ANNE SABOURIN SABOURIN@ENST.FR

Exercice 1 (**revision** : Conditional distributions in a Gaussian vector)**.**
Let $Z = (X, Y)$ be a Guaussian vector of dimension $p + q$ (p is the dimension of X and q is that of Y), with law

$$\mathcal{N}\left(m = \begin{pmatrix} m_X \\ m_Y \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_Y \end{pmatrix}\right)$$

where Σ_Y is positive definite. Our aim is to show that the conditional distribution of X given Y is (at almosst every y) :

$$\mathcal{L}\left(X|\{Y = y\}\right) = \mathcal{N}\left(m_{X|y}, \Sigma_{X|y}\right)$$

with

$$\begin{cases} m_{X|y} = m_X + \Sigma_{XY}\Sigma_Y^{-1}(y - m_Y) \\ \Sigma_{X|y} = \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX}. \end{cases} \tag{1}$$

To do this we shall write X as

$$X = AY + U, \quad \text{where } U \perp\!\!\!\perp Y, \text{ is a random vector of dimension } p \tag{2}$$

and $A \in \mathbb{R}^{p\times q}$ is a given matrix (non random)

1. Show that problem (2) is equivalent to finding a matrix A such that $(X - AY)$ and Y are uncorrelated (null covariance matrix)
2. Show that if A and U are solutions of (2), ten U follows a multivariate Gaussian distribution. Let us denote in the sequel (m_U, Σ_U) its expectancy and covariance matrix (that we shall not determine for now)
3. Show that if (2) is satisfied, then the conditional distribution $\mathcal{L}(X|\{Y = y\})$ is a multivariate Gaussian distribution with mean $Ay + m_U$ and variance Σ_U.
4. Determine A, then m_U and Σ_U. Conclude.

Exercice 2 (**revision** : 'Augmenting' a Gausian vector)**.**
Let $X \sim \mathcal{N}(m_X, \Sigma_X)$, $Y \sim \mathcal{N}(m_Y, \Sigma_Y)$ be Gaussian vectors with respective dimensions p, q. Assume $X \perp\!\!\!\perp Y$. Let $A \in \mathbb{R}^{q\times p}$ be a fixed (non random) matrix. Show that

$$\begin{pmatrix} X \\ AX + Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} m_X \\ Am_X + m_Y \end{pmatrix}, \begin{pmatrix} \Sigma_X & \Sigma_X A^\top \\ A\Sigma_X & \Sigma_Y + A\Sigma_X A^\top \end{pmatrix}\right)$$

Exercice 3 (Linear transforms and conditioning, inverse formula)**.**

1. (linear algebra) Let $\Lambda \in \mathbb{R}^{p\times p}$ and $L \in \mathbb{R}^{q\times q}$ be invertible matrices and let $A \in \mathbb{R}^{q\times p}$. Define

   $$K = A\Lambda^{-1}A^\top + L^{-1}$$

   Assuming that K is invertible, show that $\Lambda + A^\top LA$ is so, with inverse

   $$\left(\Lambda + A^\top LA\right)^{-1} = \Lambda^{-1} - \Lambda^{-1}A^\top K^{-1}A\Lambda^{-1} \tag{3}$$

2. Let $X \sim \mathcal{N}(\mu; \Lambda^{-1})$ and $\epsilon \sim \mathcal{N}(0, L^{-1})$ be independent from each other and let $b \in \mathbb{R}^q$ be a constant vector. Let $Y = AX + b + \epsilon$. Define $S = (\Lambda + A^\top LA)^{-1}$. Show that the conditional distribution of X given Y is given by

   $$\mathcal{L}(X|Y = y) = \mathcal{N}(\mu_{X|y}, \Sigma_{X|y})$$

   with

   $$\begin{cases} \mu_{X|y} = S\big(\Lambda\mu + A^\top L(y - b)\big) \\ \Sigma_{X|y} = S \end{cases} \tag{4}$$