# ADAPTIVE NONPARAMETRIC STRATEGIES FOR CENSORED LIFETIMES WITH UNKNOWN SAMPLING BIAS.

E. BRUNEL[1], F. COMTE[2], AND A. GUILLOUX[3]

Abstract. In addition to the right-censoring of the lifetimes, sampling bias possibly occurs in many situations while collecting the data. We provide a model selection strategy for estimating the hazard and the density functions in presence of both censoring and unknown sampling bias. We use a new sampling scheme description based on the Lexis diagram including time-window or cohort studies. Adaptive projection estimators on trigonometric bases are developed by contrast penalization and optimal nonparametric rates of convergence are given. Both estimators are practically studied through simulation experiments and real data in the case of the time-window study.

AMS Classification (2001): 62G07; 62N01.
**Keywords.** Adaptive estimation. Biased data. Right-Censoring. Hazard rate. Nonparametric penalized contrast estimator. Nonhomogeneous Poisson process.

## 1. Introduction

In this paper, we consider independent identically distributed (i.i.d.) lifetime data $X_1, \dots, X_n$ subject to both bias and censoring. Our aim is to provide nonparametric estimators of the probability density function (p.d.f.) $f$ and the hazard rate $\lambda$. The originality here is that the bias is not assumed to be known.

Censoring occurs when lifetimes are not completely observed. It is generally modelized by a sequence $C_1, \dots, C_n$ of i.i.d. censoring times independent of the lifetime sequence $X_1, \dots, X_n$. Estimation procedures have to take into account that the observed variables are $X_1 \wedge C_1, \dots, X_n \wedge C_n$ where $x \wedge y = \inf(x, y)$. Density and hazard estimation in this setting has been studied with nonparametric techniques by many authors (see Tanner & Wong (1983), Mielniczuk (1985), Marron & Padgett (1987), Lo *et al.* (1989), Antoniadis *et al.* (1999) and Brunel & Comte (2005)).

On the other hand, it may be impossible to draw a direct sample from the distribution of $X$ because of some "selection bias". Heuristically, selection bias arises when the event $\{X = x\}$ is observed with a probability proportional to a weight $w(x)$. The observed random variables (r.v.) have thus a common p.d.f. given by $w(x)f(x)/\int w(u)f(u)du$, for all $x > 0$. The relevance of selection bias in statistical inference has been first pointed out by Fisher (1934). Since then, many authors

(1) IUT de Paris V, MAP5, UMR CNRS 8145. email: elodie.brunel@univ-paris5.fr.
(2) Université Paris V, MAP5, UMR CNRS 8145. email: fabienne.comte@univ-paris5.fr.
(3) L.S.T.A., Université Paris VI. email: aguillou@ccr.jussieu.fr.

noticed its presence in data from a wide range of fields. We refer to Chakraborty & Rao (2000) for biomedical applications, and Heckman (1985) in Economics, among many others. The review by Patil & Rao (1977) gives numerous practical examples of weighted distributions. The problem of nonparametric estimation from a biased sample has also been widely studied (see Gill *et al.* (1988), Efromovich (2004a), Efromovich (2004b), Vardi (1982), de Uña-Àlvarez (2002) and Asgharian *et al.* (2002)). The key point in these works is that the bias function $w$ is assumed to be known, an assumption which is certainly questionable.

Note that bias and censoring have been simultaneously considered in de Uña-Àlvarez (2002) and Brunel *et al.* (2005) but still under known bias. Let us also mention the work of Uzunogullari & Wang (1992) on kernel hazard estimation for left truncated (corresponding to a length bias) and right censored data.

Our aim here is to work in a context where both selection bias and censoring occur, but without assuming the bias is known. This problem is not identifiable in the usual setting. This is why we resort to the new sampling model recently proposed by Guilloux (2006). It is based on Lexis (1875) diagrams and also exploited for analogous modelization purpose by Keiding (1990) and Lund (2000).

Let $(\sigma_i)_{i\in I}$ be the birth dates of the individuals in a given population $I$ and $(X_i)_{i\in I}$ their corresponding lifetimes. In the Lexis diagram, an individual can be represented by his life-line, $\mathcal{L}(\sigma, X) = \{(\sigma+y, y),\ 0 \le y \le X\}$, which is a unit-slope line whose points have as coordinates the calendar time $(\sigma + y)$ and the age $(y)$. Let $\mathcal{S}$ be a deterministic Borel set in the Lexis diagram. Individuals with life-lines intersecting $\mathcal{S}$ are included in the study, i.e. pairs $(\sigma, X)$ such that $\mathcal{L}(\sigma, X)\cap\mathcal{S} \neq \emptyset$. Time-window or cohort studies can be described by such a sampling pattern with a particular choice of $\mathcal{S}$ (see Section 2.2). Let $\sigma_{\mathcal{S}}$ denote the birth time and $X_{\mathcal{S}}$ the lifetime for the included individuals. The pair $(\sigma_{\mathcal{S}}, X_{\mathcal{S}})$ is the observable r.v. as opposed to the unobservable pair $(\sigma, X)$. Then, we can prove that the r.v. $X_{\mathcal{S}}$ suffer from a selection bias which can be described through the other processes involved in the model (see Section 2.3).

In what follows, we construct and study adaptive estimators of the hazard rate $\lambda$ and the density $f$ in Guilloux (2006)'s model, taking advantage of its probabilistic properties. We develop nonparametric projection methods following the model selection lines of Barron *et al.* (1999).

The paper is organized as follows. In Section 2, the bias and censoring model is described, some of its properties are recalled. Section 3 defines the penalized estimators of the functions $\lambda$ and $f$, and provides nonasymptotic bounds for their mean integrated squared errors (MISE). In Section 4, simulation experiments illustrate the penalized estimators in the case of the time-window study. An example based on Hyde (1977)'s channing house data compares in Section 5 our hazard estimator with Uzunogullari & Wang (1992)'s kernel-type one. Most proofs are gathered in Section 6.

## 2. Bias and censoring model

In this section, we present the sampling mechanism and explain how it results in a selection bias.

### 2.1. **The sampling model.**

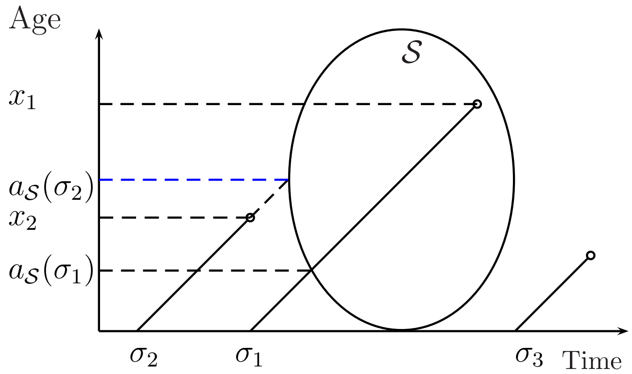



Figure 2.1. Sampling scheme

Let $\mathcal{S}$ be a Borel set and denote by $a_{\mathcal{S}}(\sigma)$ the age at inclusion for a birth-time $\sigma$:

$$\begin{cases} a_{\mathcal{S}}(\sigma) = \inf\{y \geq 0,\ (\sigma + y, y) \in \mathcal{S}\} \\ a_{\mathcal{S}}(s) = \infty \text{ if the infimum does not exist.} \end{cases}$$

The individual $i$ with birth date $\sigma_i$ and lifetime $X_i$ is included in the sample if:

$$\mathcal{L}(\sigma_i, X_i) \cap \mathcal{S} \neq \emptyset \Leftrightarrow a_{\mathcal{S}}(\sigma_i) < \infty \text{ and } X_i \geq a_{\mathcal{S}}(\sigma_i). \tag{1}$$

Figure 2.1 gives a representation of the Lexis diagram and the sampling process described above : individual 1 is included in the sample, individual 2 could have been included but died before his inclusion since $x_2 < a_{\mathcal{S}}(\sigma_2)$, whereas individual 3 is not included because $a_{\mathcal{S}}(\sigma_3) = \infty$.

### 2.2. **Two examples for $\mathcal{S}$.**

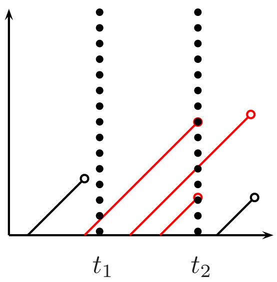
Let us first give two examples of sampling patterns, which can be described through a Borel set in $\mathcal{B}_{\mathbb{R}\times\mathbb{R}_+}$: the time-window and cohort studies.

*Time-window study.* The individuals alive at time $t_1$ or born between $t_1$ and $t_2$ are included in the study, $t_1$ and $t_2$ are fixed. The Borel set to consider is :

$$\mathcal{S}_{tw} = \{(s, y), t_1 \leq s \leq t_2, y \in \mathbb{R}_+\}.$$

The age $a_{\mathcal{S}_{tw}}$ at inclusion is given by:

$$\begin{cases} a_{\mathcal{S}_{tw}}(\sigma) &= t_1 - \sigma & \text{if} \quad -\infty < \sigma \leq t_1 \\ a_{\mathcal{S}_{tw}}(\sigma) &= 0 & \text{if} \quad t_1 < \sigma \leq t_2 \\ a_{\mathcal{S}_{tw}}(\sigma) &= +\infty & \text{if} \quad \sigma > t_2. \end{cases}$$



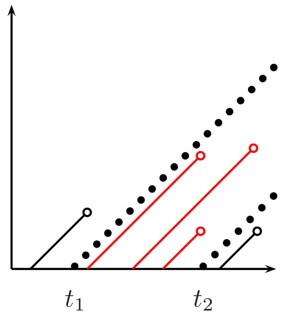

Example 1: Time-window study

*Cohort study.* The individuals born between times $t_1$ and $t_2$ are included in the study, $t_1$ and $t_2$ are fixed. In case, the Borel set $\mathcal{S}_c$ is given by:

$$\mathcal{S}_c = \{(s+u,u), t_1 \leq s \leq t_2, u \in \mathbb{R}_+\}.$$

The age $a_{\mathcal{S}_c}$ at inclusion is then given by:

$$\begin{cases} a_{\mathcal{S}_c}(\sigma) &= +\infty \quad \text{if} \quad -\infty < \sigma \leq t_1 \\ a_{\mathcal{S}_c}(\sigma) &= 0 \quad \text{if} \quad t_1 < \sigma \leq t_2 \\ a_{\mathcal{S}_c}(\sigma) &= +\infty \quad \text{if} \quad \sigma > t_2. \end{cases}$$



Example 2: Cohort study

We refer to Lund (2000) for an extensive review of studies described through the Lexis diagram.

2.3. **Theoretical representation of the model and results on the involved counting processes.** In this subsection, we recall some results obtained by Guilloux (2006) useful for the present purpose.

2.3.1. *Bias.* Following Lund (2000), we consider the following modelization associated with the Lexis diagram. Let the point process $\eta = \sum_{i\in I}\varepsilon_{\sigma_i}$, with the birth times as occurrence times, be a non-homogeneous Poisson process on $\mathbb{R}$ with intensity $\varphi$ (where $\varepsilon_a$ is the Dirac measure at point $a$). Assume furthermore, that the lifetimes $X_i$, for $i \in I$, are i.i.d. with common probability density function $f$. The marking theorem (see Kingman (1993)) ensures that the point process $\mu = \sum_{i\in I}\varepsilon_{(\sigma_i,X_i)}$ is a nonhomogeneous Poisson process with intensity $\varphi f$.

Here, we are interested in the individuals whose life-lines intersect the Borel set $\mathcal{S}$. In other words, we are interested in the restriction $\mu_{|\mathcal{S}}$ of the process $\mu$ to the Borel set $\mathcal{S}$. The restriction theorem (see Kingman (1993)) ensures that the restriction $\mu_{|\mathcal{S}}$ is a Poisson process with mean measure $\int_{B\cap\mathcal{S}}\varphi f/\int_{\mathcal{S}}\varphi f$, for any Borel set $B$ in $\mathcal{B}_{\mathbb{R}\times\mathbb{R}_+}$. Finally, by the order statistics property for Poisson processes, given the number $\mu(\mathcal{S})$ of points in the Borel set $\mathcal{S}$, the points of Poisson process $\mu_{|\mathcal{S}}$ look exactly like independent random variables, with common probability measure $\mathbb{P}(\cdot) = \int_{\cdot\cap\mathcal{S}}\varphi f/\int_{\mathcal{S}}\varphi f$ on Borel subsets of $\mathbb{R}\times\mathbb{R}_+$.

As a consequence, using (1), we have, for all $s \in \mathbb{R}$ and $x \in \mathbb{R}_+$:

$$\begin{aligned} \mathbb{P}(\sigma_{\mathcal{S}} \leq s, X_{\mathcal{S}} \leq x) &= \mathbb{P}(\sigma \leq s, X \leq x | \mathcal{L}(\sigma, X) \cap \mathcal{S} \neq \emptyset) \\ &= \frac{\iint_{]-\infty,s]\times[0,x]} \mathbb{1}_{\{(u,v)\in\mathcal{S}\}}\varphi(u)f(v)dudv}{\iint_{\mathbb{R}\times\mathbb{R}_+} \mathbb{1}_{\{(u,v)\in\mathcal{S}\}}\varphi(u)f(v)dudv} \\ &= \frac{1}{\mu_{\mathcal{S}}}\iint_{]-\infty,s]\times[0,x]}\varphi(u)f(v)\mathbb{1}_{\{a_{\mathcal{S}}(u)<\infty\}}\mathbb{1}_{\{a_{\mathcal{S}}(u)\leq v\}}dudv, \end{aligned} \tag{2}$$

where $\mathbf{I}_{\{.\}}$ stands for the indicator function and

$$\mu_{\mathcal{S}} = \iint_{\mathbb{R}\times\mathbb{R}_+} \mathbf{I}_{\{a_{\mathcal{S}}(u)<\infty\}}\mathbf{I}_{\{a_{\mathcal{S}}(u)\leq v\}}\varphi(u)f(v)dudv.$$

Hence the marginal distribution of the r.v. $X_{\mathcal{S}}$ is given, for all $x\in\mathbb{R}_+$, by :

$$F_{\mathcal{S}}(x) = \mathbb{P}\left(X_{\mathcal{S}}\leq x\right) = \frac{1}{\mu_{\mathcal{S}}}\int_0^x w(v)f(s)ds, \tag{3}$$

with

$$w(x) \quad = \quad \int_{-\infty}^{\infty}\mathbf{I}_{\{a_{\mathcal{S}}(u)\leq x\}}\varphi(u)du. \tag{4}$$

For example, in the time-window study (see Section 2.2), the weight function $w$ is given, for $x\geq 0$, by:

$$w(x) = \int_{t_1-x}^{t_2}\varphi(u)du.$$

In the particular case where $t_1 = t_2$ and $\varphi$ is a constant, such a sample is called a "length-biased sample", see Asgharian *et al.* (2002) and de Uña-Àlvarez (2002).

In the example of the cohort study (see Section 2.2), the weight function $w$ is constant and given, for $x\geq 0$, by $w(x) = \int_{t_1}^{t_2}\varphi(u)du$.

2.3.2. *Censoring.* Now only the individuals whose life-lines intersect the Borel set $\mathcal{S}$ are included in the study. For an included individual $i$, with birth date $\sigma_{\mathcal{S},i}$ and lifetime $X_{\mathcal{S},i}$, we assume that his age at inclusion $a_{\mathcal{S}}(\sigma_{\mathcal{S},i})$ is observable. The lifetime $X_{\mathcal{S},i}$ can straightforwardly be written as follows:

$$X_{\mathcal{S},i} \quad = \quad \underbrace{a_{\mathcal{S}}(\sigma_{\mathcal{S},i})}_{\text{age at inclusion}} \quad + \quad \underbrace{(X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i}))}_{\text{time spent in the study}}.$$

As the time spent in the study is given by $X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i})$, we shall assume that this time can be censored. It is the case, for example, if an individual $i$ leaves the study before his death, see Asgharian (2003) and Winter & Földes (1988).

For that matter, we introduce a non-negative r.v. $C$ with p.d.f. $h$ and c.d.f. $H$, independent of $X_{\mathcal{S}}$ and $a_{\mathcal{S}}(\sigma_{\mathcal{S}})$, such that the observable time for individual $i$ is

$$Z_i = a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) \quad + \quad (X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i}))\wedge C_i.$$

As usual, we assume furthermore that the r.v. $\mathbf{I}_{\{X_{\mathcal{S},i}-a_{\mathcal{S}}(\sigma_{\mathcal{S},i})\leq C\}}$ is observable. As a consequence, the available data are i.i.d. replications of:

$$\begin{cases} \sigma_{\mathcal{S},i} \\ Z_i = a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) + (X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i}))\wedge C_i, \quad \text{for i=1,...,n.} \\ \mathbf{I}_{\{X_{\mathcal{S},i}-a_{\mathcal{S}}(\sigma_{\mathcal{S},i})\leq C_i\}} \end{cases} \tag{5}$$

We seek to estimate the density function $f$ and the hazard rate $\lambda = f/(1-F)$ of the unbiased r.v. $X$ with the available data described by (5).

2.3.3. *Counting processes for the estimation.* In this context, Guilloux (2006) introduces the following counting processes. For all $x \geq 0$, let

$$D(x) = \sum_{i=1}^{n} \mathbb{I}_{\{Z_i \leq x, X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) \leq C_i\}}. \tag{6}$$

For $x \geq 0$, the r.v. $D(x)$ is the "number of observed deaths before age $x$" in the sample. Let furthermore the process $O$ be defined, for all $x \geq 0$, by:

$$O(x) = \sum_{i=1}^{n} \mathbb{I}_{\{a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) \leq x \leq Z_i\}} = \sum_{i=1}^{n} \mathbb{I}_{\{a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) \leq x \leq X_{\mathcal{S},i}, x \leq a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) + C_i\}} \tag{7}$$

The r.v. $O(x)$ represents the "number of individuals at risk at age $x$". In the sampling situation considered here, to be at risk at age $x$ for an individual means that he was included in the study at age less than $x$ and is neither dead nor censored before age $x$.

Let $\Lambda$ denote the cumulative hazard function of the r.v. $X$ and be defined as:

$$\Lambda(x) = \int_0^x \frac{f(s)ds}{1 - F(s)},$$

for all $x \geq 0$. As in classical survival analysis, it seems natural to define the estimator $\widehat{\Lambda}_n^\epsilon$ by:

$$\widehat{\Lambda}_n^\epsilon(x) = \int_0^x \frac{dD(s)}{O(s) + n\epsilon}, \tag{8}$$

for all $x \geq 0$. Guilloux (2006) studies $\widehat{\Lambda}_n^{\epsilon_n}(.)$ where $(\epsilon_n)_{n\geq 1}$ is a sequence of positive numbers such that $\epsilon_n \to 0$ and $\sqrt{n}\varepsilon_n \to 0$ as $n \to \infty$.

We can also define the estimator $\widehat{F}_n$ for the c.d.f. $F$ of the r.v. $X$. Mimicking the construction of the Kaplan-Meier estimator in classical survival analysis, we define, for all $x \geq 0$:

$$\hat{F}_n(x) = 1 - \prod_{i:Z_i \leq x} \left(1 - \frac{\mathbb{I}_{\{X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i}) \leq C_i\}}}{O(Z_i) + n\epsilon_n}\right). \tag{9}$$

Let us define

$$\tau = \inf(\tau_F, \tau_H) \text{ with } \tau_L = \{x \in \mathbb{R}^+ : (1 - L)(x) = 0\}, \text{ for any c.d.f. } L. \tag{10}$$

It is useful to mention the following result proved in Guilloux (2006).

**Theorem 2.1.** *Assume that there exist $w_1$ and $w_2$ such that $w_1 \leq w(x) \leq w_2$, for all $x \geq 0$. Then, for all $u > 0$:*

$$\mathbb{P}\left(\sqrt{n} \sup_{x\in[0,\tau]} \left| \left(\hat{F}_n(x) - F(x)\right)(1 - H)^2(x) w_1 \right| > u\right) \leq 2.5 \exp(-2u^2 + Cu),$$

*where $C$ is an universal constant (which does not depend on $F$, $H$ nor $w$).*

## 3. Estimation procedure

Now, gathering all the tools presented in Section 2, we are in position to define the projection estimators and to study their properties.

### 3.1. Assumptions on the model.

The estimators are built on a compact set $A$. We assume $A \subsetneq [0,\tau]$ with $\tau$ defined by (10). Up to a scaling change, we set $A = [0,1]$ without loss of generality. First, we assume that

$$\exists w_1, w_2 > 0, \text{ such that } 0 < w_1 \leq w(x) \leq w_2 < +\infty, \ \forall x \in A. \tag{11}$$

This is a very mild condition since $A$ is a compact set.
It follows from (3) that $F_{\mathcal{S}}(x) = \int_0^{\tau_F \wedge x} w(s)f(s)ds/\mu_{\mathcal{S}}$ with $\mu_{\mathcal{S}} = \int_0^{\tau_F} w(s)f(s)ds$, so that $\tau_{F_{\mathcal{S}}} \leq \tau_F$. Moreover, the function $w$ is nondecreasing and by (11) it is lower bounded, thus $\tau_F = \tau_{F_{\mathcal{S}}}$.
Secondly, the following property holds for $H$:

$$\exists c_H > 0, \forall x \in A, c_H \leq 1 - H(x) \leq 1, \tag{12}$$

by taking $c_H = \inf_{x\in A}[1 - H(x)]$.
Lastly, the same property as (12) holds also for $F$ with $c_H$ replaced by $c_F$.
In the following, estimators are computed over the compact set $A \subsetneq [0,\tau]$. Note that this does not imply any practical restriction since, for estimation purpose, we can choose the interval $A$ such that the largest uncensored $X_{\mathcal{S},i}$ belongs to it.

### 3.2. Description of the projection bases.

Let $\psi_0(x) = 1$, $\psi_{2j}(x) = \sqrt{2}\cos(2\pi jx)$, $\psi_{2j-1}(x) = \sqrt{2}\sin(2\pi jx)$ and consider

$$S_m = \mathrm{Span}(\psi_j(x), j = 0, \dots, m-1).$$

Then $S_m$ is a finite dimensional subspace with dimension $m$ of $\mathbb{L}^2([0,1])$ endowed with the scalar product $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$. The functions $(\psi_j)_{0\leq j\leq m-1}$ constitute an orthonormal basis of $S_m$ so that $t = \sum_{j=0}^{m-1} a_j\psi_j$, for any function $t$ in $S_m$ and with $a_j = \langle t, \psi_j\rangle$. A key property of the trigonometric basis relies on the link between the supremum norm and the $\mathbb{L}^2$ norm:

$$\forall m, \forall t \in S_m, \|t\|_\infty \leq \sqrt{2m}\|t\| \quad \text{and} \quad \|\sum_{j=0}^{m-1} \psi_j^2\|_\infty \leq 2m. \tag{13}$$

For $m, m' \in \mathcal{M}_n$, $m \leq m'$ implies that $S_m \subset S_{m'}$ which means that the subspaces are nested. Then $\mathcal{S}_n = \cup_{m\in\mathcal{M}_n} S_m$ stands for the largest subspace of the collection and we denote by $N_n$ the dimension of $\mathcal{S}_n$.

### 3.3. Definition of the estimators.

Using the observations described by (5), we can define contrast functions to estimate both the density $f$ and the hazard rate $\lambda$.

3.3.1. *Estimation of the hazard rate.* Let us define

$$\Phi(x) = \mathbb{P}(a_{\mathcal{S}}(\sigma_{\mathcal{S},1}) \le x \le Z_1),$$

which stands for the probability for an individual to be at risk at age $x$. This means clearly that the individual was included in the study at age less than $x$ and is neither dead nor censored before $x$, so that we can write

$$\Phi(x) = \mathbb{P}(a_{\mathcal{S}}(\sigma_{\mathcal{S},1}) \le x \le X_{\mathcal{S},1}, x \le a_{\mathcal{S}}(\sigma_{\mathcal{S},1}) + C_1). \tag{14}$$

From definition (7) of the process $O(x)$, it is obvious that $\mathbb{E}(O(x)/n) = \Phi(x)$. This is why we take

$$\Phi_n(x) = \frac{O(x)}{n}\,, \tag{15}$$

as an estimator of $\Phi(x)$. Note that

$$\forall n \in \mathbb{N}^*,\ \Phi_n(Z_i) \ge 1/n\,,\ \text{for } i = 1, \dots, n. \tag{16}$$

The following lemma states useful properties of $\Phi$ and $\Phi_n$ with first a useful consequence of Theorem 2.1.

**Lemma 3.1.** *1) Assume that there exists $w_1$ such that, for all $x \ge 0$, $w_1 \le w(x)$. For all $k \in \mathbb{N}^*$, there exists a constant $C_F(k)$ depending on $k$, $w$ and $c_H$ such that*

$$\mathbb{E}\left(\sup_{x \in A} |\hat{F}_n(x) - F(x)|^{2k}\right) \le C_F(k) n^{-k}.$$

*2) There exists a constant $c_\Phi$ such that, for all $x \in A$, $\Phi(x) \ge c_\Phi$.*
*3) For all $k \in \mathbb{N}^*$, there exists a constant $C_\Phi(k)$ such that*

$$\mathbb{E}\left(\sup_{x \in \mathbb{R}^+} |\Phi_n(x) - \Phi(x)|^{2k}\right) \le C_\Phi(k) n^{-k}.$$

**Remark 3.1.** *The condition $w_1 \le w(x)$ holds for the time-window and the cohort studies as soon as the interior of $\mathcal{S} \cap \{(x,0), x \in \mathbb{R}\}$ is non empty. It means that a death can occur immediately at the inclusion of an individual.*

Moreover, the definition of the contrast for estimating $\lambda$ is deduced from the following property:

**Proposition 3.1.** *For any function $t$ in $S_m$,*

$$\mathbb{E}\left(\frac{\delta_1 t(Z_1)}{\Phi(Z_1)}\right) = \langle t, \lambda\rangle \text{ where } \langle u, v\rangle = \int_0^1 u(x)v(x)dx.$$

Therefore, by replacing $\Phi(x)$ by its natural estimator, we obtain the following projection contrast function, for $t \in S_m$,

$$\gamma_n(t) = \|t\|^2 - \frac{2}{n}\sum_{i=1}^n \frac{\delta_i t(Z_i)}{\Phi_n(Z_i)}. \tag{17}$$

Clearly, $\gamma_n(t)$ is the empirical version of $\|t\|^2 - 2\langle t, \lambda\rangle = \|t-\lambda\|^2 - \|\lambda\|^2$, where $\|t\|^2 = \int_0^1 t^2(x)dx$. Moreover, the contrast function can also be seen as

$$\gamma_n(t) = \|t\|^2 - 2\int_0^1 t(x)d\widehat{\Lambda}_n^0(x)$$

where $\widehat{\Lambda}_n^0$ is defined by (8).
Then we can define the projection estimator

$$\hat{\lambda}_m = \arg\min_{t\in S_m} \gamma_n(t)$$

which gives an estimator of $\lambda_m$, the orthogonal projection of $\lambda$ on $S_m$. Lastly, we select

$$\hat{m} = \arg\min_{m\in\mathcal{M}_n} \left\{\gamma_n(\hat{\lambda}_m) + \mathrm{pen}(m)\right\}$$

which gives an estimator $\hat{\lambda}_{\hat{m}}$ of $\lambda$ on a random space $S_{\hat{m}}$. Here, pen(.) is a penalty function that ensures the squared-bias variance compromise, it is given by

$$\mathrm{pen}(m) = \kappa\left(\frac{1}{n}\sum_{i=1}^n \frac{\delta_i}{\Phi_n^2(Z_i)}\right)\frac{m}{n}. \tag{18}$$

Note that

$$\lambda_m = \sum_{j=0}^{m-1} a_j\psi_j \text{ with } a_j = \langle\lambda, \psi_j\rangle,$$

and

$$\hat{\lambda}_m = \sum_{j=0}^{m-1} \hat{a}_j\psi_j, \text{ with } \hat{a}_j = \frac{1}{n}\sum_{i=1}^n \frac{\delta_i\psi_j(Z_i)}{\Phi_n(Z_i)}.$$

Moreover $\gamma_n(\hat{\lambda}_m) = -\sum_{j=1}^{m-1}\hat{a}_j^2$.

3.3.2. *Estimation of the density.* We can deduce from Proposition 3.1 and from $f = \lambda(1-F) = \lambda\bar{F}$, the following projection contrast for the estimation of $f$:

$$\tilde{\gamma}_n(t) = \|t\|^2 - \frac{2}{n}\sum_{i=1}^n \delta_i t(Z_i)\frac{1-\hat{F}_n(Z_i)}{\Phi_n(Z_i)}, \tag{19}$$

where $\hat{F}_n$ and $\Phi_n$ are the estimators of $F$ and $\Phi$ defined by (9) and (15). Again $\tilde{\gamma}_n(t)$ is an empirical version of $\|t\|^2 - 2\int_0^1 t(x)f(x)dx = \|t-f\|^2 - \|f\|^2$. Then we can define

$$\hat{f}_m = \arg\min_{t\in S_m} \tilde{\gamma}_n(t)$$

and $\tilde{m} = \arg\min \tilde{\gamma}_n(\hat{f}_m) + \widetilde{\mathrm{pen}}(m)$. As previously, $\widetilde{\mathrm{pen}}(.)$ is a penalty function that ensures the squared-bias variance compromise, it is given by

$$\widetilde{\mathrm{pen}}(m) = \kappa\left(\frac{1}{n}\sum_{i=1}^n \frac{\delta_i(1-\hat{F}_n(Z_i))^2}{\Phi_n^2(Z_i)}\right)\frac{m}{n}. \tag{20}$$

We also have, $f_m = \sum_{j=0}^{m-1} b_j\psi_j$ with $b_j = \langle f, \psi_j\rangle$, and $\hat{f}_m = \sum_{j=0}^{m-1} \hat{b}_j\psi_j$, with $\hat{b}_j = (1/n)\sum_{i=1}^n \delta_i(1-\hat{F}_n(Z_i))\psi_j(Z_i)/\Phi_n(Z_i)$. Moreover $\tilde{\gamma}_n(\hat{f}_m) = -\sum_{j=1}^{m-1} \hat{b}_j^2$.

3.4. **MISE bounds for the adaptive estimators of the hazard rate and the density.** The study of the estimator relies on the following decompositions. For $\lambda$, we have

$$\gamma_n(t) - \gamma_n(s) = \|t-\lambda\|^2 - \|s-\lambda\|^2 - 2\nu_n(t-s) - 2R_n(t-s)$$

where

$$\nu_n(t) = \frac{1}{n}\sum_{i=1}^n \left[\frac{\delta_i t(Z_i)}{\Phi(Z_i)} - \langle t, \lambda\rangle\right], \quad R_n(t) = \frac{1}{n}\sum_{i=1}^n \delta_i t(Z_i)\frac{\Phi(Z_i)-\Phi_n(Z_i)}{\Phi(Z_i)\Phi_n(Z_i)}.$$

For $f$, we have

$$\tilde{\gamma}_n(t) - \tilde{\gamma}_n(s) = \|t-f\|^2 - \|s-f\|^2 - 2\tilde{\nu}_n(t-s) - 2R_{n,1}(t-s) - 2R_{n,2}(t-s) \tag{21}$$

where $\tilde{\nu}_n$ is a centered empirical process

$$\tilde{\nu}_n(t) = \frac{1}{n}\sum_{i=1}^n \left[\frac{\delta_i t(Z_i)\bar{F}(Z_i)}{\Phi(Z_i)} - \langle t, f\rangle\right]$$

and $R_{n,1}$ and $R_{n,2}$ are residual terms:

$$R_{n,1}(t) = \frac{1}{n}\sum_{i=1}^n \delta_i t(Z_i)\frac{F(Z_i)-\hat{F}_n(Z_i)}{\Phi_n(Z_i)}, \quad R_{n,2}(t) = \frac{1}{n}\sum_{i=1}^n \delta_i t(Z_i)\bar{F}(Z_i)\frac{\Phi(Z_i)-\Phi_n(Z_i)}{\Phi_n(Z_i)\Phi(Z_i)}. \tag{22}$$

By proving that the residual terms are negligible and applying Talagrand (1996) inequality to $\nu_n$ or $\tilde{\nu}_n$, we obtain the following results.

**Theorem 3.1.** *Consider the estimators $\hat{\lambda}_{\hat{m}}$ and $\hat{f}_{\tilde{m}}$ defined in Section 3.3.1 and 3.3.2 and the collection of models $(S_m)_{m\in\mathcal{M}_n}$ with $N_n \le c/\sqrt{n}$. Assume that $f(x) \le f_1$ and that $w_1 \le w(x) \le w_2$ for all $x \in A$, then*

$$\mathbb{E}(\|\hat{g}_{\hat{m}} - g\|^2) \le C_g \inf_{m\in\{1,\ldots,n\}} \left(\|g-g_m\|^2 + \frac{m}{n}\right) + \frac{C'_g\sqrt{\ln(n)}}{n}, \tag{23}$$

*where $C_g$ and $C'_g$ are constant depending on $f$, $w$, $h$ and $g$, $g_m$ and $\hat{g}_{\hat{m}}$ stands successively for $\lambda$, $\lambda_m$ and $\hat{\lambda}_{\hat{m}}$ and for $f$, $f_m$ and $\hat{f}_{\tilde{m}}$.*

Theorem 3.1 states that the adaptive estimator automatically realizes a compromises between the squared-bias $\|f-f_m\|^2$ (or $\|\lambda-\lambda_m\|^2$) and the variance order $m/n$, given by the penalties.

Moreover, when the function $g$ (i.e. $\lambda$ or $f$) has regularity $\alpha$ (for instance belongs to a Besov space on the interval with indexes $\alpha, \infty$) then it is well known that $\|g-g_m\|$ is of order $m^{-\alpha}$. Therefore, the automatic optimization performed in $\inf_{m\in\{1,\ldots,n\}}(\|g-g_m\|^2 + m/n)$ yields an order $n^{-2\alpha/(2\alpha+1)}$. This is the standard optimal rate in the minimax sense for hazard or density estimation.

## 4. Simulations

4.1. **Simulated sampling pattern.** We consider the time-window study with fixed values $t_1$ and $t_2$, for the implementation of the algorithm. We have to build a $n$-sample of $(\sigma_{\mathcal{S}}, Z, \delta)$ and for this purpose, the steps are the following:

*1*) Given $t_1$, $t_2 > 0$ with $t_1 < t_2$, draw $N$ birth-times $(\sigma_i)_{i=1,\dots,N}$ on $\mathbb{R}^+$ as realizations of a Poisson process with given intensity $\varphi(s) \propto 1$ (homogeneous Poisson process) which corresponds to unbiased case (denoted by UB hereafter) or linear bias (denoted by LB) or $\varphi(s) \propto s^{-2/3}$ (inhomogeneous Poisson process) resulting in a power bias (denoted by PB) $w(t) \propto t^{1/3}$. For the simulations of inhomogeneous Poisson process, we refer to Devroye (1986). Moreover, the intensity $\varphi$ is "calibrated" in function of the values $t_1$ and $t_2$ such that the generated birth-times satisfy both conditions: with probability near of 1, at least $n$ lifetimes $X_i$'s fall in the time-window $[t_1, t_2]$ and the largest simulated birth-time $\sigma_N$ would exceed $t_2$. Note that $N >> n$, but for example, $N = 1000$ ensure that $n = 200$ biased lifetimes will be included in the study after step 4).

*2*) For each birth-time $\sigma_i$, draw the associated lifetime $X_i$ with given c.d.f $F$.

*3*) Inclusion in the study: the age at inclusion $\xi_i$ in the study is given by $\xi_i = t_1 - \sigma_i$ if $\sigma_i \le t_1$ and $\xi_i = 0$ if $t_1 \le \sigma_i \le t_2$. So, choose the $X_i = X_{\mathcal{S},i}$ such that $\xi_i \le t_1$ and $X_i \ge \xi_i$. This mechanism of inclusion in the study involves a bias selection and we keep only selected couples $(\xi_i, X_{\mathcal{S},i})$.

*4*) Draw a $n$-sample $(\sigma_{\mathcal{S},1}, X_{\mathcal{S},1}), \dots, (X_{\mathcal{S},n}, \sigma_{\mathcal{S},n})$ by uniform random sampling among all the individuals falling in the time-window and selected in step 3).

*5*) Draw the censoring variables $C_1, \dots, C_n$ with exponential c.d.f. $\mathcal{E}xp(c)$ for several values of $c > 0$ involving different censoring proportions.

*6*) Put $Z_i = (t_1 - \sigma_{\mathcal{S},i}) + (X_{\mathcal{S},i} - (t_1 - \sigma_{\mathcal{S},i})) \wedge C_i$ if $\sigma_{\mathcal{S},i} \le t_1$ and $Z_i = X_{\mathcal{S},i} \wedge C_i$ if $t_1 \le \sigma_{\mathcal{S},i} \le t_2$.

Note that in the UB case, the step 3) is skipped.

4.2. **Practical implementation of the estimators.** Two models are investigated as in Antoniadis *et al.* (1999), Reynaud-Bouret (2006) and Brunel & Comte (2005):

(a) The first example is the *Gamma case*: the $X_i$'s are generated following a Gamma distribution with shape parameter 5 and scale 1. The hazard rate is a monotone curve.

(b) The second data set is called the *Bimodal case*: The $X_i$'s are distributed from a bimodal density defined by $f = 0.8u + 0.2v$ where $u$ is the p.d.f of $\exp(Z/2)$ with $Z \sim \mathcal{N}(0, 1)$ and $v = 0.17Z + 2$.

The expression of the hazard estimator on $A = [a, b]$ is the following:

$$\hat{\lambda}_m(x) = \frac{\hat{a}_0}{\sqrt{b-a}} + \sqrt{\frac{2}{b-a}} \left( \sum_{j=1}^{[(m-1)/2]} \hat{a}_{2j-1} \sin(\frac{2\pi j(x-a)}{b-a}) + \sum_{j=1}^{[m/2]} \hat{a}_{2j} \cos(\frac{2\pi j(x-a)}{b-a}) \right)$$

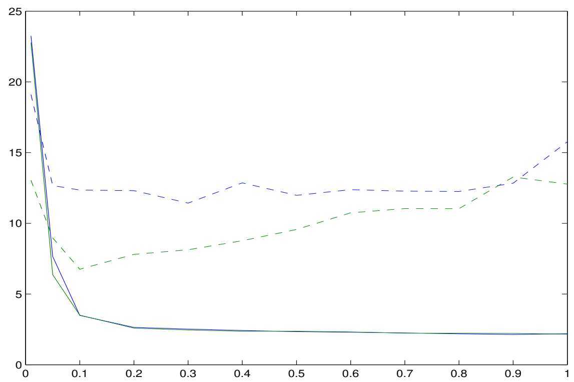



FIGURE 4.1. Choice of $\kappa$: the dimension (solid line) and the MSE2$*10^4$ (dashed line) in function of $\kappa$, the two curves correspond to unbiased and power-biased cases.

| $n$ | | Gamma 200 | | Gamma 500 | | Bimodal 200 | | Bimodal 500 | |
|---|---|---|---|---|---|---|---|---|---|
| Censoring | | 0% | 40-50% | 0% | 40-50% | 0% | 40-50% | 0% | 40-50% |
| UB | MSE | 0.11 | 0.11 | 0.12 | 0.13 | 1.65 | 1.80 | 1.18 | 1.50 |
| | MSE2 | 0.0018 | 0.004 | 0.011 | 0.029 | 0.19 | 0.17 | 0.11 | 0.08 |
| LB | MSE | 0.12 | 0.11 | 0.13 | 0.12 | 1.45 | 1.50 | 1.11 | 1.27 |
| | MSE2 | 0.0014 | 0.0017 | 0.0008 | 0.0013 | 0.15 | 0.09 | 0.08 | 0.04 |
| PB | MSE | 0.12 | 0.101 | 0.13 | 0.10 | 1.37 | 1.32 | 1.02 | 1.07 |
| | MSE2 | 0.0012 | 0.0014 | 0.0007 | 0.0009 | 0.17 | 0.05 | 0.08 | 0.02 |

TABLE 4.1. Monte-Carlo results for the estimator of $\lambda$, for $J = 200$ replications.

with $\hat{a}_0 = \sum_{i=1}^{n} \delta_i/n$ and for $\tilde{Z}_i = (Z_i - a)/(b-a)$,

$$\hat{a}_{2j} = \frac{\sqrt{2}}{n\sqrt{b-a}} \sum_{i=1}^{n} \frac{\delta_i \cos(2\pi j \tilde{Z}_i)}{\Phi_n(\tilde{Z}_i)}, \hat{a}_{2j+1} = \frac{\sqrt{2}}{n\sqrt{b-a}} \sum_{i=1}^{n} \frac{\delta_i \sin(2\pi j \tilde{Z}_i)}{\Phi_n(\tilde{Z}_i)}.$$

The optimal dimension $\hat{m}$ is chosen to minimize the penalized contrast $-\sum_{j=0}^{m-1} \hat{a}_j^2 + \kappa\,(n^{-1}\sum_{i=1}^{n} \delta_i/\Phi_n^2(Z_i))\,m/n$. Here, the universal constant has been approximated by simulation experiments as shown in Figure 4.1 and taken as $\kappa = 0.15$. We also took a modified version of the estimator $1/\Phi_n(x) = \max(1.5; \min(1/(1+(O(x)-n)/(n+5)); \sqrt{n}))$ for numerical reasons.

Plots of both estimators are given in Figure 4.2. The estimators are only illustrated in the PB case since Tables 4.1 and 4.2 show that the estimation errors are quite the same whatever the form of the bias is. As expected, the behaviour of the

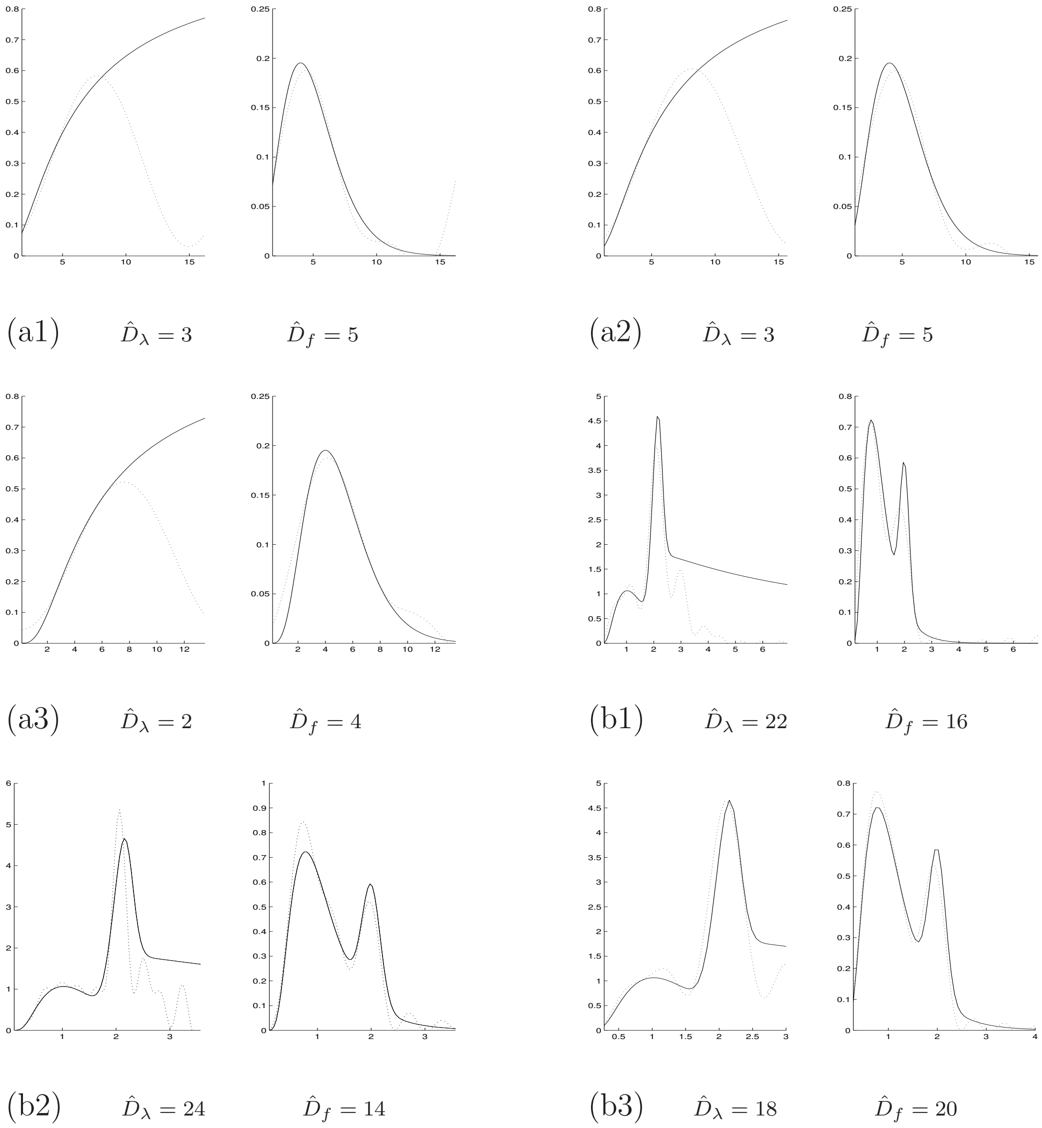



FIGURE 4.2. Penalized estimators of $\lambda$ and $f$ with power bias. Model (a): (a1) $n = 200$, no censure; (a2) $n = 500$, no censure; (a3) $n = 500$, censure 38%. Model (b): (b1) $n = 500$, no censure; (b2) $n = 500$, censure 33%; (b3) $n = 1000$, no censure. For each picture, on the left, the estimator of $\lambda$ and on the right the estimator of $f$, $\hat{D}_\lambda$ and $\hat{D}_f$ denote the selected dimension for $\lambda$ and $f$ respectively. True curve (full) and estimate (dotted).

| | | Gamma | | | | Bimodal | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | | 200 | | 500 | | 200 | | 500 | |
| Censoring | | 0% | 40-50% | 0% | 40-50% | 0% | 40-50% | 0% | 40-50% |
| UB | MSE | 0.0008 | 0.0011 | 0.0004 | 0.0008 | 0.02 | 0.02 | 0.01 | 0.01 |
| | MSE2 | 0.0005 | 0.0003 | 0.0003 | 0.0002 | 0.01 | 0.01 | 0.008 | 0.004 |
| LB | MSE | 0.0005 | 0.0005 | 0.0002 | 0.0003 | 0.01 | 0.01 | 0.005 | 0.003 |
| | MSE2 | 0.0002 | 0.0003 | 0.0001 | 0.0002 | 0.007 | 0.008 | 0.003 | 0.002 |
| PB | MSE | 0.0006 | 0.0005 | 0.0003 | 0.0004 | 0.01 | 0.009 | 0.005 | 0.003 |
| | MSE2 | 0.0002 | 0.0003 | 0.0001 | 0.0002 | 0.006 | 0.006 | 0.003 | 0.002 |

TABLE 4.2. Monte-Carlo results for the estimator of $f$, for $J = 200$ replications

hazard estimator at the end of the interval is erratic due to the sparsity of the observations there. This problem vanishes when considering the density estimator. Note that the dimensions selected by the algorithm for both $\lambda$ and $f$ are much smaller in the Gamma case (a) than in the Bimodal case (b). This reflects that the algorithm makes automatically the squared Bias/Variance compromise and is able to adapt to various forms of curves. Now, to study the quality of the estimation procedure, we compute over $J$ replications of samples of size $n = 200$ and $n = 500$ the mean squared errors (MSE) over a grid of $K = 64$ regularly spaced points $t_1, \dots, t_K$ of $[a, b]$:

$$\mathrm{MSE}_j = \frac{1}{n}\sum_{k=1}^{K}[g(t_k) - \hat{g}_{\hat{m}_j}(t_k)]^2$$

where $\hat{g}_{\hat{m}_j}(t_k)$ is the penalized estimator of $\lambda$ or $f$ computed for the $j$th replication for $j = 1, \dots, J$. Then, the MSE given in Table 4.1 is the arithmetic mean of the $J$ $\mathrm{MSE}_j$. In order to take into account the sparsity of the observations at the end of the interval, ($\mathbb{P}(X > 6) = 0.25$ in the Gamma case and $\mathbb{P}(X > 2) = 0.16$ in the Bimodal case), we also compute an error MSE2 defined by the same kind of mean squared error but with a truncated mean over the $t_k$'s less than 6 in the Gamma case and 2 in the Bimodal case. In the unbiased case with censoring, we get very conforming results to those in Brunel & Comte (2005) so that we can consider the estimator as a generalized version of the estimator therein. We can also compare our results to those in Brunel *et al.* (2005) for density estimation in presence of bias but without censoring. Indeed, the Gamma case is also studied therein and the MSE's up to the interval length factor ($b-a$ is about 15 here) are of the same order. The MSE2's for the hazard estimators are really better than the full MSE's (from 10 to 100 times smaller) which is not very surprising since the estimation breaks up at the end of the interval. This improvement still occurs for the density but in a less striking way. Besides, the censoring effect for both density and hazard does not seem to degrade the estimation error. We can even notice that for hazard estimation in the Bimodal case, the censoring (with significant rate about $40 - 50\%$) systematically seems to improve the MSE2's.

## 5. An example

In this section, the procedure described above is applied to the Channing House data of Hyde (1977) and compared with the method proposed by Uzunogullari & Wang (1992). As we have already mentioned in the introduction, truncated data can also be modelled as length-biased data and this explains why the procedures can be compared in this case. Uzunogullari & Wang (1992) propose adaptive kernel estimators for the hazard rate. The observations consist of survival data for members of the Channing House retirement community in Palo Alto, California. There are 462 observations among which 97 men and 365 women. Both Hyde (1977) and Uzunogullari & Wang (1992) consider the data for the men only. In Uzunogullari & Wang (1992)'s representation, the variable of interest $X_i$ is the lifetime in months of the male retired members, and it is considered that the truncation time $T_i$ is the age in months at entry into study plus one. Then the observation $Y_i = X_i \wedge C_i$ is the age in months when last seen in the study. As for us, the data can be considered as resulting from a time-window study, provided that it is assumed that all individuals are included in the study at the same date (1974). Then the data involve the age at inclusion $a_{\mathcal{S}}(\sigma_{\mathcal{S},i})$ and the time spent in the study $X_{\mathcal{S},i} - a_{\mathcal{S}}(\sigma_{\mathcal{S},i})$, together with the birth date $\sigma_{\mathcal{S},i}$ and the censoring time $C_i$; thus the observations required by our model are available.

The estimators are plotted in Figure 5.1 and show that in the case of female data, both methods lead to the same type of curves. In the case of the smaller sample of male data, the comparison is more interesting. In Uzunogullari & Wang (1992), the plot is given on the interval $[825, 1125]$ only, because there are too few data at the beginning and the end of the interval to obtain anything but boundary effects. We can observe that our method considerably reduces boundary effects at the beginning of the interval. However, both estimators reveal poor performances at the end of the period. For such a type of data, a growing hazard rate with increasing trend would be expected, which is observed on none of the curves. Note that, for the larger female sample, the boundary effects are less visible at the beginning of the interval. For male data, Uzunogullari & Wang (1992) observe three periods: a rapid increase below 900 months followed by a stable period between 900 and 960 and lastly a faster increase. On the opposite, we distinguish on our estimator a global low increase from 880 to 1050.

As a conclusion, our estimator provides convincing estimates, with smoothness in accordance with the trends expected for such hazard rate data. It is worth emphasizing that the observations are processed under a different framework, with assumptions on the way of collecting the data, rather than on the bias type. In particular, the Channing House example also shows that the modelization allows a new glance to deal with truncated data.

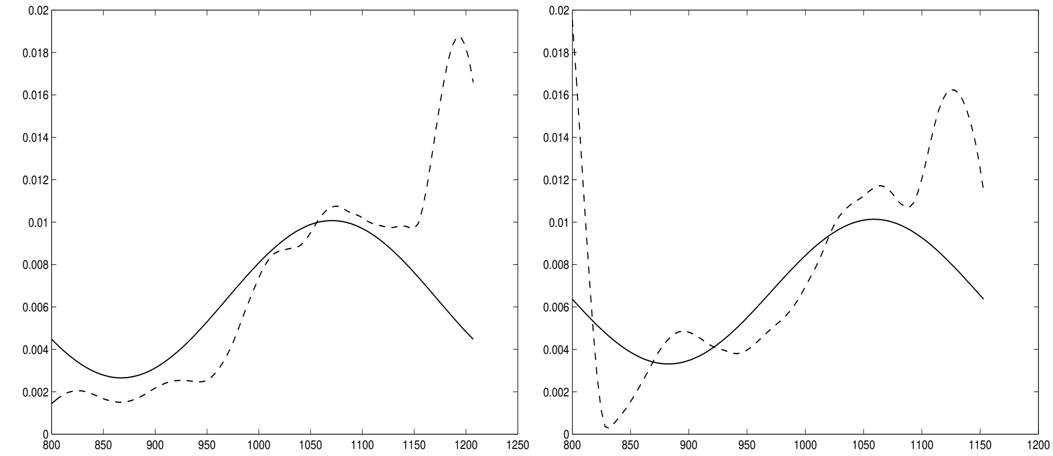



FIGURE 5.1. Plots of the estimators of the hazard rate for the Channing House data (hazard rate in function of the age in months). Dotted: Uzunogullari & Wang (1992)'s kernel estimate with fixed bandwidth (50), full line: our adaptive estimate, with selected dimension 6 in both cases. Left: female data. Right: male data.

## 6. PROOFS

6.1. **Proof of Lemma 3.1.** *Proof of 1).* The result follows by simply integrating the deviation given in Theorem 2.1.

$$
\begin{aligned}
\mathbb{E}\left[\left(\sup_{x\in[0,1]}|\hat{F}_n(x)-F(x)|\right)^{2k}\right] &= 2k\int_0^{+\infty} u^{2k-1}\,\mathbb{P}(\sup_{x\in[0,1]}|\hat{F}_n(x)-F(x)|>u)\,du \\
&\le 2k\int_0^{+\infty} u^{2k-1}\,\mathbb{P}(\sqrt{n}\|(1-H)^2\,(\hat{F}_n-F)w_1\|_\infty > c_H^2 w_1\sqrt{n}\,u)\,du \\
&\le 5ke^{C^2/8}\int_0^{\infty} u^{2k-1}\exp\left(-2(c_H^2w_1)^2 n\left[u-\frac{C}{4\sqrt{n}c_H^2 w_1}\right]^2\right)du \\
&\le \frac{5e^{C^2/8}k}{2^k(c_H^2w_1)^{2k}}\left(\int_{-C/(2\sqrt{2})}^{+\infty}\left(z+\frac{1}{2\sqrt{2}}\right)^{2k-1}e^{-z^2}dz\right)n^{-k} = C_F(k)n^{-k}.
\end{aligned}
$$

*Proof of 2).* First, using the definition of $\Phi$ given by (14) and the expression of the density probability $f_{\sigma_{\mathcal{S}},X_{\mathcal{S}}}$ of $(\sigma_{\mathcal{S}},X_{\mathcal{S}})$ deduced from (2), we write that

$$
\begin{aligned}
\Phi(x) &= \iiint \mathbb{1}_{\{a_{\mathcal{S}}(\sigma)\le x\}}\mathbb{1}_{\{y\ge x\}}\mathbb{1}_{\{x\le a_{\mathcal{S}}(\sigma)+c\}}f_{\sigma,X_{\mathcal{S}}}(\sigma,y)h(c)d\sigma dy dc \\
&= \frac{1}{\mu_{\mathcal{S}}}\int \mathbb{1}_{\{y\ge x\}}f(y)\left[\int h(c)\left(\int \mathbb{1}_{\{x-c\le a_{\mathcal{S}}(\sigma)\le x\}}\varphi(\sigma)d\sigma\right)dc\right]dy.
\end{aligned}
$$

By reminding that $w(x) = \int \mathbb{I}_{\{a_{\mathcal{S}}(u)\le x\}}\varphi(u)du$, we get

$$\int \mathbb{I}_{\{x-c\le a_{\mathcal{S}}(\sigma)\le x\}}\varphi(\sigma)d\sigma = \begin{cases} w(x)-w(x-c) & \text{if } x-c>0 \\ w(x) & \text{otherwise,}\end{cases}$$

since $w(0^-)=0$. It follows that

$$\begin{aligned}\Phi(x) &= \frac{\bar{F}(x)}{\mu_{\mathcal{S}}}\left[\int_0^x h(c)(w(x)-w(x-c))dc + \int_x^{+\infty} h(c)w(x)dc\right] \\ &= \frac{\bar{F}(x)}{\mu_{\mathcal{S}}}\left(w(x) - \int_0^x h(c)w(x-c)dc\right). \end{aligned} \tag{24}$$

For $x \ge s$, we have $w(x)-w(s) = \int \mathbb{I}_{\{s<a_{\mathcal{S}}(\sigma)\le t\}}\varphi(\sigma)d\sigma \ge 0$. Thus, for $c$ in $[0,x]$, we have $w(x-c)\le w(x)$ and

$$\Phi(x) \ge \frac{\bar{F}(x)}{\mu_{\mathcal{S}}}w(x)\left(1-\int_0^x h(c)dc\right) = \frac{\bar{F}(x)}{\mu_{\mathcal{S}}}w(x)\bar{H}(x).$$

This implies that $\forall x\in A,\ \Phi(x)\ge c_F c_H w_1/\mu_{\mathcal{S}} := c_\Phi$.
*Proof of 3).* We just have to write that $\Phi(x) = \mathbb{P}(a_{\mathcal{S}}(\sigma_{\mathcal{S},1})\le x\le Z_1)$ is a difference of two distribution functions $\Phi(x) = \mathbb{P}(a_{\mathcal{S}}(\sigma_{\mathcal{S},1})\le x) - \mathbb{P}(Z_1\le x)$. Then we can use the exponential deviation inequality between theoretical and empirical distribution functions for i.i.d. random variables, as given in Massart (1990) (which gives that $\forall\lambda>0, \mathbb{P}(\sqrt{n}\|F_{n,Z}-F_Z\|_\infty \ge \lambda)\le 2e^{-2\lambda^2}$, where $F_{n,Z}(x) = (1/n)\sum_{i=1}^n \mathbb{I}_{\{Z_i\le x\}}$ and $F_Z(x) = \mathbb{P}(Z_1\le x)$). Then we can integrate the resulting exponential probabilities as above (see the proof of 1)).

6.2. **Proof of Proposition 3.1.** Using equations (5) and (2), we obtain:

$$\begin{aligned}\mathbb{E}\left(\frac{\delta_1 t(Z_1)}{\Phi(Z_1)}\right) &= \mathbb{E}\left(\mathbb{I}_{\{X_{\mathcal{S},1}-a_{\mathcal{S}}(\sigma_{\mathcal{S},1})\le C_1\}}\frac{t(X_{\mathcal{S},1})}{\Phi(X_{\mathcal{S},1})}\right) \\ &= \iiint \mathbb{I}_{\{x-a_{\mathcal{S}}(\sigma)\le c\}}\mathbb{I}_{\{a_{\mathcal{S}}(\sigma)\le x\}}\frac{t(x)}{\Phi(x)}h(c)\varphi(\sigma)\frac{f(x)}{\mu_{\mathcal{S}}}dc d\sigma dx \\ &= \int \frac{t(x)f(x)}{\mu_{\mathcal{S}}\Phi(x)}\left[\int h(c)\left(\int \mathbb{I}_{\{x-c\le a_{\mathcal{S}}(\sigma)\le x\}}\varphi(\sigma)d\sigma\right)dc\right]dx.\end{aligned}$$

Then it follows from (24) that

$$\int h(c)\left(\int \mathbb{I}_{\{x-c\le a_{\mathcal{S}}(\sigma)\le x\}}\varphi(\sigma)d\sigma\right)dc = \frac{\Phi(x)}{\bar{F}(x)}\mu_{\mathcal{S}}.$$

Thus

$$\mathbb{E}\left(\frac{\delta_1 t(Z_1)}{\Phi(Z_1)}\right) = \int \frac{t(x)f(x)}{\Phi(x)\mu_{\mathcal{S}}}\frac{\Phi(x)\mu_{\mathcal{S}}}{\bar{F}(x)}dx = \int t(x)\frac{f(x)}{1-F(x)}dx = \langle t,\lambda\rangle.$$

6.3. **Proof of Theorem 3.1.** For the sake of brevity, we prove the result only in the more complicated case, that is for the estimation of $f$. The proof for the estimation of $\lambda$ would follow the same line.

For the sake of simplicity, we also prove the result with a deterministic penalty function,

$$\widetilde{\text{pen}}(m) = \kappa \mathbb{E}\left(\frac{\delta_1 \bar{F}^2(Z_1)}{\Phi^2(Z_1)}\right)\frac{m}{n}.$$

Going from the above penalty to the random one given in the definitions of the estimators is now standard (see Comte and Brunel (2005, p.465)).

*Step 1: Decomposition of the* $\mathbb{L}^2$ *risk.* Let $B_{m,m'}(0,1) = \{t \in S_m + S_{m'},\ \|t\| = 1\}$. We deduce from (21) and the definition of $\hat{f}_{\tilde{m}}$ that

$$\begin{aligned}
\|\hat{f}_{\tilde{m}} - f\|^2 &\leq \|f - f_m\|^2 + 2\tilde{\nu}_n(\hat{f}_{\tilde{m}} - f_m) + 2R_{n,1}(\hat{f}_{\tilde{m}} - f_m) + 2R_{n,2}(\hat{f}_{\tilde{m}} - f_m) \\
&\quad + \widetilde{\text{pen}}(m) - \widetilde{\text{pen}}(\tilde{m}) \\
&\leq \|f - f_m\|^2 + 2\|\hat{f}_{\tilde{m}} - f_m\| \left(\sup_{t \in B_{m,\tilde{m}}(0,1)} (|\tilde{\nu}_n(t)| + |R_{n,1}(t)| + |R_{n,2}(t)|)\right) \\
&\quad + \widetilde{\text{pen}}(m) - \widetilde{\text{pen}}(\tilde{m})
\end{aligned}$$

We use that for $x, y \geq 0$, $2xy \leq (1/4)x^2 + 4y^2$ and $(x+y+z)^2 \leq 2x^2 + 4y^2 + 4z^2$ and we obtain

$$\begin{aligned}
\|\hat{f}_{\tilde{m}} - f\|^2 &\leq \|f - f_m\|^2 + \frac{1}{4}\|\hat{f}_{\tilde{m}} - f_m\|^2 + 8 \sup_{t \in B_{m,\tilde{m}}(0,1)} \tilde{\nu}_n^2(t) + 16 \sup_{t \in B_{m,\tilde{m}}(0,1)} R_{n,1}^2(t) \\
&\quad + 16 \sup_{t \in B_{m,\tilde{m}}(0,1)} R_{n,2}^2(t) + \widetilde{\text{pen}}(m) - \widetilde{\text{pen}}(\tilde{m}) \\
&\leq \|f - f_m\|^2 + \frac{1}{4}\|\hat{f}_{\tilde{m}} - f_m\|^2 + 8\left(\sup_{t \in B_{m,\tilde{m}}(0,1)} (\tilde{\nu}_n)^2(t) - p(m,\tilde{m})\right)_+ \\
&\quad + 16 \sup_{t \in B_{m,\tilde{m}}(0,1)} R_{n,1}^2(t) + 16 \sup_{t \in B_{m,\tilde{m}}(0,1)} R_{n,2}^2(t) \\
&\quad + 8p(m,\tilde{m}) + \widetilde{\text{pen}}(m) - \widetilde{\text{pen}}(\tilde{m})
\end{aligned}$$

Next $p(m, m')$ is chosen such that

$$\mathbb{E}\left(\sup_{t \in B_{m,\tilde{m}}(0,1)} \tilde{\nu}_n^2(t) - p(m,\tilde{m})\right)_+ \leq \frac{C}{n}, \tag{25}$$

and the penalty function $\widetilde{\text{pen}}(m)$ is deduced from $p(.,.)$ by setting the constraint

$$\forall m, m' \in \mathcal{M}_n,\ 8p(m,m') \leq \widetilde{\text{pen}}(m) + \widetilde{\text{pen}}(m'). \tag{26}$$

*Step 2: Use of Talagrand (1996)'s Inequality.* >From the linear centered empirical process $\tilde{\nu}_n(t)$, we compute the following constants in view of applying Talagrand's

Inequality. The connection between the norms stated in (13) is very useful in all the following.

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\in B_{m,m'}(0,1)} \tilde{\nu}_n^2(t)\right) &\leq \frac{1}{n}\sum_{j=0}^{m\vee m'-1} \mathrm{Var}\left(\frac{\psi_j(X_{\mathcal{S},1})\bar{F}(X_{\mathcal{S},1})}{\Phi(X_{\mathcal{S},1})}\right) \\
&\leq \frac{1}{n}\mathbb{E}\left(\frac{\sum_{j=0}^{m\vee m'-1}\psi_j^2(X_{\mathcal{S},1})\bar{F}^2(X_{\mathcal{S},1})}{\Phi^2(X_{\mathcal{S},1})}\right) \leq \frac{2(m\vee m')}{n}\sup_{x\in A}\frac{\bar{F}^2(x)}{\Phi^2(x)} \qquad (27)\\
&\leq \frac{2\mu_{\mathcal{S}}^2}{w_1^2 c_H^2}\frac{m\vee m'}{n} := H^2. \qquad (28)
\end{aligned}
$$

Indeed, it follows from (24) that $\Phi(t)/\bar{F}(t) \geq w(t)\bar{H}(t)/\mu_{\mathcal{S}}$.

$$
\begin{aligned}
\sup_{t\in B_{m,m'}(0,1)} \mathrm{Var}\left(\frac{t(X_{\mathcal{S},1})\bar{F}(X_{\mathcal{S},1})}{\Phi(X_{\mathcal{S},1})}\right) &\leq \sup_{t\in B_{m,m'}}\int \frac{t^2(x)\bar{F}^2(x)}{\Phi^2(x)}\frac{w(x)f(x)}{\mu_{\mathcal{S}}}dx \\
&\leq \frac{\mu_{\mathcal{S}} w_2 f_1}{c_H^2 w_1^2}\sup_{t\in B_{m,m'}}\int_A t^2(x)dx \leq \frac{\mu_{\mathcal{S}} w_2 f_1}{c_H^2 w_1^2} := v \qquad (29)
\end{aligned}
$$

$$
\sup_{t\in B_{m,m'}(0,1)}\sup_{x\in A}\left|\frac{t(x)\bar{F}(x)}{\Phi(x)}\right| \leq \frac{\mu_{\mathcal{S}}}{w_1 c_H}\sqrt{2(m\vee m')} := M_1. \qquad (30)
$$

Thus, we obtain, with $C(\epsilon) = (\sqrt{1+\epsilon}-1)\wedge 1$ and $K_1$ standing for numerical constant,

$$
\mathbb{E}\left(\sup_{t\in B_{m,m'}(0,1)} \tilde{\nu}_n^2(t) - 2(1+2\epsilon)H^2\right)_+ \leq \frac{6}{K_1}\left(\frac{v}{n}e^{-K_1\epsilon nH^2/v} + \frac{8M_1^2}{K_1 n^2 C^2(\epsilon)}e^{-\frac{K_1 C(\epsilon)\sqrt{\epsilon}}{\sqrt{2}}\frac{nH}{M_1}}\right)
$$

Now, by taking $\epsilon = 1/2$ and with the values of $H^2$, $v$ and $M_1$ computed above, we can write the resulting upper bound in this context, for all $m, m'$

$$
\mathbb{E}\left(\sup_{t\in B_{m,m'}(0,1)} \tilde{\nu}_n^2(t) - 4H^2\right)_+ \leq \frac{6/K_1}{n}\left(ae^{-b(m\vee m')} + ce^{-d\sqrt{n}}\right)
$$

with $a = \mu_{\mathcal{S}} w_2 f_1/(c_H^2 w_1^2)$, $b = K_1\mu_{\mathcal{S}}/(w_2 f_1)$, $c = 16\mu_{\mathcal{S}}^2/(K_1 w_1^2 c_H^2 C^2(1/2))$ and $d = K_1 C(1/2)/2$. Finally, with the choice $p(m,m') = 4H^2$

$$
\mathbb{E}\left(\sup_{t\in B_{m,\tilde{m}}(0,1)} \tilde{\nu}_n^2(t) - p(m,\tilde{m})\right)_+ \leq \sum_{m'\leq n}\mathbb{E}\left(\sup_{t\in B_{m,m'}(0,1)} \tilde{\nu}_n^2(t) - p(m,m')\right)_+ \leq \frac{C}{n}.
$$

Then, if (25) and (26) hold, then we have, since $\|\hat{f}_{\tilde{m}}-f_m\|^2 \leq 2\|\hat{f}_{\tilde{m}}-f\|^2+2\|f_m-f\|^2$,

$$\begin{aligned}\frac{1}{2}\mathbb{E}(\|\hat{f}_{\tilde{m}}-f\|^2) &\leq \frac{3}{2}\|f-f_m\|^2+2\widetilde{\mathrm{pen}}(m)+\frac{8C}{n}\\ &+16\mathbb{E}\left(\sup_{t\in B_{m,\tilde{m}}(0,1)} R_{n,1}^2(t)\right)+16\mathbb{E}\left(\sup_{t\in B_{m,\tilde{m}}(0,1)} R_{n,2}^2(t)\right)\end{aligned}$$

Then, for nested model collections described in Section 3.2, $S_m+S_{m'}\subset \mathcal{S}_n$ for all $m,m'$ implies that the supremun taken over the unit ball $B_{m,\tilde{m}}(0,1)$ is lower than the supremun over $\mathcal{B}_n(0,1)=\{t\in\mathcal{S}_n, \|t\|\leq 1\}$. As a consequence, it remains to study the residual terms involving $R_{n,1}$ and $R_{n,2}$.

*Step 3: Study of the two residual terms.*

**Lemma 6.1.** *Let $\Omega_\Phi=\{\omega, \Phi_n(x)\geq c_\Phi/2, \forall x\in[0,1]\}$. Assume that $f(x)\leq f_1$ and $w_1\leq w(x)\leq w_2$, for all $x\in[0,1]$,*

(31) $$L_n(\varphi)\leq N_n^2 \quad \textit{with} \quad N_n = dim\,\mathcal{S}_n,$$

*then, for $i=1,2$, we have*

(32) $$\mathbb{E}\left(\sup_{t\in\mathcal{B}_n(0,1)} R_{n,i}^2(t)\mathbb{I}_{\Omega_\Phi}\right)\leq K_{i+1}\frac{\sqrt{\ln(n)}}{n}$$

*with $\mathcal{B}_n(0,1)=\{t\in\mathcal{S}_n, \|t\|=1\}$ and for numerical constants $K_2>0$ and $K_3>0$.*

On the complementary of $\Omega_\Phi$, we use that $\Phi_n(x)\geq 1/n$ for all $i$, and that $\sup_{x\in[0,1]}|\Phi_n(x)-\Phi(x)|>c_\Phi/2$, to obtain with Lemma 3.1,

$$\mathbb{E}\left(\sup_{t\in\mathcal{B}_n(0,1)} R_{n,1}^2(t)\mathbb{I}_{\Omega_\Phi^c}\right)\leq c/n.$$

Indeed, on $\Omega_\Phi^c$, we can write

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in\mathcal{B}_n(0,1)} R_{n,1}^2(t)\mathbb{I}_{\Omega_\Phi^c}\right) &\leq 2n^2N_n\mathbb{E}\left(\|F-\hat{F}_n\|^2\mathbb{I}_{\Omega_\Phi^c}\right)\\ &\leq 2n^2N_n\mathbb{E}^{1/2}\left(\|F-\hat{F}_n\|^4\right)\mathbb{P}^{1/2}(\Omega_\Phi^c).\end{aligned}$$

With Lemma 3.1 1), we have $\mathbb{E}(\|F-\hat{F}_n\|_\infty^4)\leq C_F(2)/n^2$ and as $N_n\leq n$,

$$\mathbb{P}(\Omega_\Phi^c)\leq\mathbb{P}(\|\Phi_n-\Phi\|_\infty>c_\Phi/2)\leq\mathbb{E}\left[\left(\frac{2}{c_\Phi}\right)^{2k}\|\Phi_n-\Phi\|_\infty^{2k}\right]\leq\left(\frac{2}{c_\Phi}\right)^{2k}\frac{C_\Phi(k)}{n^k}$$

by using Lemma 3.1 2). Then choosing $k\geq 12$ gives the result.

The same approach gives the same order for $\mathbb{E}\left(\sup_{t\in\mathcal{B}_n(0,1)} R_{n,2}^2(t)\mathbb{I}_{\Omega_\Phi^c}\right)$.

6.4. **Proof of Lemma 6.1.**

$$\mathbb{E}\left(\sup_{t\in\mathcal{B}_n(0,1)} R_{n,1}^2(t)\mathbb{I}_{\Omega_\Phi}\right) \le \frac{1}{c_\Phi^2}\mathbb{E}\left[\sup_{x\in[0,1]}|\hat{F}_n(x)-F(x)|^2 \sup_{t\in\mathcal{B}_n(0,1)}\left(\frac{1}{n}\sum_{i=1}^n |t(X_{\mathcal{S},i})|\right)^2\right]$$

$$\le \frac{1}{c_\Phi^2}\mathbb{E}\left\{\sup_{x\in[0,1]}|\hat{F}_n(x)-F(x)|^2\left[\sup_{t\in\mathcal{B}_n(0,1)}[|\nu_n^*(t^2)|+\mathbb{E}(t^2(X_{\mathcal{S},1}))]\right]\right\}$$

$$\le \frac{1}{c_\Phi^2}\mathbb{E}\left\{\sup_{x\in[0,1]}|\hat{F}_n(x)-F(x)|^2\left[\sup_{t\in\mathcal{B}_n(0,1)}|\nu_n^*(t^2)|+\frac{f_1 w_2}{\mu_{\mathcal{S}}}\right]\right\},$$

with $\nu_n^*(t)=\frac{1}{n}\sum_{i=1}^n[t(X_{\mathcal{S},i})-\mathbb{E}(t(X_{\mathcal{S},i}))]$. Now, by applying Schwartz Inequality and Lemma 3.1 1) with $k=2$, we obtain

$$\mathbb{E}\left(\sup_{t\in\mathcal{B}_n(0,1)} R_{n,1}^2(t)\mathbb{I}_{\Omega_\Phi}\right) \le \frac{C_F^{1/2}(2)}{c_\Phi^2}\frac{1}{n}\left[\mathbb{E}^{1/2}\sup_{t\in\mathcal{B}_n(0,1)}[\nu_n^*(t^2)]^2+\frac{f_1 w_2}{\mu_{\mathcal{S}}}\right]$$

>From Baraud (2002), we have the following exponential inequality, for all $\rho>0$,

$$\mathbb{P}\left(\sup_{t\in\mathcal{B}_n(0,1)}|\nu_n^*(t^2)|\ge\rho\right)\le|\Lambda_n|^2\exp\left(-\frac{n\rho^2}{4\tilde{f}_1 L_n(\varphi)}\right)$$

where we denote by $\tilde{f}_1=f_1 w_2/\mu_{\mathcal{S}}$ and $L_n(\varphi)$ is a quantity associated to the orthonormal basis $\psi=(\psi_j)_{j\in\{0,\dots,N_n-1\}}$ of the largest space $\mathcal{S}_n=\bigcup_{m\in\mathcal{M}_n}S_n$ of the (nested) collection satisfying (31). Moreover, in the proof of Theorem 4.2. by Brunel & Comte (2005), it is checked that

$$\mathbb{E}\left(\sup_{t\in B_n(0,1)}[\nu_n^*(t^2)]^2\right)\le 2\ln(n)\quad\text{for } n\ge 2,$$

as soon as $L_n(\varphi)\le n/(16\tilde{f}_1)$. Therefore, if $N_n\le\sqrt{n}/(4\sqrt{\tilde{f}_1})$, we easily deduce Inequality (32).

## References


Antoniadis, A., Grégoire, G. & Nason, G. (1999). Density and hazard rate estimation for right-censored data by using wavelet methods. *J. Roy. Statist. Soc. Ser. B* **61**, 63-84.

Asgharian, M., M'Lan, C.E. & Wolfson, D.B. (2002). Length-biased sampling with right censoring: an unconditional approach. *J. Amer. Statist. Assoc.* **97**, 201-209.

Asgharian, M. (2003). Biased sampling with right censoring: a note on Sun, Cui & Tiwari (2002). *Canad. J. Statist.* **30**, 475-490.

Baraud, Y. (2002). Model selection for regression on a random design. *ESAIM Probab. Statist.* **6**, 127-146.

Barron, A.R., Birgé, L. & Massart, P. (1999). Risk bounds for model selection via penalization. *Probab. Theory Related Fields* **113**, 301-413.

Barron, A.R. & Cover, T. M. (1991). Minimum complexity density estimation. *IEEE Trans. Inform. Theory* **37**, 1034-1054.

Brunel, E. & Comte, F. (2005). Penalized contrast estimation of hazard rate with censored data. *Sankhya*, **67**, 441-475.

Brunel, E., Comte, F. & Guilloux, A. (2005). Nonparametric density estimation in presence of bias and censoring. Preprint MAP5, 2005-13, www.math-info.univ-paris5.fr/map5/.

Chakraborty, R. & Rao, C.R. (2000). Selection biases of samples and their resolutions. In *Handbook of Statistics*, Vol. 18 (P.K. Sen & C.R. Rao, eds), 675-712. Elsevier Science, North-Holland, Amsterdam.

Devroye, L. (1986). Non-Uniform Random Variate Generation (originally published with Springer-Verlag, New York).

Efromovich, S. (2004, a). Distribution estimation for biased data. *J. Statist. Plann. Inference* **124**, 1-43.

Efromovich, S. (2004, b). Density estimation for biased data. *Ann. Statist.* **32**, no. 3, 1137-1161.

Fisher, R.A. (1934). The effect of methods of ascertainment upon the estimation of frequencies. *Ann. Eugen.* **6**, 13-25.

Gill, R.D., Vardi, Y. & Wellner, J.A. (1988). Large sample theory of empirical distributions in biased sampling models. *Ann. Statist.* **16**, 1069-1172.

Guilloux, A. (2006) Nonparametric estimation for censored lifetimes suffering from an unknown selection bias. Preprint LSTA 2006-4. www.lsta.upmc.fr/prepublications.html.

Heckman, J. (1985). Selection bias and self-selection. In *The new Palgrave: A Dictionary of Economics*, 287-296. MacMillan Press, Stockton, New York.

Hyde, J. (1977). Testing survival under right censoring and left truncation. *Biometrika* **64**, 225-230.

Keiding, N. (1990). Statistical inference for the Lexis Diagram. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **332**, 487-509.

Kingman, J.F.C. (1993). *Poisson processes*. Oxford Science Publications.

Lexis, W. (1875). Einleitung in die Theorie der Bevölkerung-Statistik. In *Mathematical demography* (D. Smith & N. Keyfitz eds), Springer-Verlag, Berlin and *Biomathematics* **6**, 39-41 (1977).

Lo, S.H., Mack, Y.P. & Wang, J.L. (1989). Density and hazard rate estimation for censored data via strong representation of the Kaplan-Meier estimator. *Probab. Theory Related Fields* **80**, 461-473.

Lund, J. (2000). Sampling bias in population studies - How to use the Lexis diagram. *Scand. J. Statist.* **27**, 589-604.

Marron, J.S. & de Uña-Àlvarez, J. (2004). SiZer for length biased, censored density and hazard estimation. *J. Statist. Plann. Inference* **121**, 149-161.

Marron, J.S. & Padgett, W.J. (1987). Asymptotically optimal bandwith selection for kernel density estimators from randomly right censored samples. *Ann. Statist.* **15**, 1520-1535.

Massart, P. (1990). The tight constant in Dvoretzky-Kiefer-Wolfowitz inequality. *Ann. Probab.* **18**, 1269-1283.

Mielniczuk, J. (1985). Properties of some kernel estimators and of the adapted Loftsgarden-Quesenberry estimator of a density function for censored data. *Period. Math. Hungar.* **16**, 69-81

Patil, G.P. & Rao, C.R., (1977). The weighted distributions : A survey of their applications. In *Applications of Statistics*, (P.R. Krishnaiah, ed.), 383-405. North-Holland Publishing Co., Amsterdam.

Reynaud-Bouret, P. (2006). Penalized projection estimators of the Aalen multiplicative intensity. Forthcoming in *Bernoulli.*

Talagrand, M. (1996). New concentration inequalities in product spaces. *Invent. Math.* **126**, 505-563.

Tanner, M. A. & Wong, W.H. (1983). The estimation of the hazard function from randomly censored data by the kernel method. *Ann. Statist.* **11**, 989-993.

de Uña-Àlvarez, J. (2002). Product-limit estimation for length-biased censored data. *Test* **11**, 109-125.

Uzunoḡullari, Ü & Wang, J.-L. (1992). A comparison of hazard rate estimators for left truncated and right censored data. *Biometrika* **79**, 297–310.

Vardi, Y. (1982). Nonparametric estimation in presence of length bias. *Ann. Statist.* **10**, 616-620.

Winter, B.B. & Földes, A. (1988). A product-limit estimator for use with length-biased data. *Canad. J. Statist.* **16**, 337-355.