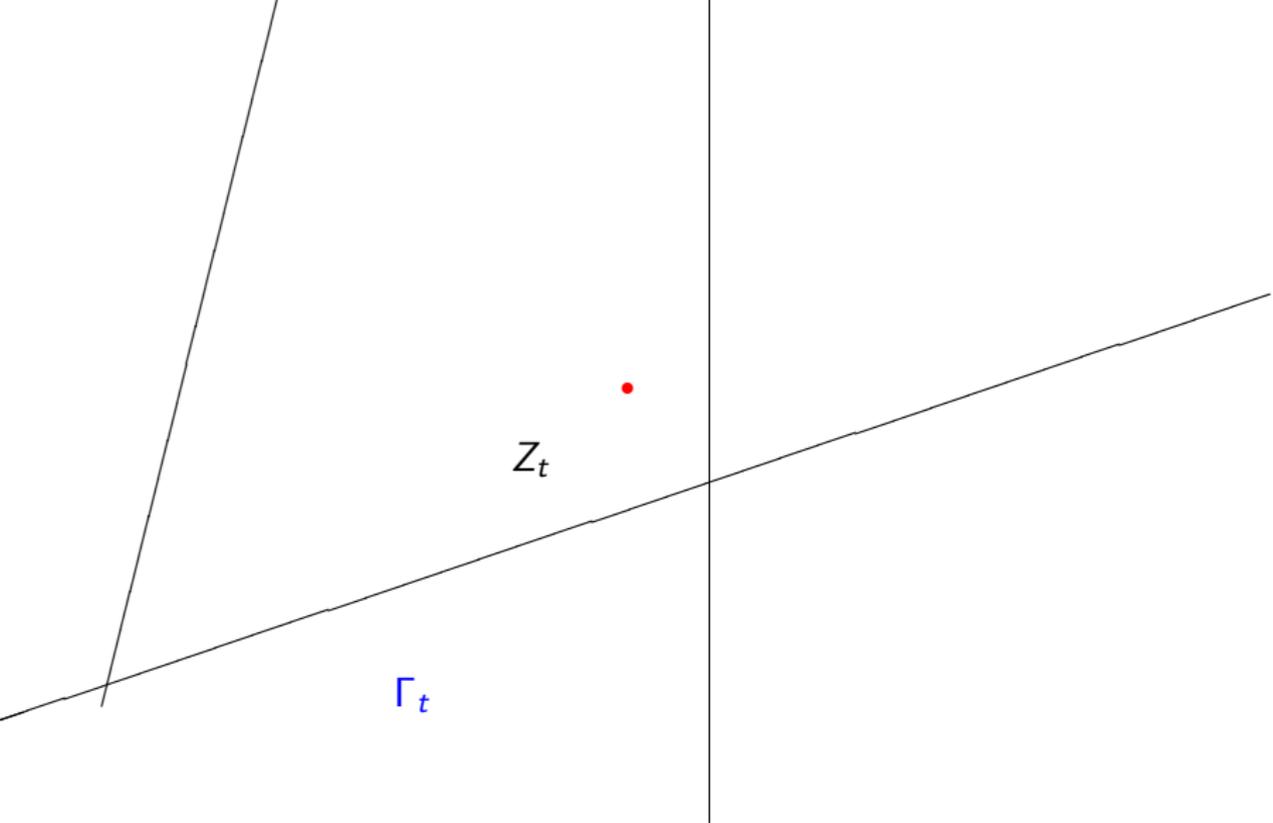
A preliminary construction

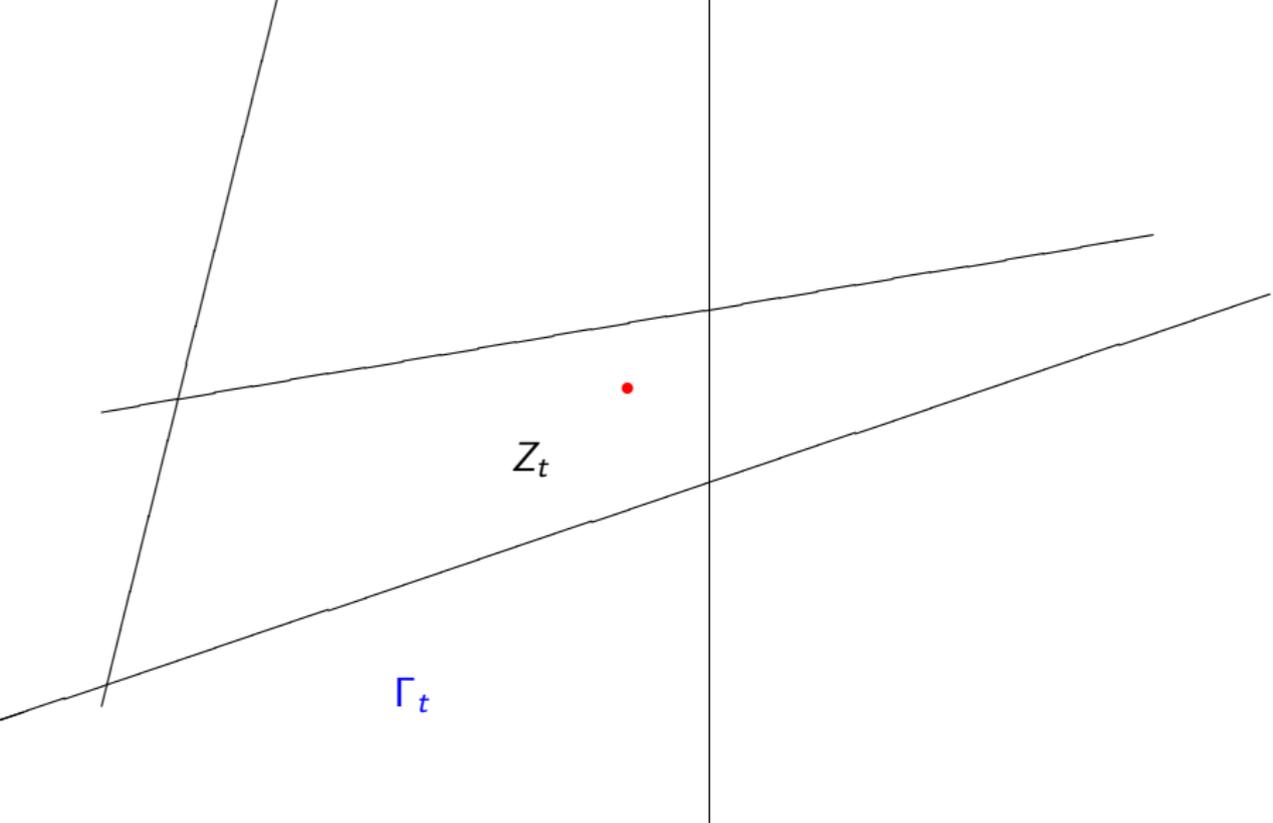
How to start a division of the whole plane when all segments have to have a finite length ?

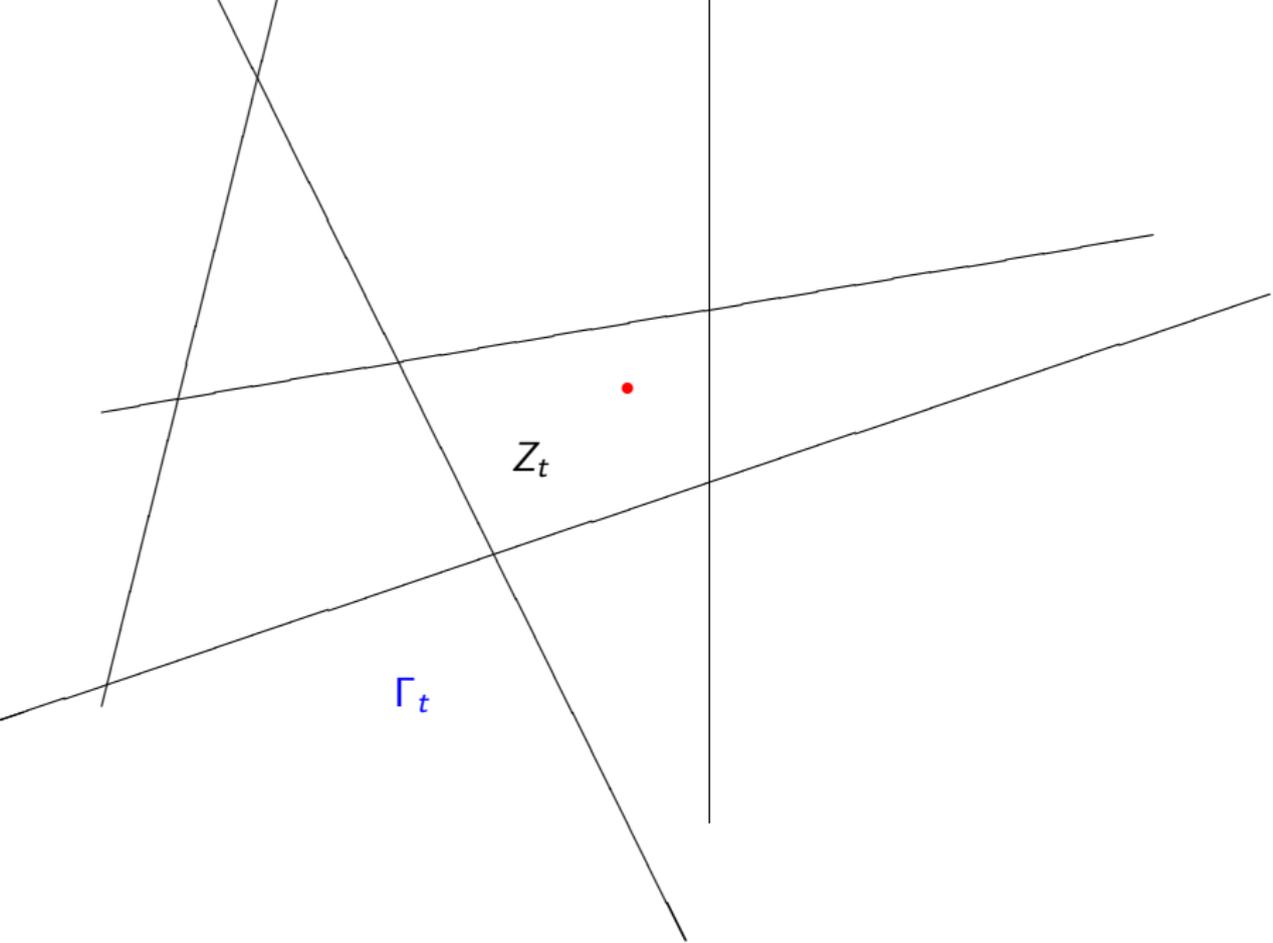
And all nodes are of T -type.

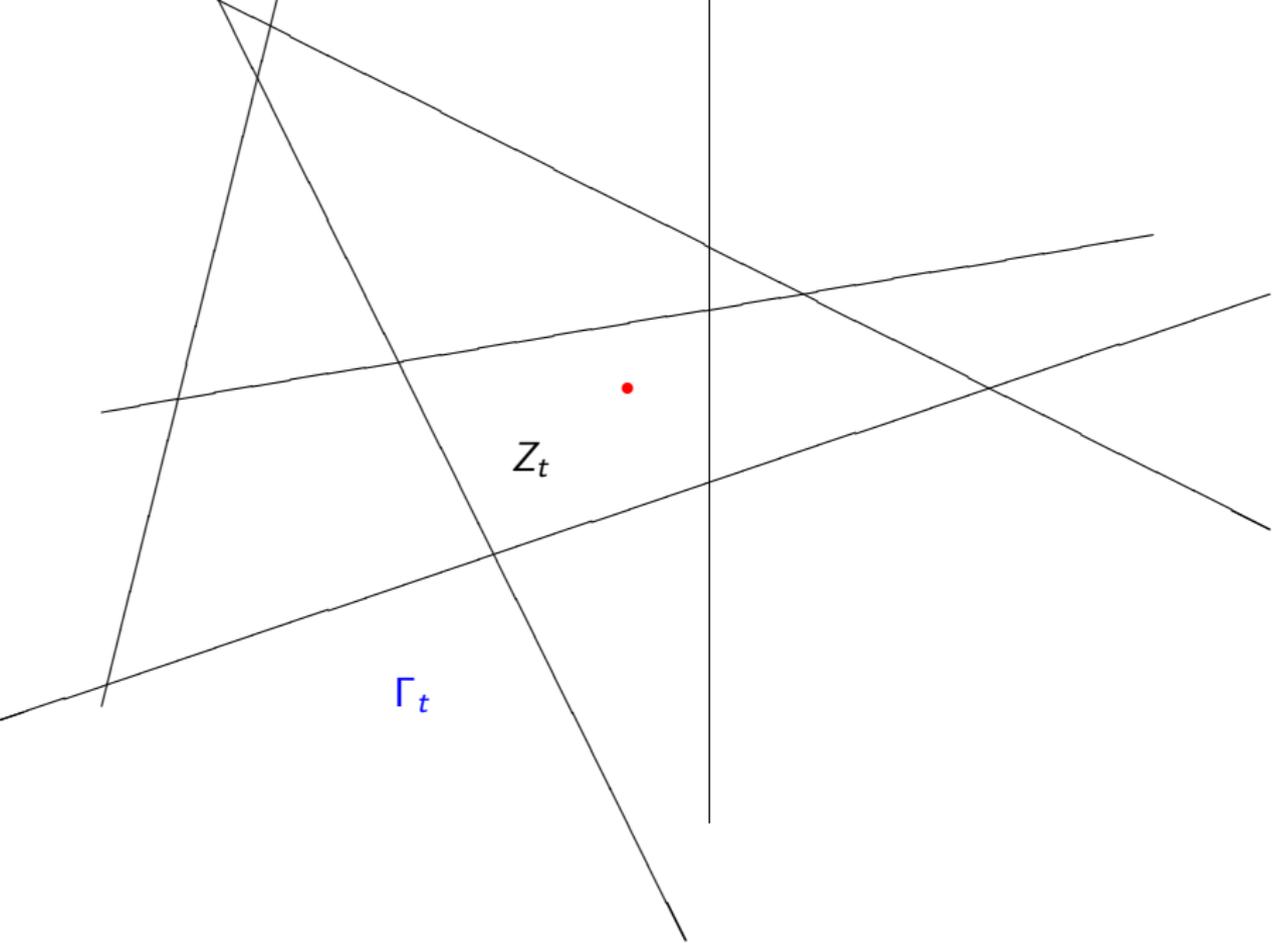
The crucial idea (by Joseph Mecke) :

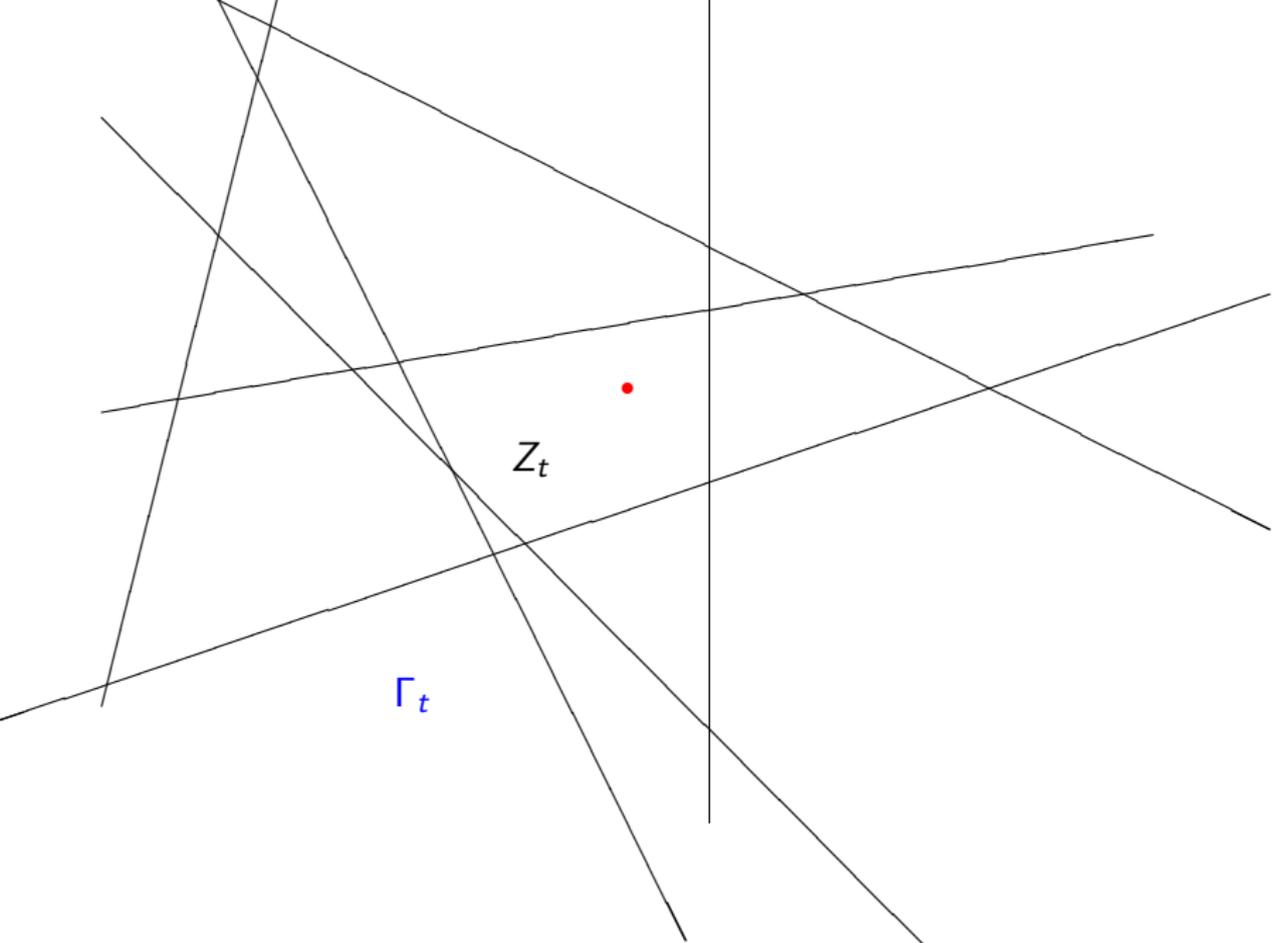
Consider the process $(Z_t)_{t>0}$ of the σ -cells of $(\Gamma_t)_{t>0}$.











For all $t > 0$ the Poisson line process Γ_t is a.s. not empty.

Assume that the directional distribution \mathcal{R} of the lines is not concentrated in a single point.

Then Γ_t generates a tessellation with a compact convex polygon Z_t that a.s. contains the origin o in its interior.

\implies Stochastic process $(Z_t)_{t>0}$ of o -cells of $(\Gamma_t)_{t>0}$.

The isotony

$$\Gamma_{t_1} \subseteq \Gamma_{t_2} \quad \text{for} \quad t_1 < t_2$$

implies

$$Z_{t_1} \supseteq Z_{t_2} \quad \text{for} \quad t_1 < t_2.$$

The process $(Z_t)_{t>0}$ is piecewise constant.

\mathbb{Z} ... the set of all integers.

Monotonic sequence $(\sigma_k)_{k \in \mathbb{Z}}$ of times where $(Z_t)_{t>0}$ changes its state.

σ_k is the time when the interior *int* $Z_{\sigma_{k-1}}$ is hit by a line from Γ .

$$\dots < \sigma_{-2} < \sigma_{-1} < \sigma_0 < \sigma_1 < 1 < \sigma_2 < \dots$$

We obtain

$$\lim_{k \rightarrow -\infty} \sigma_k = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \sigma_k = \infty$$

Crucial for the construction

$$Z_{\sigma_k} \uparrow \mathbb{R}^2 \quad \text{a.s.} \quad \text{if} \quad k \downarrow -\infty$$

Also

$$Z_{\sigma_k} \downarrow \{o\} \quad \text{a.s.} \quad \text{if} \quad k \uparrow \infty$$

A preliminary tessellation of \mathbb{R}^2

For $t > 0$ we define a tessellation Ψ_t with the cells

$$Z_t \quad \text{and} \quad \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}, \quad \sigma_k < t.$$

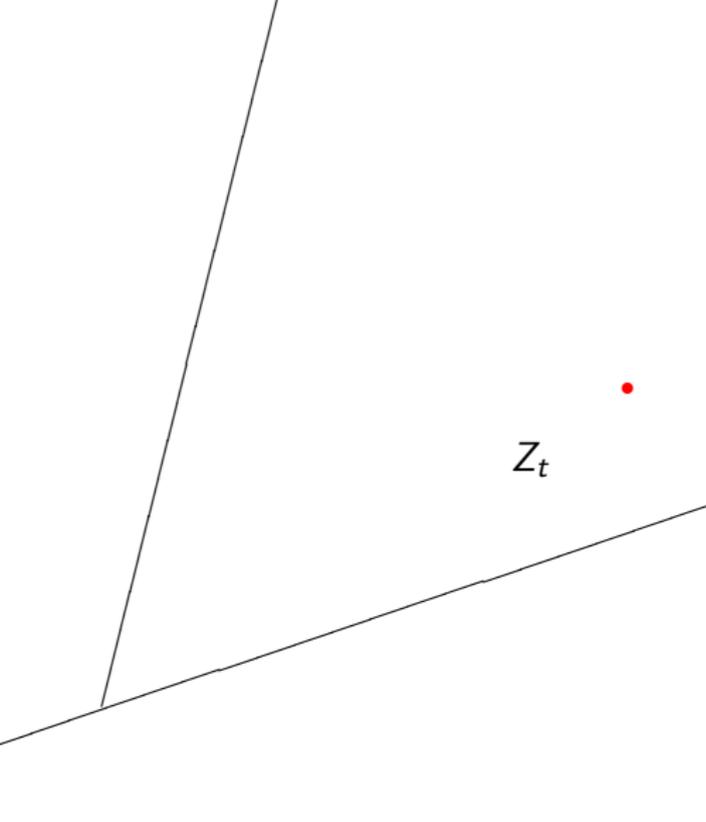
All these cells are compact, convex and have a pairwise disjoint interior.

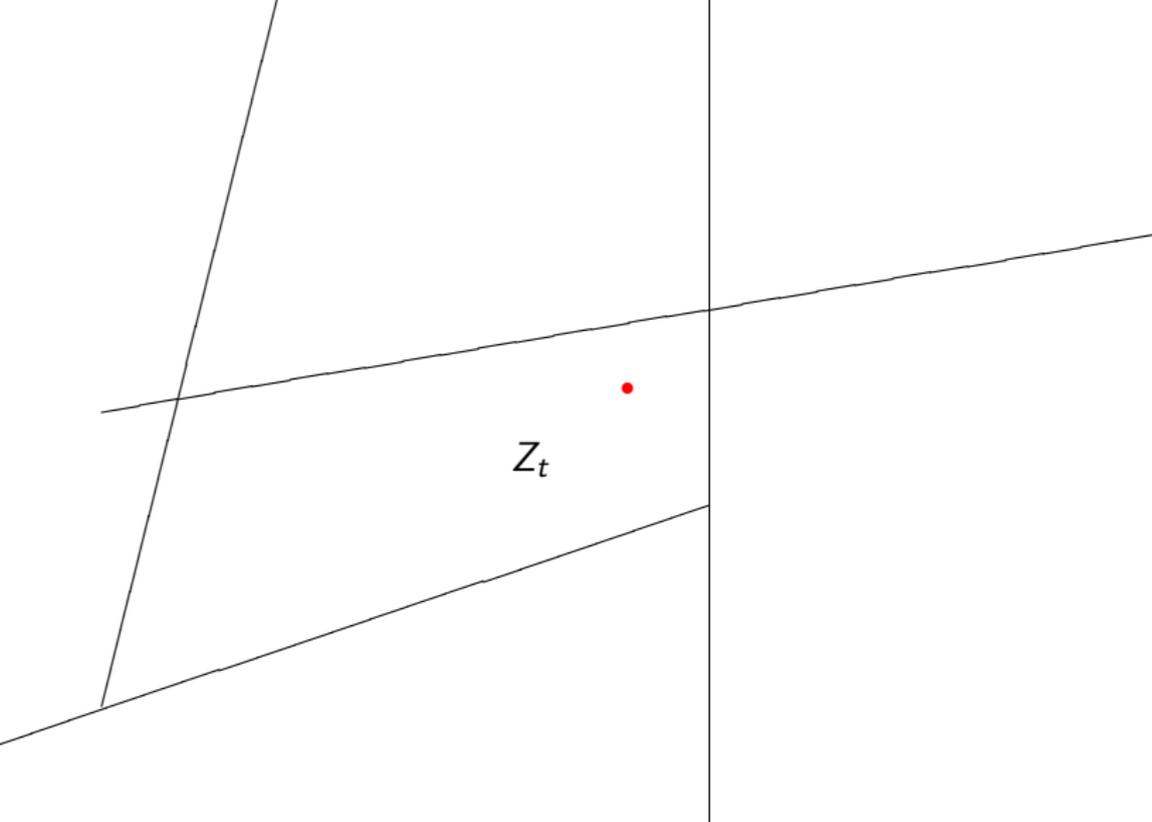
Due to

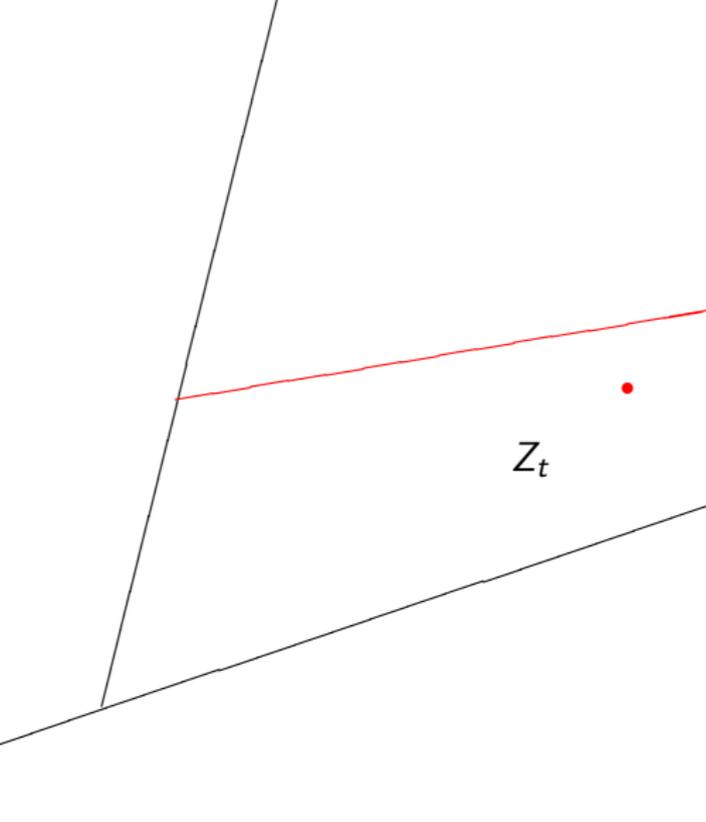
$$Z_{\sigma_k} \uparrow \mathbb{R}^2 \quad \text{a.s.} \quad \text{if} \quad k \downarrow -\infty$$

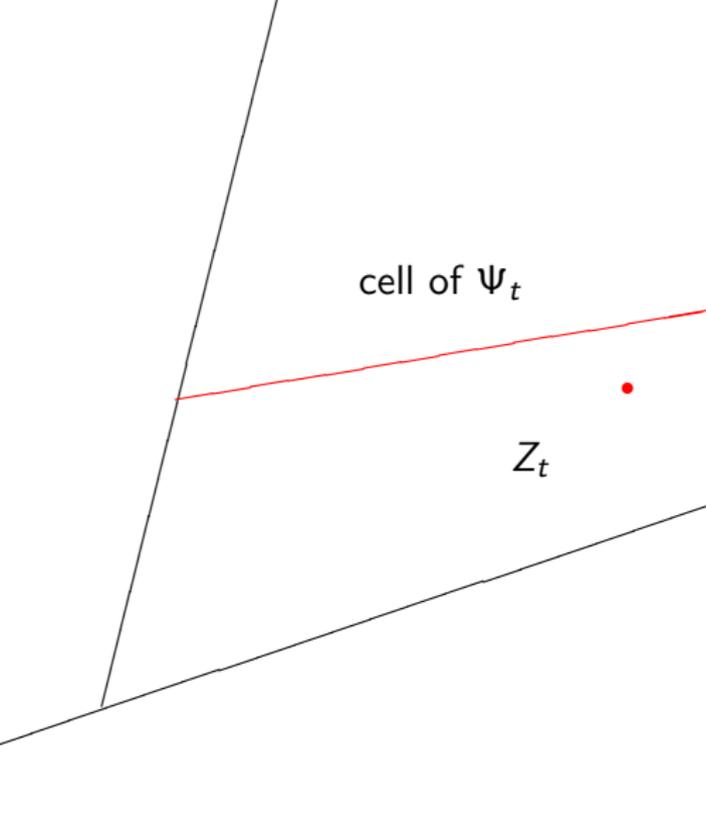
the cells fill the plane, i.e. for all $t > 0$

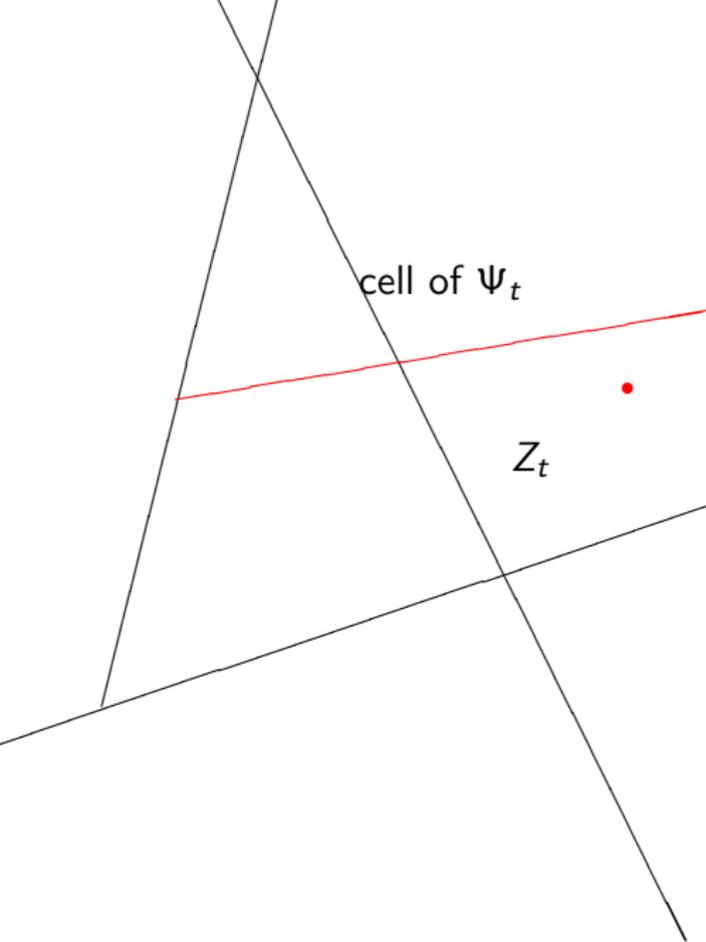
$$Z_t \cup \bigcup_{\sigma_k < t} \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} = \mathbb{R}^2$$

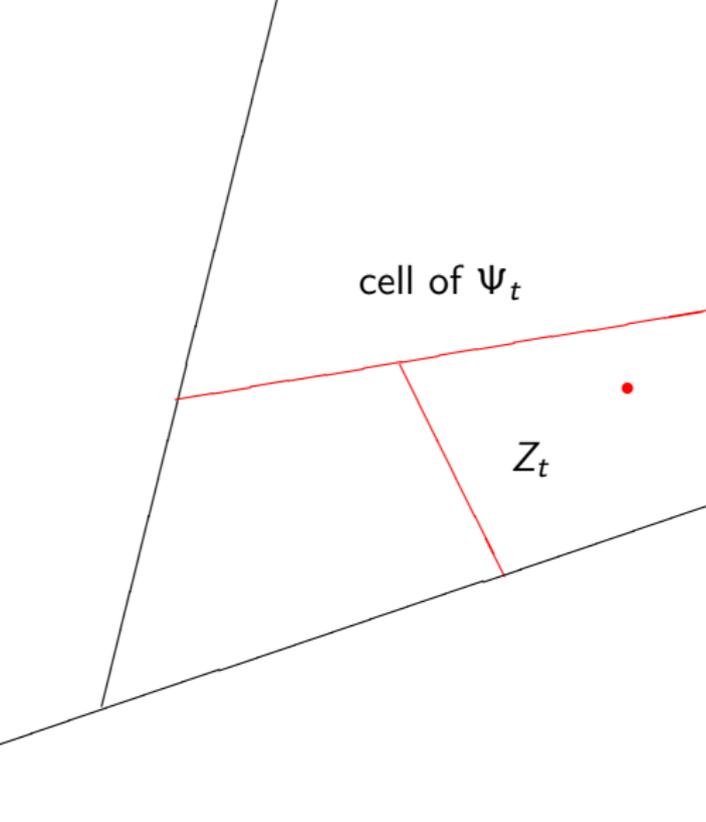


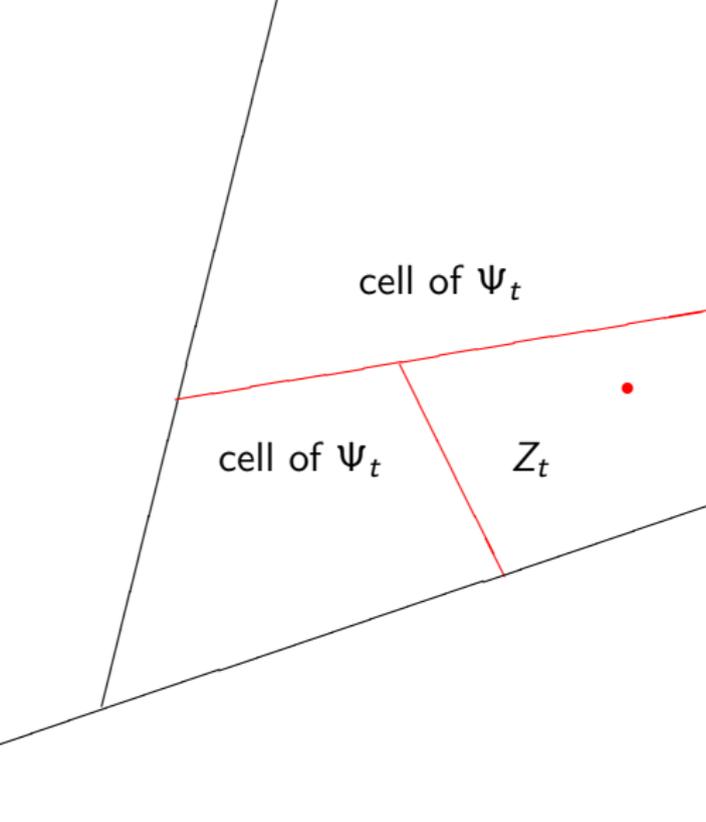


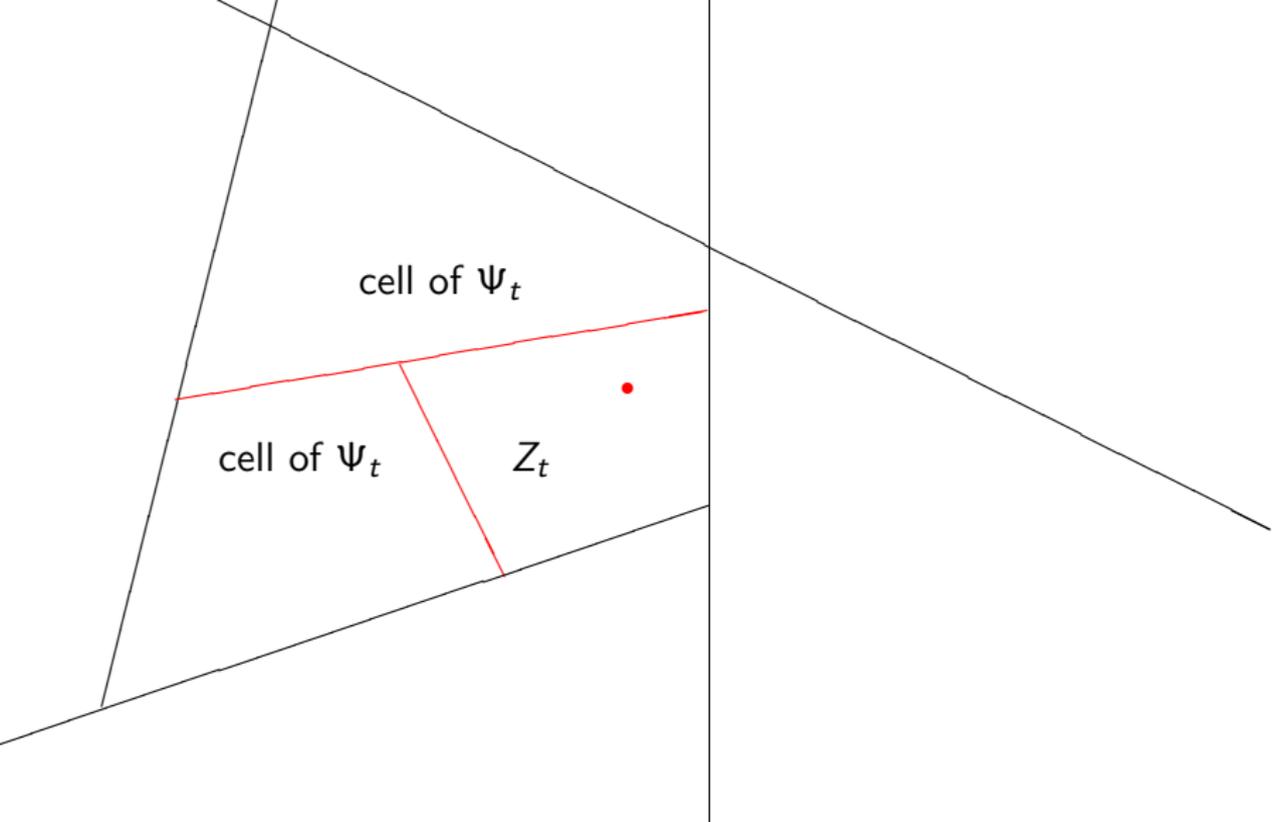


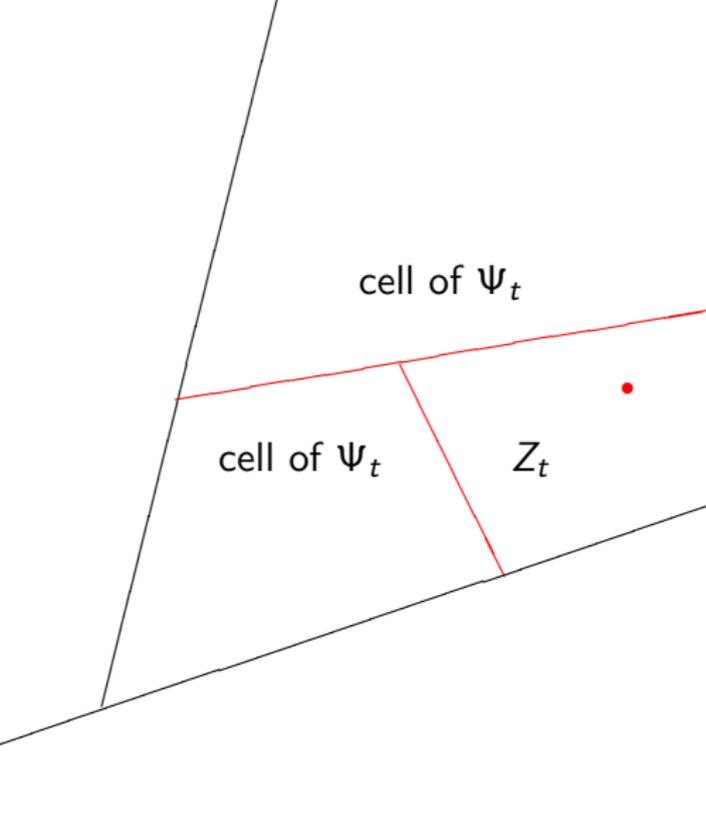












The random tessellation Ψ_t is non-homogeneous (spatially non-stationary).

Intuitively, the older cells of Ψ_t ,

$\overline{Z_{\sigma_{k-1}}} \setminus \overline{Z_{\sigma_k}}$ with σ_k close to the time 0

(the

moment of the 'Big Bang')

are very far from the origin $o \in \mathbb{R}^2$

and they tend to be larger than the younger ones.

The final steps of the construction – generating a spatially homogeneous random tessellation

For $t > 0$: non-homogeneous tessellation Ψ_t with the cells

$$Z_t \quad \text{and} \quad \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}, \quad \sigma_k < t.$$

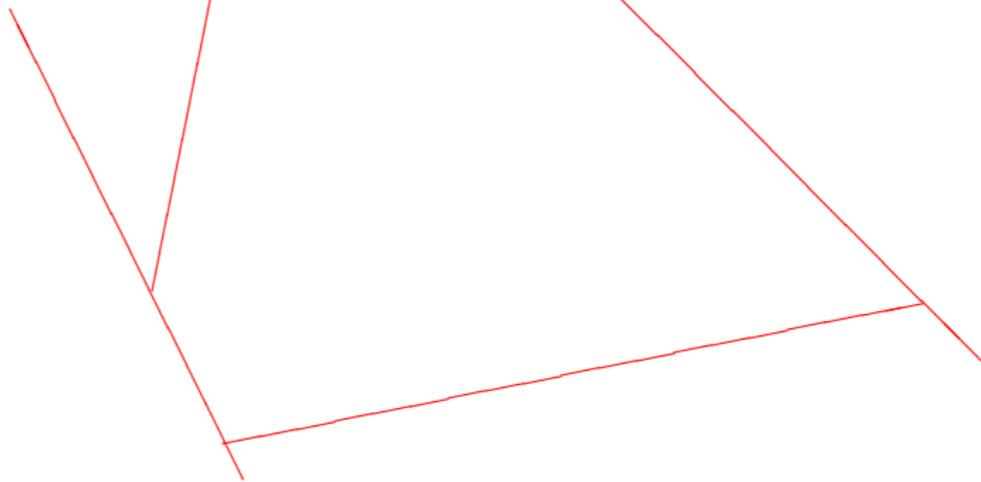
A cell

$$\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$$

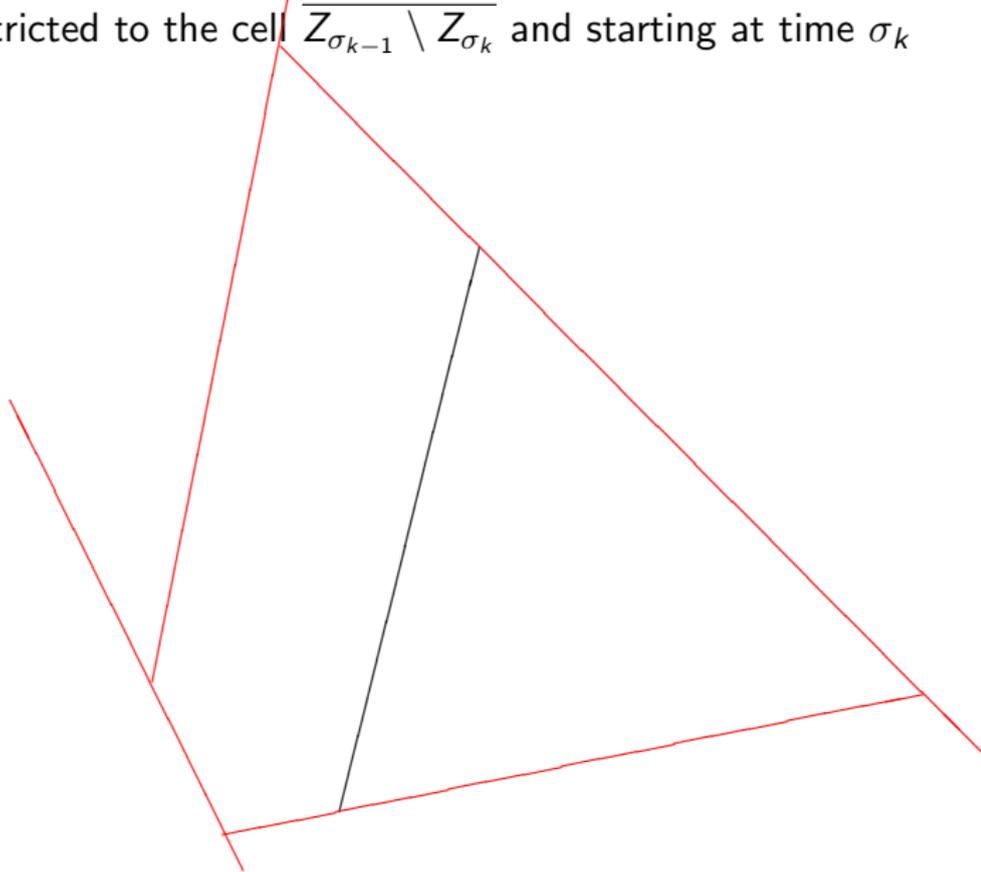
is born at the time $\sigma_k < t$.

During the time interval (σ_k, t) this bounded cell is divided by random chords as described in the beginning.

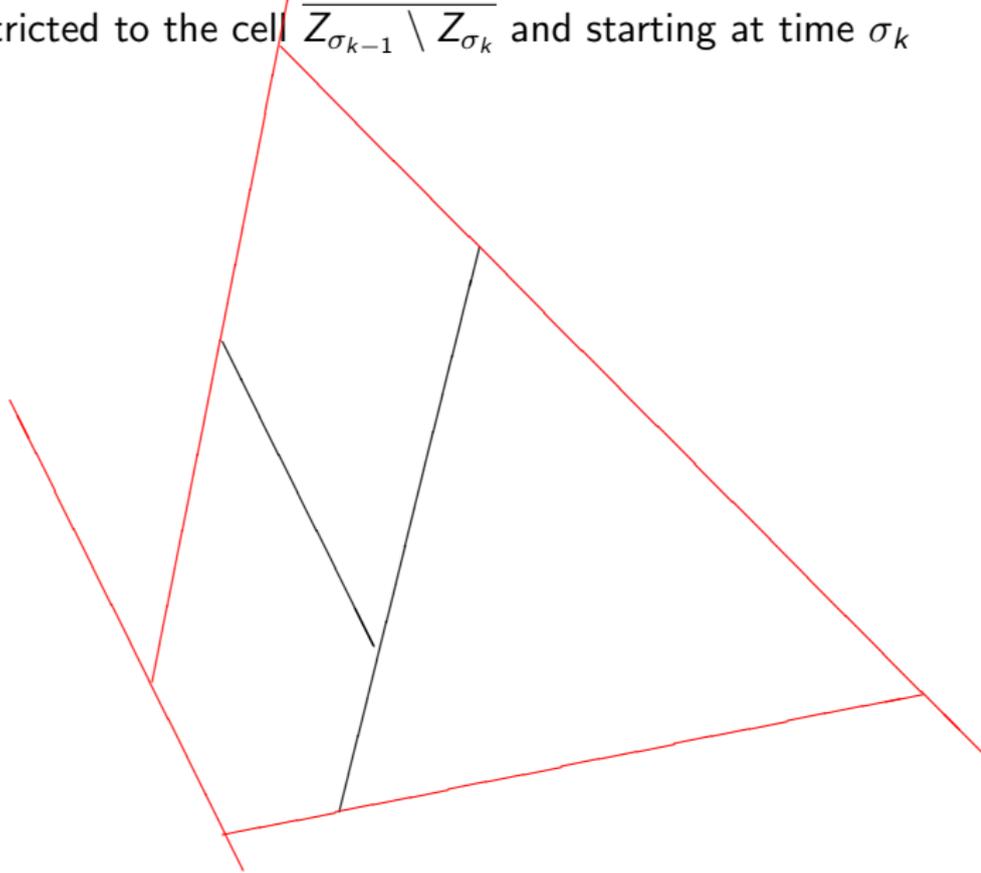
Restricted to the cell $\overline{Z_{\sigma_{k-1}}} \setminus Z_{\sigma_k}$ and starting at time σ_k



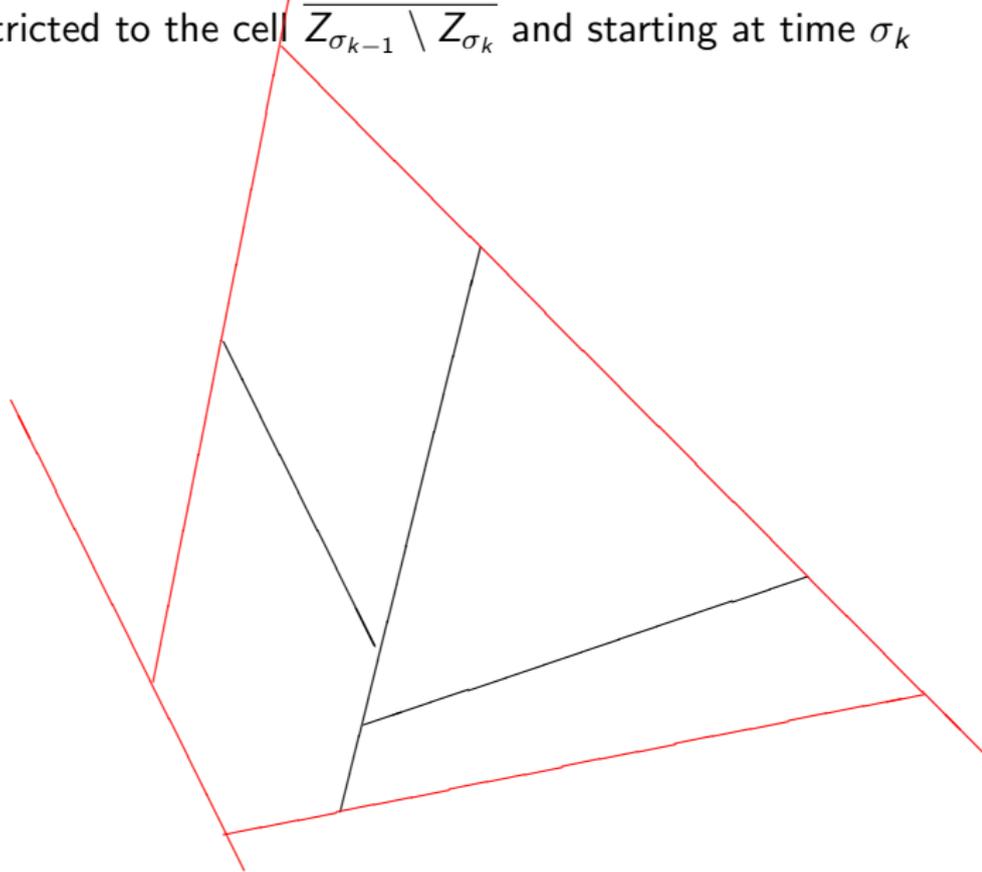
Restricted to the cell $\overline{Z_{\sigma_{k-1}}} \setminus Z_{\sigma_k}$ and starting at time σ_k



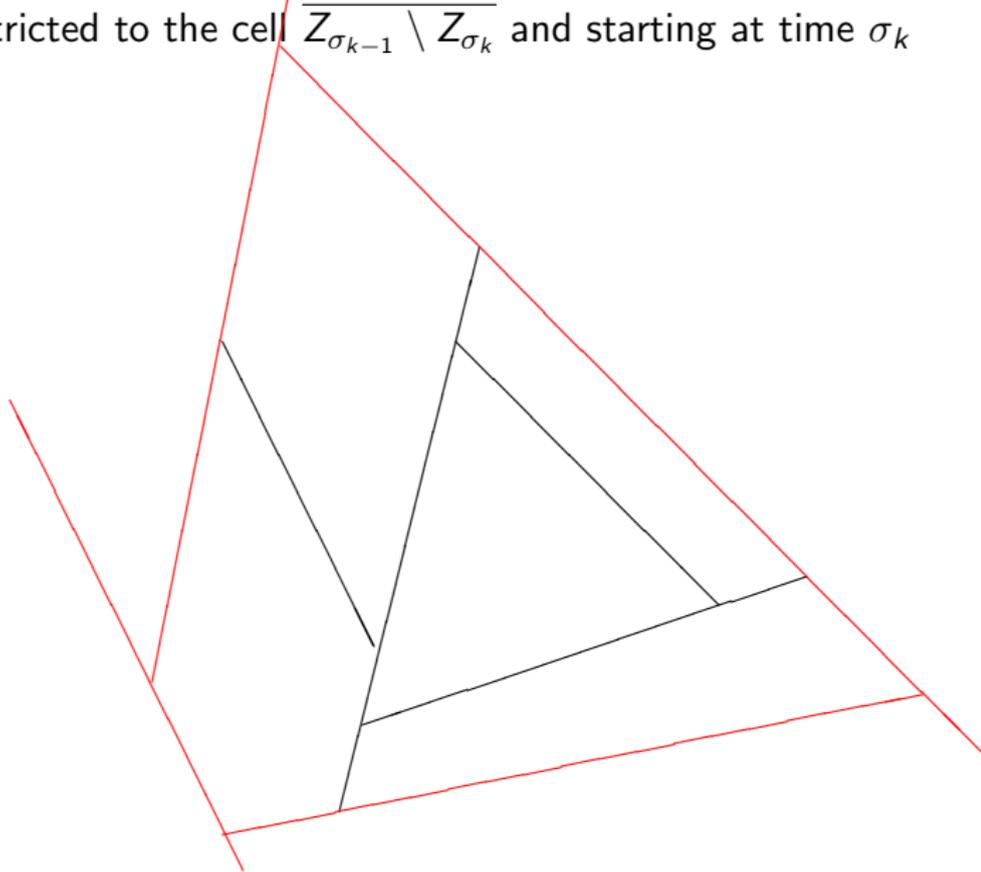
Restricted to the cell $\overline{Z_{\sigma_{k-1}}} \setminus Z_{\sigma_k}$ and starting at time σ_k



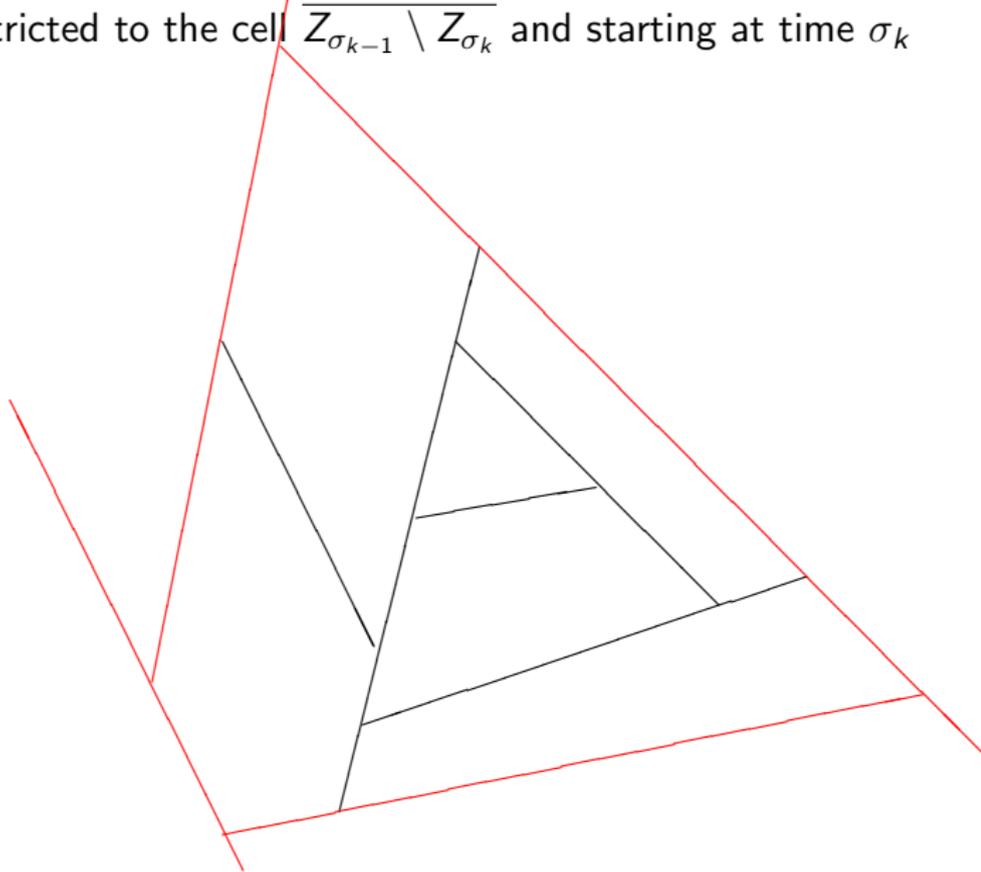
Restricted to the cell $\overline{Z_{\sigma_{k-1}}} \setminus Z_{\sigma_k}$ and starting at time σ_k



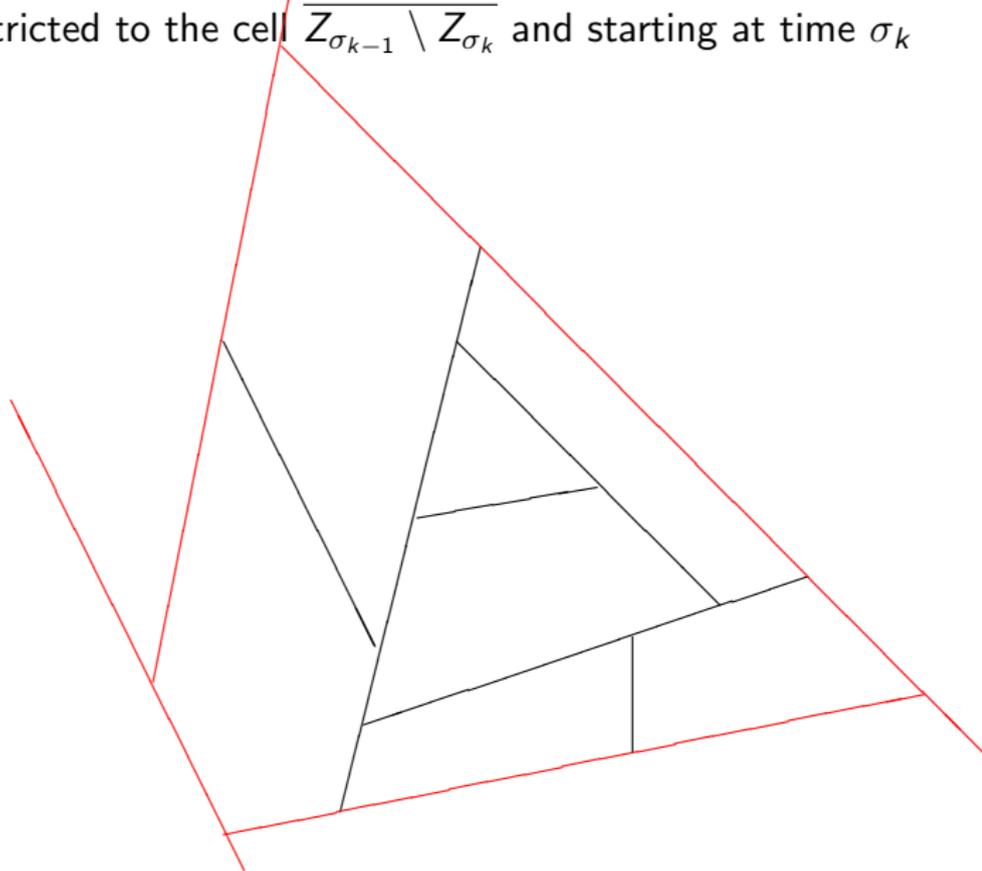
Restricted to the cell $\overline{Z_{\sigma_{k-1}}} \setminus Z_{\sigma_k}$ and starting at time σ_k



Restricted to the cell $\overline{Z_{\sigma_{k-1}}} \setminus Z_{\sigma_k}$ and starting at time σ_k



Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time σ_k



... and finishing at time t .

Thus the cells $\overline{Z_{\sigma_{k-1}}} \setminus \overline{Z_{\sigma_k}}$ are filled during the time interval (σ_k, t) such that the resulting tessellation Φ_t

- is spatially homogeneous,
- STIT, i.e. stable under iteration/nesting of tessellations.

Mecke, J., Nagel, W., Weiß, V. (2008). A global construction of homogeneous random planar tessellations that are stable under iteration. *Stochastics : An Int. J. of Prob. and Stoch. Proc.* **80**, 51-67.

Mecke, J., Nagel, W., Weiß, V. (2008). The iteration of random tessellations and a construction of a homogeneous process of cell divisions. *Advances in Applied Probability* **40**, 49-59.

Nagel, W., Weiß, V. (2005). Crack STIT tessellations – characterization of the stationary random tessellations which are stable with respect to iteration. *Advances in Applied Probability* **37**, 859-883.

Lachièze-Rey R., Mixing properties of STIT tessellations. *Advances in Applied Probability* **43.1** (march 2011)