

# Derivation of Maxwell-Stefan and Fick cross-diffusion models from the multi-species Boltzmann equation in the diffusive scaling

Bérénice GREC<sup>1</sup>

in collaboration with L. BOUDIN, M. BRIANT, V. PAVAN

<sup>1</sup>MAP5 – Université de Paris, France

*Third Workshop on Compressible Multiphase Flows, Strasbourg, June 21st, 2021*



# Outline of the talk

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- 1 Introduction
  - Context of the study
  - Gaseous mixtures: macroscopic models
  - Kinetic setting
- 2 Moment method
- 3 Perturbative method
- 4 Stiff dissipative hyperbolic formalism
- 5 Conclusion and prospects

# Context of the study

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- ▶ Non-reactive **mixture** of  $p$  monoatomic gases
- ▶ Isothermal setting  $T > 0$  uniform and constant
- ▶ Two different scales for the description of the mixture
  - ▶ **mesoscopic scale** (kinetic model): species  $i$  described by its distribution function  $f_i(t, x, v)$
  - ▶ **macroscopic scale**: species  $i$  described by the physical observables
    - ▶ number density  $n_i(t, x)$
    - ▶ velocity  $u_i(t, x)$

↪ average velocity of species  $i$  :  $J_i(t, x) = n_i(t, x)u_i(t, x)$

↪ vectorial quantities  $\mathbf{n} = \begin{bmatrix} n_1 \\ \vdots \\ n_p \end{bmatrix}$ ,  $\mathbf{J} = \begin{bmatrix} J_1 \\ \vdots \\ J_p \end{bmatrix}$

# Diffusion models for mixtures: Maxwell-Stefan/Fick

Mass conservation:

$$\partial_t \mathbf{n} + \nabla \cdot \mathbf{J} = 0$$

Diffusion process (link between  $\mathbf{J}$  and  $\nabla \mathbf{n}$ ):

Maxwell-Stefan equations

$$-\nabla \mathbf{n} = A(\mathbf{n}) \mathbf{J}$$

Fick equations

$$\mathbf{J} = -B(\mathbf{n}) \nabla \mathbf{n}$$

- ▶  $A(\mathbf{n})$  and  $B(\mathbf{n})$  are not invertible (rank  $p - 1$ )
- ▶ Using Moore-Penrose pseudo-inverse: structural similarity

Formal analogy of the two systems,  
but Fick and Maxwell-Stefan are not obtained in the same way

# Maxwell-Stefan vs. Fick (macroscopic point of view)

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## Obtention of the Maxwell-Stefan equations

- ▶ Mechanical considerations on forces (balance of pressure and friction forces)
- ▶ Assumption: different species have different macroscopic velocities on macroscopic time scales
- ▶ Stems from momentum equations

## Obtention of the Fick equations

- ▶ Thermodynamics of irreversible processes (entropy decay) [Onsager]
- ▶ Thermodynamical considerations on fluxes, written (close to equilibrium) as linear combinations of potential gradients
  - ▶ nonreactive isothermal setting  $\rightsquigarrow$  chemical potential gradients
  - ▶ ideal gases  $\rightsquigarrow$  (number) density gradients
- ▶ Stems from mass equations

# Maxwell-Stefan vs. Fick (mesoscopic point of view)

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## Two main questions

- ▶ How to justify these two (different) equations from the (same) kinetic description?
- ▶ Is there a link between the Maxwell-Stefan and the Fick diffusion coefficients?
- ▶ In which regime are these macroscopic models valid?

## Moment method (Maxwell-Stefan)

- ▶ Based on the ansatz that the distribution functions are at local Maxwellian states [Levermore], [Müller, Ruggieri]

## Perturbative method (Fick)

- ▶ Based on the Chapman-Enskog expansion [Bardos, Golse, Levermore], [Bisi, Desvillettes]

# Kinetic setting

- ▶ Elastic collision rules, for  $\sigma \in \mathbb{S}^{d-1}$

$$\begin{cases} v' = (m_i v + m_k v_* + m_k |v - v_*| \sigma) / (m_i + m_k), \\ v'_* = (m_i v + m_k v_* - m_i |v - v_*| \sigma) / (m_i + m_k) \end{cases}$$

- ▶ Boltzmann collision operator, for  $v \in \mathbb{R}^d$

$$Q_{ij}(f_i, f_j)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathcal{B}_{ij}(v, v_*, \sigma) \left[ f_i(v') f_j(v'_*) - f_i(v) f_j(v_*) \right] d\sigma dv_*$$

- ▶ Cross sections  $\mathcal{B}_{ij} = \mathcal{B}_{ji} > 0$

- ▶ Boltzmann equations for mixtures

$$\partial_t f_i + v \cdot \nabla_x f_i = \sum_{j=1}^p Q_{ij}(f_i, f_j), \quad \text{on } \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \quad 1 \leq i \leq p$$

[DESUILLETES, MONACO, SALVARANI, '05]

# Properties of the collision operator

- ▶ Equilibrium: Maxwellian with same bulk velocity and temperature

$$n_i(t, x) \left( \frac{m_i}{2\pi k_B T} \right)^{d/2} \exp \left( - \frac{m_i |v - u(t, x)|^2}{2k_B T} \right)$$

- ▶ Conservation properties of the collision operator for  $1 \leq i, j \leq p$ .

$$\int_{\mathbb{R}^d} Q_{ij}(f_i, f_j)(v) dv = 0, \quad \int_{\mathbb{R}^d} Q_{ji}(f_i, f_j)(v) v dv = 0,$$
$$\int_{\mathbb{R}^d} \left( m_i Q_{ij}(f_i, f_j)(v) + m_j Q_{ji}(f_j, f_i)(v) \right) \frac{v^2}{2} dv = 0$$

## Diffusive scaling

Small mean free path and Mach number:  $\text{Kn} \sim \text{Ma} \sim \varepsilon$

$$\varepsilon \partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon} \sum_{j=1}^p Q_{ij}(f_i^\varepsilon, f_j^\varepsilon), \quad 1 \leq i \leq p$$



# Properties of the collision operator & Diffusive scaling

- ▶ Equilibrium: Maxwellian with same bulk velocity and temperature

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# Moment method

## Moments of the distribution functions

- ▶ Number density of species  $i$

$$n_i^\varepsilon(t, x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t, x, v) dv$$

- ▶ Flux of species  $i$

$$J_i^\varepsilon(t, x) = n_i^\varepsilon(t, x) u_i^\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} v f_i^\varepsilon(t, x, v) dv$$

## Ansatz

The distribution function of each species  $i$  is at a **local Maxwellian state** with a **small velocity of order  $\varepsilon$**  for any  $(t, x) \in \mathbb{R}_+ \times \Omega$

$$f_i^\varepsilon(t, x, v) = n_i^\varepsilon(t, x) \left( \frac{m_i}{2\pi k_B T} \right)^{d/2} \exp \left( - \frac{m_i |v - \varepsilon u_i^\varepsilon(t, x)|^2}{2k_B T} \right)$$

# Macroscopic diffusion equations

$$\varepsilon \partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon} \sum_j Q_{ij}(f_i^\varepsilon, f_j^\varepsilon), \quad \forall i \in \llbracket 1, p \rrbracket$$

- **Mass conservation:** moment of order 0

$$\varepsilon \frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3} f_i^\varepsilon(v) dv \right) + \nabla_x \cdot \left( \int_{\mathbb{R}^3} v f_i^\varepsilon(v) dv \right) = 0,$$

where the collision term vanishes (conservation property).

- **Momentum equation:** moment of order 1

$$\varepsilon \frac{\partial}{\partial t} \int_{\mathbb{R}^3} v f_i^\varepsilon(v) dv + \int_{\mathbb{R}^3} v (v \cdot \nabla_x f_i^\varepsilon(v)) dv = \frac{1}{\varepsilon} \sum_{j \neq i} \int_{\mathbb{R}^3} v Q_{ij}(f_i^\varepsilon, f_j^\varepsilon)(v) dv$$

where the mono-species collision term vanishes (conservation property).

# Computation of the divergence term

$$\varepsilon \frac{\partial}{\partial t} \int_{\mathbb{R}^3} v f_i^\varepsilon(v) dv + \boxed{\int_{\mathbb{R}^3} v (v \cdot \nabla_x f_i^\varepsilon(v)) dv} = \frac{1}{\varepsilon} \sum_{j \neq i} \int_{\mathbb{R}^3} v Q_{ij}(f_i^\varepsilon, f_j^\varepsilon)(v) dv$$

- Use of the Ansatz, translation in  $v$  + parity argument

$$\nabla_x \cdot \left( \int v \otimes v f_i^\varepsilon(v) dv \right) \propto \nabla_x \cdot \left( n_i^\varepsilon \int \left( v \otimes v + \varepsilon^2 u_i^\varepsilon \otimes u_i^\varepsilon \right) e^{-m_i |v|^2 / 2kT} dv \right)$$

- In terms of macroscopic quantities

$$\nabla_x \cdot \left( \int v \otimes v f_i^\varepsilon(v) dv \right) = \frac{k_B T}{m_i} \nabla_x n_i^\varepsilon + \varepsilon^2 \nabla_x \cdot \left( n_i^\varepsilon u_i^\varepsilon \otimes u_i^\varepsilon \right)$$

# Computation of the collision term

- ▶ For Maxwell molecules: weak form, collision rules, symmetry and parity arguments:

$$\int v Q_{ij}(f_i^\varepsilon, f_j^\varepsilon)(v) dv = \frac{m_j}{m_i + m_j} \int b_{ij}(\cos \theta) f_i^\varepsilon f_j^\varepsilon (v_* - v + |v - v_*| \sigma) d\sigma dv_* dv$$

In terms of macroscopic quantities

$$\frac{1}{\varepsilon} \sum_{j \neq i} \int v Q_{ij}(f_i^\varepsilon, f_j^\varepsilon)(v) dv = \sum_{j \neq i} \underbrace{\frac{2\pi m_i m_j \|b_{ij}\|_{L^1}}{(m_i + m_j) k_B T}}_{D_{ij}^{-1}} \frac{k_B T}{m_i} (n_i^\varepsilon n_j^\varepsilon u_j^\varepsilon - n_j^\varepsilon n_i^\varepsilon u_i^\varepsilon)$$

- ▶ For general cross sections: algebraic arguments [BOUDIN, G., PAVAN, '17]

$$D_{ij} \propto \iiint B_{ij}(v, v_*, \sigma) \exp \left[ -\frac{m_i}{2k_B T} v^2 - \frac{m_j}{2k_B T} v_*^2 \right] (v' - v)^2$$

# Formal asymptotics and limit equation

## Formal limit

$$n_i(t, \mathbf{x}) = \lim_{\varepsilon \rightarrow 0} n_i^\varepsilon(t, \mathbf{x}), \quad J_i(t, \mathbf{x}) = \lim_{\varepsilon \rightarrow 0} n_i^\varepsilon(t, \mathbf{x}) u_i^\varepsilon(t, \mathbf{x})$$

- ▶ Mass conservation (order  $\varepsilon^1$ )

$$\partial_t n_i + \nabla_x \cdot J_i = 0$$

- ▶ Momentum equation (order  $\varepsilon^0$ )

$$\varepsilon^2 \frac{m_i}{k_B T} \left( \partial_t (n_i^\varepsilon u_i^\varepsilon) + \nabla_x \cdot (n_i^\varepsilon u_i^\varepsilon \otimes u_i^\varepsilon) \right) + \nabla_x n_i^\varepsilon = \sum_{j \neq i} \frac{n_j^\varepsilon n_i^\varepsilon u_j^\varepsilon - n_j^\varepsilon n_i^\varepsilon u_i^\varepsilon}{D_{ij}} + o(1)$$

$$-\nabla_x n_i = \sum_{j \neq i} \frac{n_j J_j - n_i J_j}{D_{ij}}, \quad \text{i.e.} \quad -\nabla_x \mathbf{n} = A(\mathbf{n}) \mathbf{J}.$$

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# Perturbative method

$$\varepsilon \partial_t f_i^\varepsilon + \mathbf{v} \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon} \sum_j Q_{ij}(f_i^\varepsilon, f_j^\varepsilon), \quad \forall i \in \llbracket 1, p \rrbracket$$

- Expansion around the global Maxwellian with zero bulk velocity (equilibrium) with number density  $n_i$

$$f_i^\varepsilon = n_i \mu_i + \varepsilon g_i^\varepsilon$$

$$\mu_i = (m_i / 2\pi k_B T)^{d/2} e^{-m_i |\mathbf{v}|^2 / 2k_B T}$$

- Moments of the distribution function

$$J_i(t, \mathbf{x}) = \int \mathbf{v} f_i^\varepsilon(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} = \int \mathbf{v} g_i^\varepsilon(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$$

- Mass conservation: zero-th moment in velocity, order  $\varepsilon^1$

$$\partial_t n_i + \nabla_x \cdot J_i = 0$$

- Inject expansion in the Boltzmann equation, order  $\varepsilon^0$

$$\mu_i \mathbf{v} \cdot \nabla_x n_i = \sum_j Q_{ij}(n_i \mu_i, g_j^\varepsilon) + Q_{ji}(g_i^\varepsilon, n_j \mu_j) =: \mathcal{L}_i(\mathbf{g}^\varepsilon),$$

where  $\mathbf{g}^\varepsilon = (g_i^\varepsilon)_i \rightsquigarrow$  defines the linearized Boltzmann operator  $\mathbf{L} = (\mathcal{L}_i)_i$

- ▶ In a vectorial form, defining  $W_i = \mu_i \mathbf{v} \cdot \nabla_x n_i$  and  $\mathbf{W} = (W_i)_i$

$$\mathbf{W} = \mathbf{L}(\mathbf{g}^\varepsilon) \quad \rightsquigarrow \quad \mathbf{g}^\varepsilon = \mathbf{L}^{-1}\mathbf{W}$$

(\*)

- ▶ Inject this expression for  $g_i^\varepsilon$  in the definition of  $J_i$

$$J_i = \int v [\mathbf{L}^{-1}\mathbf{W}]_i dv = \int n_i \mu_i v [\mathbf{L}^{-1}\mathbf{W}]_i (n_i \mu_i)^{-1} dv$$

- ▶ With  $\mathbf{C}_i = (\mu_j v \delta_{ij})_j$ , we get

$$J_i = n_i \langle \mathbf{C}_i, \mathbf{L}^{-1}\mathbf{W} \rangle_{L^2((n\mu)^{-1/2})}$$

- ▶  $\mathbf{L}^{-1}$  is self-adjoint on  $(\text{Ker } \mathbf{L})^\perp$ . Let  $\Gamma$  be the projection of  $\mathbf{C}$  on  $\text{Ker } \mathbf{L}$ . Thus

$$J_i = n_i \sum_j \langle [\mathbf{L}^{-1}(\mathbf{C} - \Gamma)]_j, W_j \rangle_{L^2((n\mu)^{-1/2})}$$

- ▶ Since  $W_j = \mu_j \mathbf{v} \cdot \nabla_x n_j = \mathbf{C}_j \cdot \nabla_x n_j$

$$J_i = \sum_j n_i \underbrace{\langle [\mathbf{L}^{-1}(\mathbf{C} - \Gamma)]_j, \mathbf{C}_j \rangle_{L^2((n\mu)^{-1/2})}}_{b_{ij}(n_i)} \nabla_x n_j$$

⇒ Fick equation:  $\mathbf{J} = -B(\mathbf{n})\nabla_x \mathbf{n}$

- In a vectorial form, defining  $W_i = \mu_i \mathbf{v} \cdot \nabla_x n_i$  and  $\mathbf{W} = (W_i)_i$

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$$J_i = \int \mathbf{v} [\mathbf{L}^{-1}\mathbf{W}]_i d\mathbf{v} = \int n_i \mu_i \mathbf{v} [\mathbf{L}^{-1}\mathbf{W}]_i (n_i \mu_i)^{-1} d\mathbf{v}$$

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# Closure relation (1)

## Maxwell-Stefan equation $-\nabla \mathbf{n} = A(\mathbf{n})\mathbf{J}$

- ▶ Summing over  $i$  the equations ( $A$  has rank  $p - 1$ ) implies that  $\nabla_x \sum_i n_i = 0$
- ▶ Ansatz (local Maxwellian) implies

$$\int m_i |v|^2 f_i^\varepsilon dv = 3k_B T n_i^\varepsilon + o(\varepsilon), \quad \int m_i |v|^2 v f_i^\varepsilon dv = 5\varepsilon k_B T J_i^\varepsilon + o(\varepsilon).$$

- ▶ Moment of order 2 (order  $\varepsilon^1$ ), summing over  $i$ , and taking the limit  $\varepsilon \rightarrow 0$

$$3\partial_t \sum_i n_i + 5\nabla_x \cdot \sum_i J_i = 0,$$

where the collision operator disappears by symmetry when summing over  $i$ .

- ▶ Combining with mass conservation implies

$$\partial_t \sum_i n_i = \nabla_x \cdot \sum_i J_i = 0$$

- ▶ Constant total number of molecules  $\sum_i n_i$
- ▶ Compatible with equimolar diffusion  $\sum_i J_i(t, x) = 0$

## Closure relation (2)

Fick equation  $\mathbf{J} = -B(\mathbf{n})\nabla\mathbf{n}$

- ▶ Summing over  $i$  the equations ( $B$  has rank  $p - 1$ ) implies that  $\sum_i m_i J_i = 0$
- ▶ Inversion giving the perturbation  $\mathbf{g}^\varepsilon$  (relation  $(\star)$ ) only valid if the RHS  $W_i = \mu_i \mathbf{v} \cdot \nabla_x n_i \in (\text{Ker } \mathbf{L})^\perp$ .
- ▶  $\text{Ker } \mathbf{L}$  spanned by  $(\sqrt{n_i} \mu_i \mathbf{e}_i)_i, m_i n_i \mu_i \mathbf{v}, m_i n_i \mu_i |\mathbf{v}|^2$
- ▶ Orthogonality

$$0 = \sum_i \int \mu_i \mathbf{v} \cdot \nabla_x n_i m_i \mathbf{v} d\mathbf{v} = \nabla_x \sum_i m_i n_i$$

- ▶ Mass conservation for each species implies (when summing with weights  $m_i$ )

$$0 = \frac{d}{dt} \int \sum_i m_i n_i dx$$

- ▶ Constant mass  $\sum_i m_i n_i$



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# Stiff dissipative model for mixtures

For any species  $i$  with mass density  $\rho_i$  and velocity  $\mathbf{u}_i$ , we write mass and momentum conservation

$$(*) \begin{cases} \partial_t \rho_i + \nabla_{\mathbf{x}} \cdot (\rho_i \mathbf{u}_i) = 0, \\ \partial_t (\rho_i \mathbf{u}_i) + \nabla_{\mathbf{x}} \cdot (\rho_i \mathbf{u}_i \otimes \mathbf{u}_i + P_i(\rho_i)) + \frac{1}{\varepsilon} R_i = 0 \end{cases}$$

- ▶ Ideal gas law for the partial pressure  $P_i(\rho_i) = \rho_i k_B T / m_i$
- ▶ Relaxation term: friction force exerted by the mixture on species  $i$

$$R_i = \sum_{j \neq i} a_{ij} \rho_i \rho_j (\mathbf{u}_j - \mathbf{u}_i) = \sum_j \alpha_{ij} \mathbf{u}_j$$

Using the formalism of Chen, Levermore, Liu, CPAM, '94

Obtain a reduced system when  $\varepsilon$  remains small

- ▶ Derive an approximation of the local equilibrium and its first-order correction
  - ▶ Build a relevant entropy which ensures...
  - ▶ ... the hyperbolicity of the local equilibrium approximation...
  - ▶ ... and the dissipativity of its first-order correction

# Steps of the computations

- ▶ Internal energy  $E_i''(\rho_i) = P_i'(\rho_i)/\rho_i$
- ▶ (Strictly convex) entropy  $\eta = \sum_{j=1}^P \frac{1}{2} \rho_j \mathbf{u}_j^2 + E_j(\rho_j)$
- ▶  $(p + d)$  independent conserved quantities :  $[\rho_1, \dots, \rho_p, \sum_{j=1}^P \rho_j \mathbf{u}_j]$
- ▶ Equilibrium:  $[\rho_1, \dots, \rho_p, \rho_1 \mathbf{u}, \dots, \rho_p \mathbf{u}]$  for some  $\mathbf{u}$

## Formal expansion around the equilibrium & linearization

↪ expression of the **correction** provided (**pseudo-**)**inversion** of “the gradient of the relaxation term”, involving the “**flux terms**”

$$\sum_{j=1}^P \alpha_{ij} \frac{X_j}{\rho_j} = \nabla_x P_i(\rho_i) - \frac{\rho_i}{\rho} \nabla_x P$$

with  $\rho = \sum_i \rho_i$ ,  $P = \sum_i P_i$

↪ equation on the conserved quantities with the correction term

# Steps of the computations

- ▶ Internal energy  $E_i''(\rho_i) = P_i'(\rho_i)/\rho_i$
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## Formal expansion around the equilibrium & linearization

↪ expression of the **correction** provided (**pseudo-**)**inversion** of “the gradient of the relaxation term”, involving the “flux terms”

$$X_i = \sum_{j=1}^P \frac{\beta_{ij}}{\rho_j} \left( \nabla_x P_j(\rho_j) - \frac{\rho_j}{\rho} \nabla_x P \right)$$

with  $\rho = \sum_i \rho_i$ ,  $P = \sum_i P_i$

↪ equation on the conserved quantities with the correction term

# Maxwell-Stefan vs. Fick

Reduced system involving the bulk velocity  $\mathbf{u}$  for small  $\varepsilon$

Let  $\rho = \sum_i \rho_i$ , and  $\mathbf{u}$  the mass-weighted averaged (aligned) velocity.

System (\*) formally reduces to

$$\begin{cases} \partial_t \rho_i + \nabla_{\mathbf{x}} \cdot \left( \rho_i \mathbf{u} - \varepsilon \sum_{j=1}^p \beta_{ij} \frac{\nabla_{\mathbf{x}} P_j}{\rho_j} \right) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} P = \mathbf{0}. \end{cases}$$

where  $P = \sum_j P_j(\rho_j)$  is the total pressure, and  $(\beta_{ij})$  are positive constants.

- ▶ Diffusion correction term of Fick's type (on the mass equation)
  - ▶ Ideal gases  $\rightsquigarrow \nabla_{\mathbf{x}} P_j \propto \nabla \rho_j$
  - ▶ Fick equations model mass diffusion in a continuous regime
- ▶ No viscosity term on the momentum equation (convective  $\gg$  diffusive fluxes)
- ▶ Maxwell-Stefan equations needed in a moderately rarefied regime

# Justification of the Ansatz for the Maxwell-Stefan equations

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In a moderately rarefied regime (not so dominant collision process)

- ▶ Significant deviation from local equilibrium described by the moment method
- ▶ Moment method: approach to compute Galerkin solutions to the Boltzmann equation

[LEVERMORE, JSP '96]

- 1 First finite dimensional subspace  $\mathbb{M}_0 = \text{Ker } Q$  spanned by  $e_1, \dots, e_p$ ,  $[m_1 v, \dots, m_p v]$  and  $[m_1 v^2, \dots, m_p v^2]$

↪ equilibrium with one bulk velocity

- 2 Second finite dimensional subspace  $\mathbb{M}_1 \supset \mathbb{M}_0$  spanned by  $e_1, \dots, e_p$ ,  $m_1 v e_1, \dots, m_p v e_p$  and  $[m_1 v^2, \dots, m_p v^2]$

↪ local Maxwellian with different macroscopic velocities

# Outline of the talk

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- 1 Introduction
  - Context of the study
  - Gaseous mixtures: macroscopic models
  - Kinetic setting
- 2 Moment method
- 3 Perturbative method
- 4 Stiff dissipative hyperbolic formalism
- 5 Conclusion and prospects

# Conclusion and prospects

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## Conclusions

- ▶ Maxwell-Stefan and Fick equations are obtained formally from the Boltzmann equation for mixtures in the diffusive regime
- ▶ Not the same assumption on the macroscopic velocities
- ▶ Maxwell-Stefan can describe a moderate rarefied regime

## Prospects

- ▶ Compare the experimental and theoretical relaxation times
- ▶ Taking into account non isothermal effects



Thank you for your attention!

