Mathematics in Image Processing

Georges KOEPFLER

Georges.Koepfler@math-info.univ-paris5.fr

MAP5, UMR CNRS 8145

UFR de Mathématiques et Informatique

Université René Descartes - Paris 5

45 rue des Saints-Pères

75270 PARIS cedex 06, France

Outline

- *Natural and Numerical Images*
- *Mathematical Representation*
- *Image Processing Tasks*
- *Denoising*
- *Segmentation*
- *Compression*

Natural vs. Numerical Images

Natural vs. Numerical Images

- The human eye depends on a finite number of cells;

Natural vs. Numerical Images

- The human eye depends on a finite number of cells;
- Classical camera photos have a natural grain *i.e.* resolution, approached by current numerical cameras;

Natural vs. Numerical Images

natural or digital image ?

- The human eye depends on a finite number of cells;
- Classical camera photos have a natural grain *i.e.* resolution, approached by current numerical cameras;

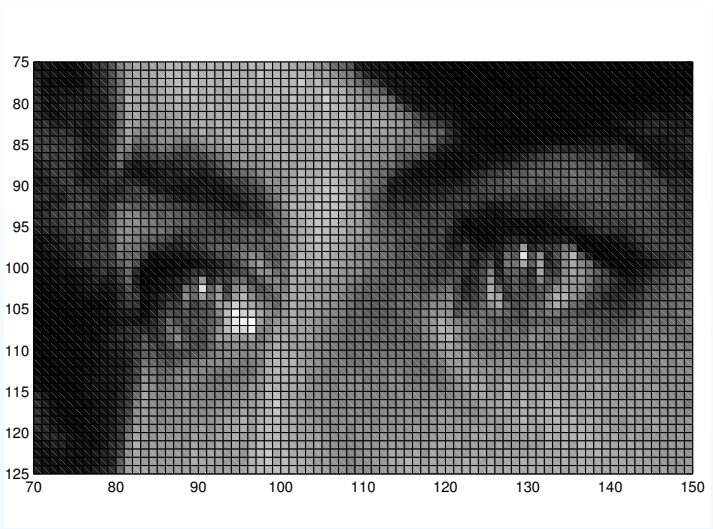
$\Downarrow$

we process only digital images

(correctly sampled)

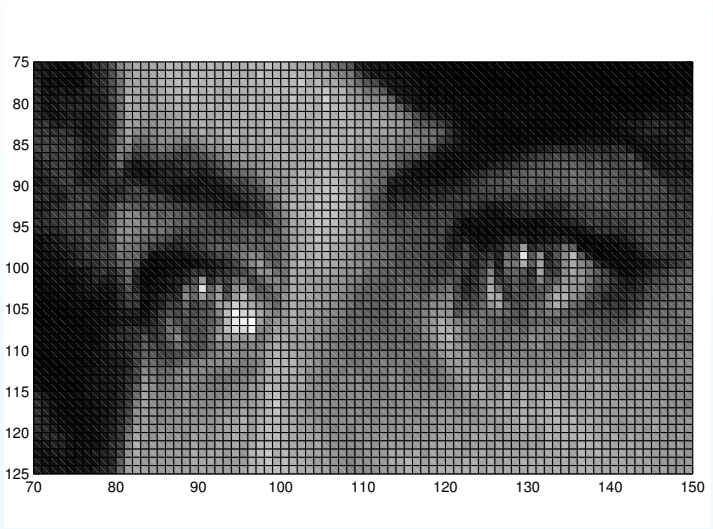
Numerical Images

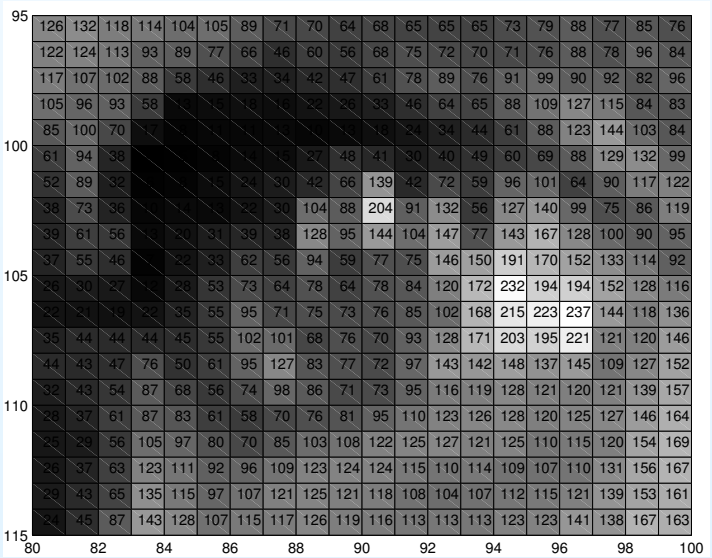
discrete grid



Numerical Images

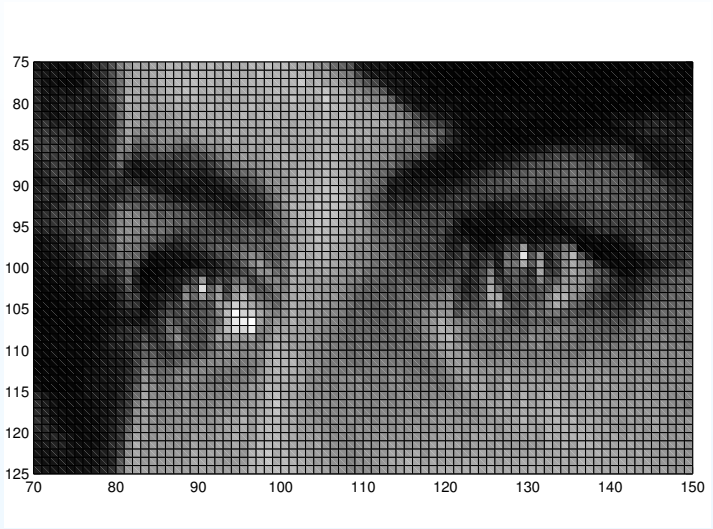
discrete grid

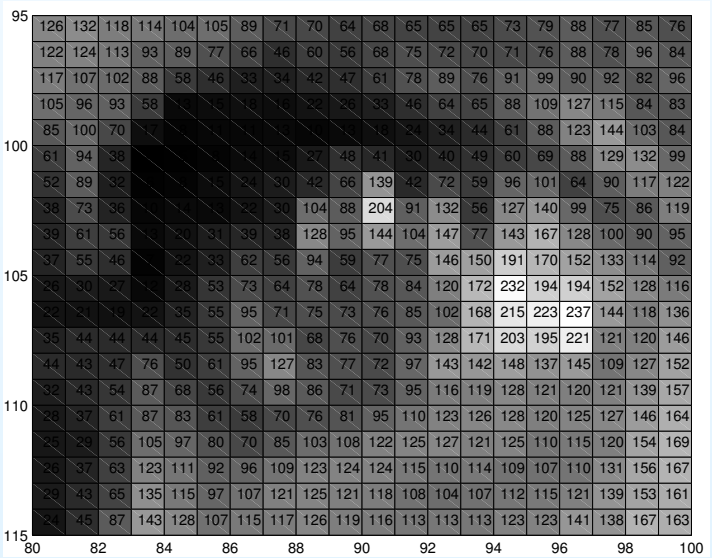


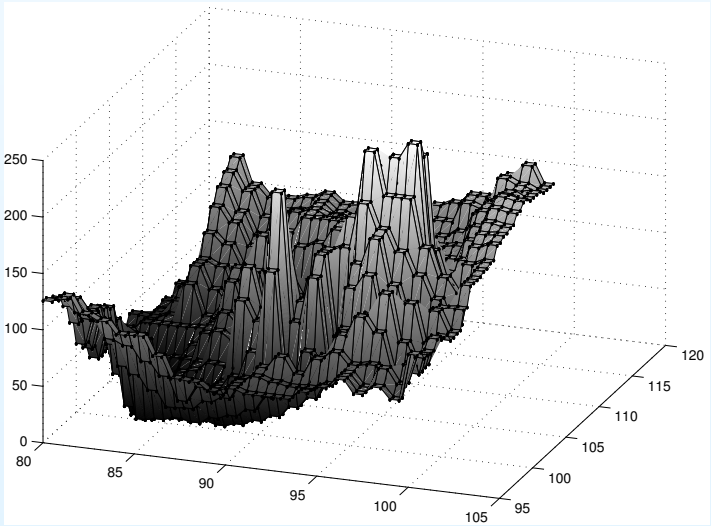


Numerical Images

discrete grid







grid of numbers or pixels

relief or function

Numerical Images (cont.)

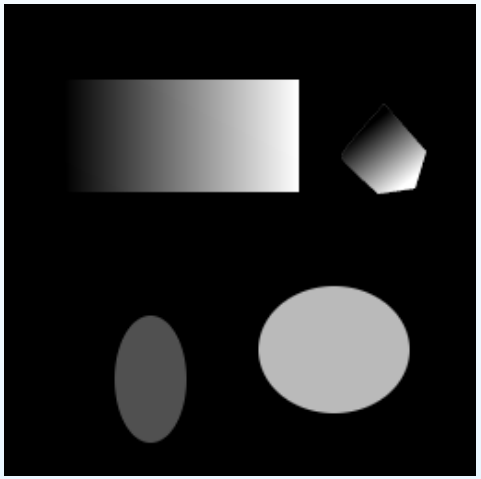
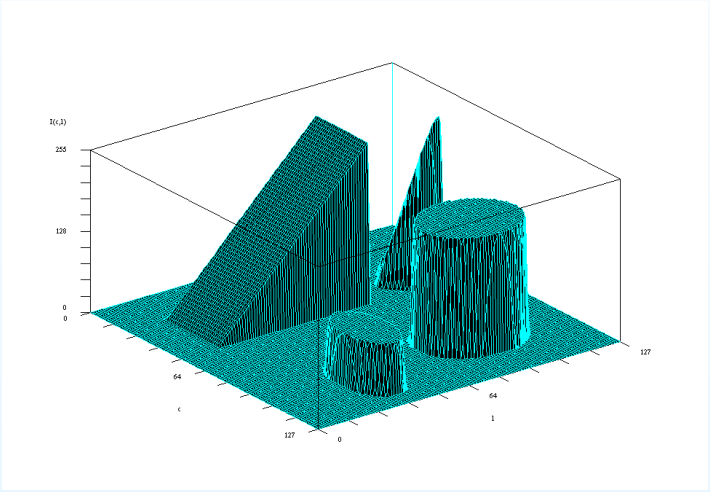
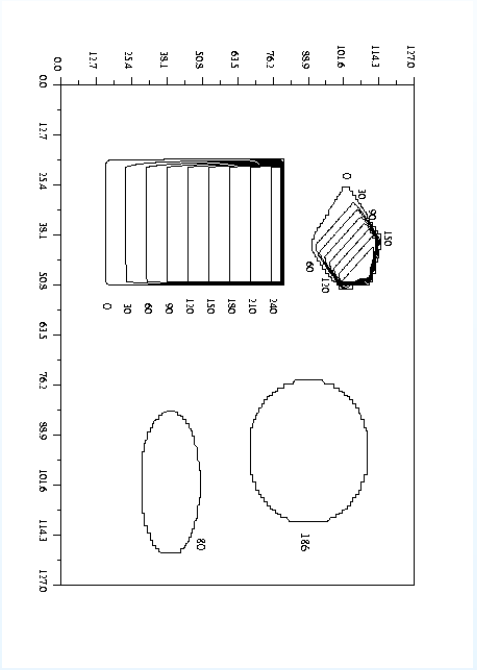


image data



relief



topographic map

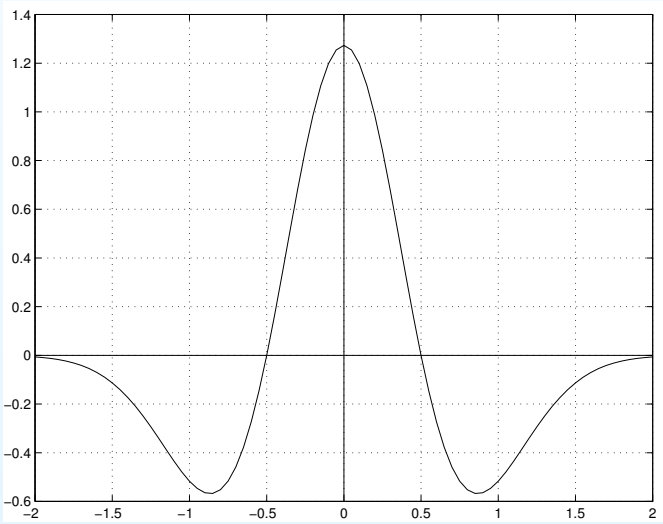
Mathematical representation

Although discrete it is useful to represent an image with an infinite resolution, as a function on real numbers.

Mathematical representation

Although discrete it is useful to represent an image with an infinite resolution, as a function on real numbers.
Basic analysis: a real valued function of one variable

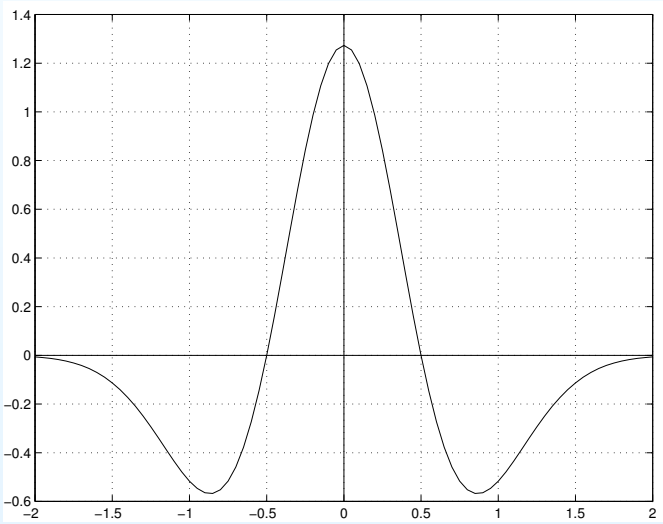
$$x \in [-2, 2] : f(x) = \alpha \left(1 - \frac{x^2}{\beta}\right) e^{-\frac{x^2}{2\beta}} .$$



Mathematical representation

Although discrete it is useful to represent an image with an infinite resolution, as a function on real numbers.
Basic analysis: a real valued function of one variable

$$x \in [-2, 2] : f(x) = \alpha \left(1 - \frac{x^2}{\beta}\right) e^{-\frac{x^2}{2\beta}} .$$

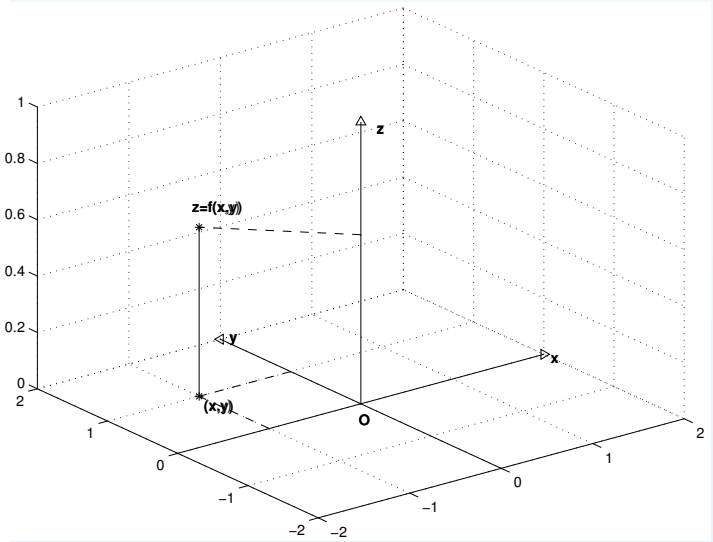


But in an image we need columns and rows:
thus we use functions of two variables.

Mathematical representation (cont.)

Consider the function of two variables $x \in [-2, 2]$, $y \in [-2, 2]$ thus $(x, y) \in [-2, 2] \times [-2, 2]$ and $z = u(x, y) \in \mathbb{R}$.

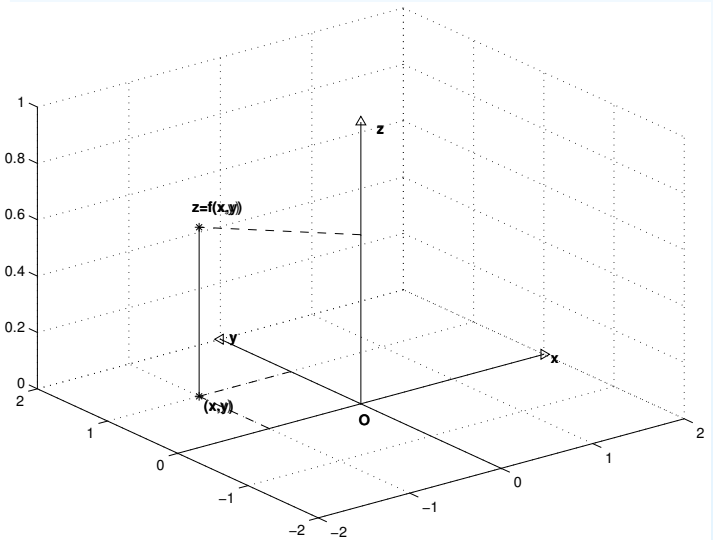
$$u(x, y) = \alpha \left(1 - \frac{x^2 + y^2}{\beta} \right) e^{-\frac{x^2+y^2}{2\beta}} .$$

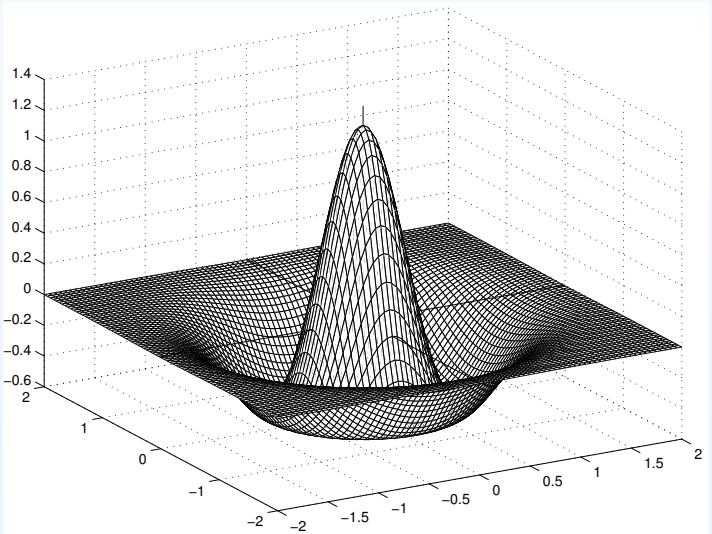


Mathematical representation (cont.)

Consider the function of two variables $x \in [-2, 2]$, $y \in [-2, 2]$ thus $(x, y) \in [-2, 2] \times [-2, 2]$ and $z = u(x, y) \in \mathbb{R}$.

$$u(x, y) = \alpha \left(1 - \frac{x^2 + y^2}{\beta}\right) e^{-\frac{x^2+y^2}{2\beta}} .$$

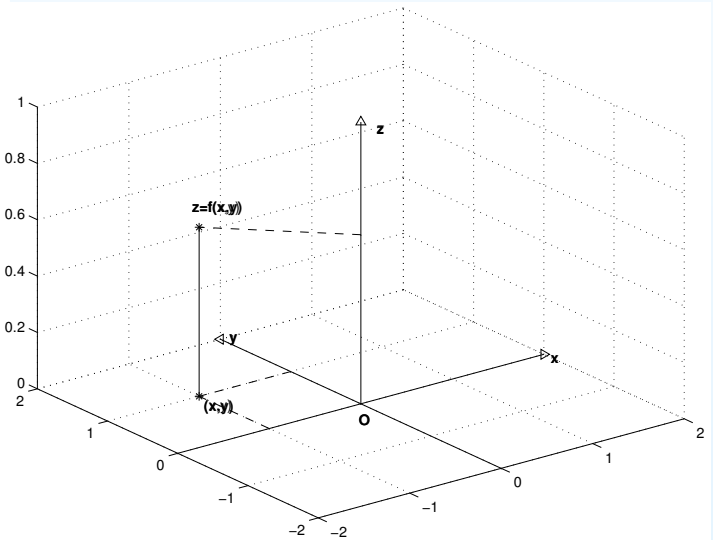


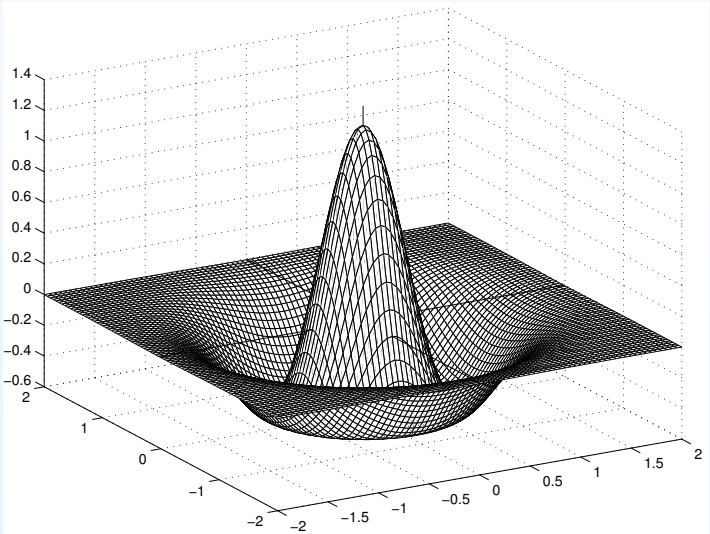


Mathematical representation (cont.)

Consider the function of two variables $x \in [-2, 2]$, $y \in [-2, 2]$ thus $(x, y) \in [-2, 2] \times [-2, 2]$ and $z = u(x, y) \in \mathbb{R}$.

$$u(x, y) = \alpha \left(1 - \frac{x^2 + y^2}{\beta}\right) e^{-\frac{x^2+y^2}{2\beta}} .$$





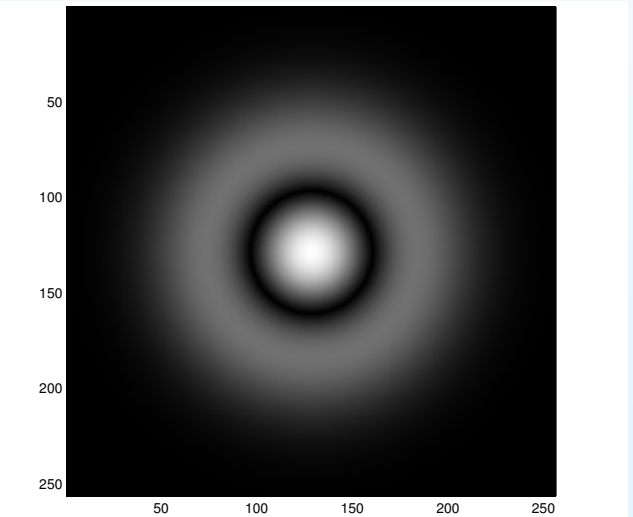
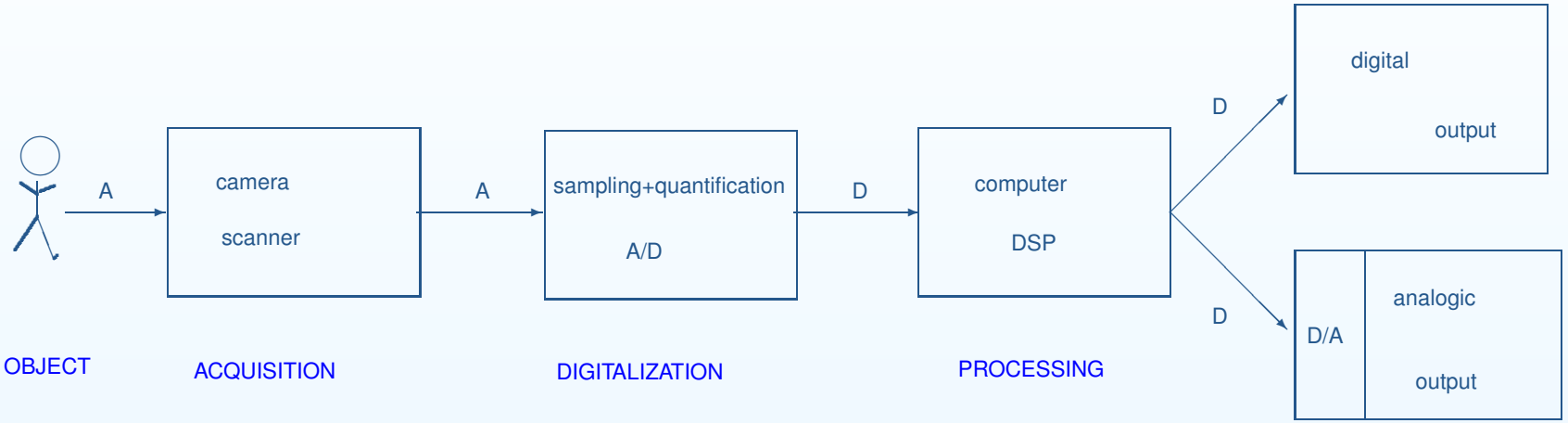


Image processing

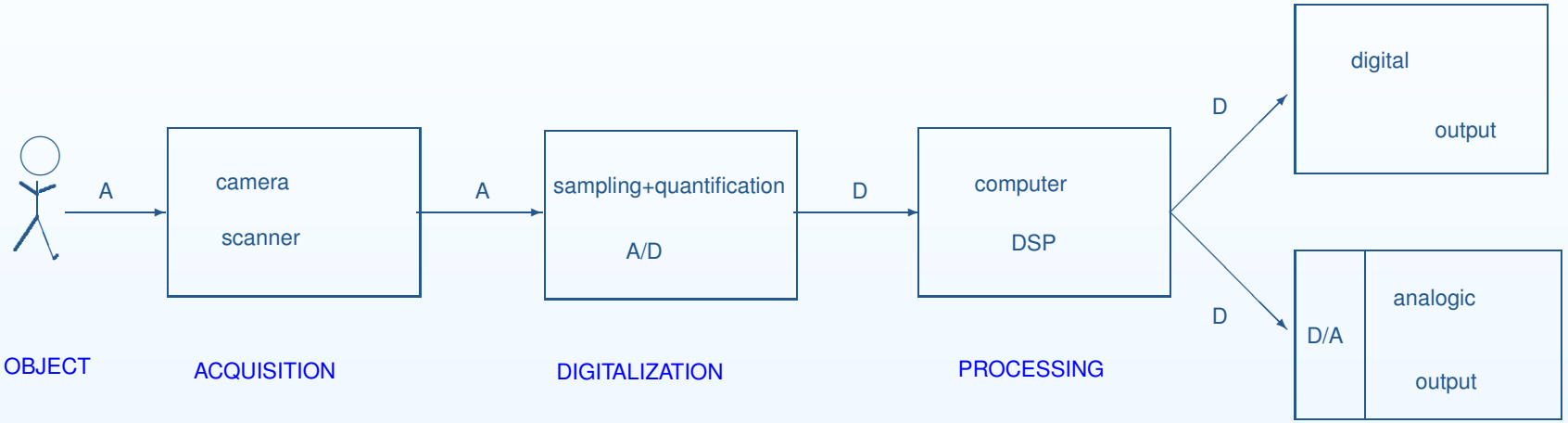
D : digital, A : analogic



- Satellite images give information about natural resources, meteorological data, . . .
- Medical images help detect anatomical pathologies, give quantitative data and functional informations. . .
- Video surveillance is an important issue for security in public transportation. . .
- Images and videos have to be stored/transmitted efficiently. . .

Image processing

D : digital, A : analogic

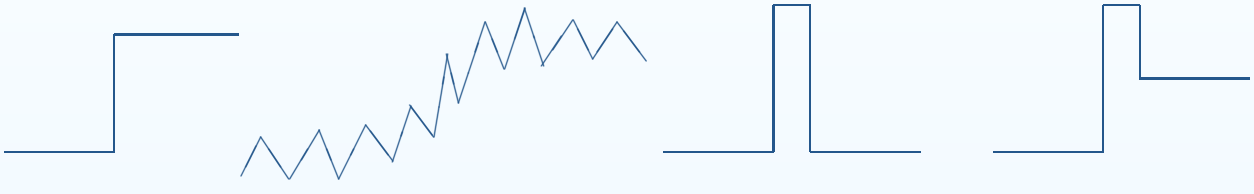


- Satellite images give information about natural resources, meteorological data, . . .
- Medical images help detect anatomical pathologies, give quantitative data and functional informations. . .
- Video surveillance is an important issue for security in public transportation. . .
- Images and videos have to be stored/transmitted efficiently. . .

Process images as a discrete grid of numbers or a function but take into account the visual content!

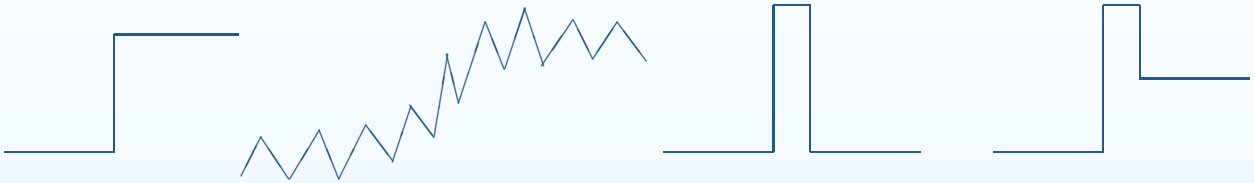
Edge/contour detection

Idea: Detect an object thanks to its boundary
and characterize boundaries by change of luminosity.



Edge/contour detection

Idea: Detect an object thanks to its boundary
and characterize boundaries by change of luminosity.



A discrete rectangular grid :

directions

NW	N	NE
W		E
SW	S	SE

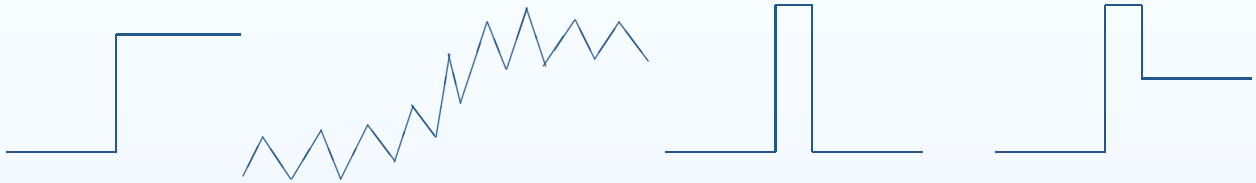
or

(line,column)

	$j-1$	j	$j+1$
$i-1$			

Edge/contour detection

Idea: Detect an object thanks to its boundary
and characterize boundaries by change of luminosity.



A discrete rectangular grid :

directions

NW	N	NE
W		E
SW	S	SE

or

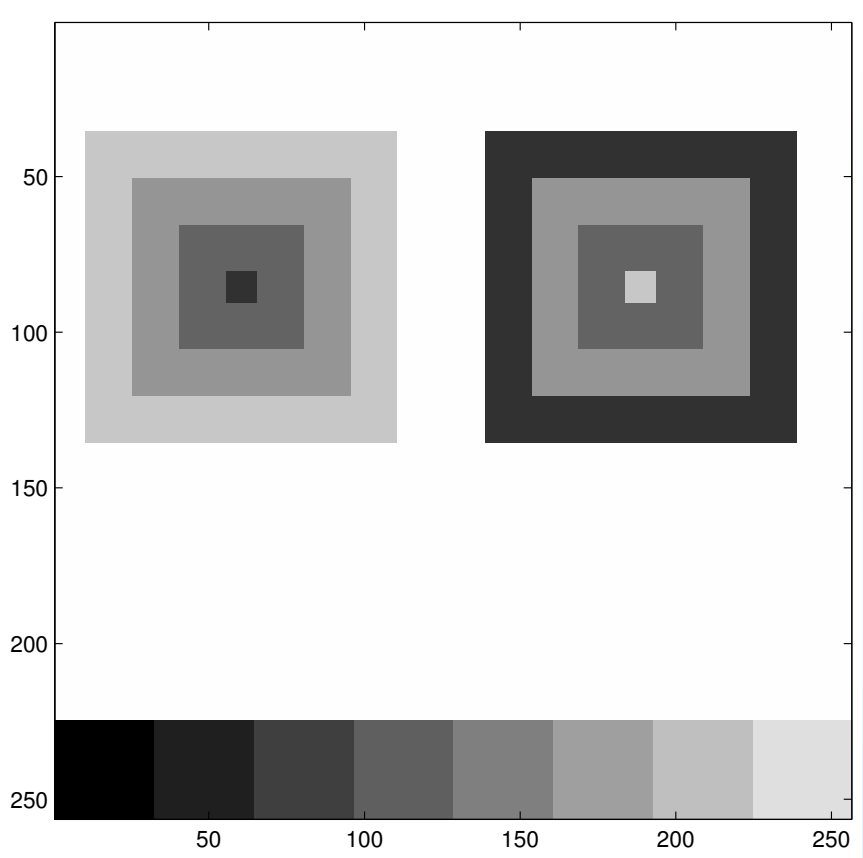
(line,column)

	$j-1$	j	$j+1$
$i-1$		●	
i	●		●
$i+1$		●	

Finite difference operators are used, for example:

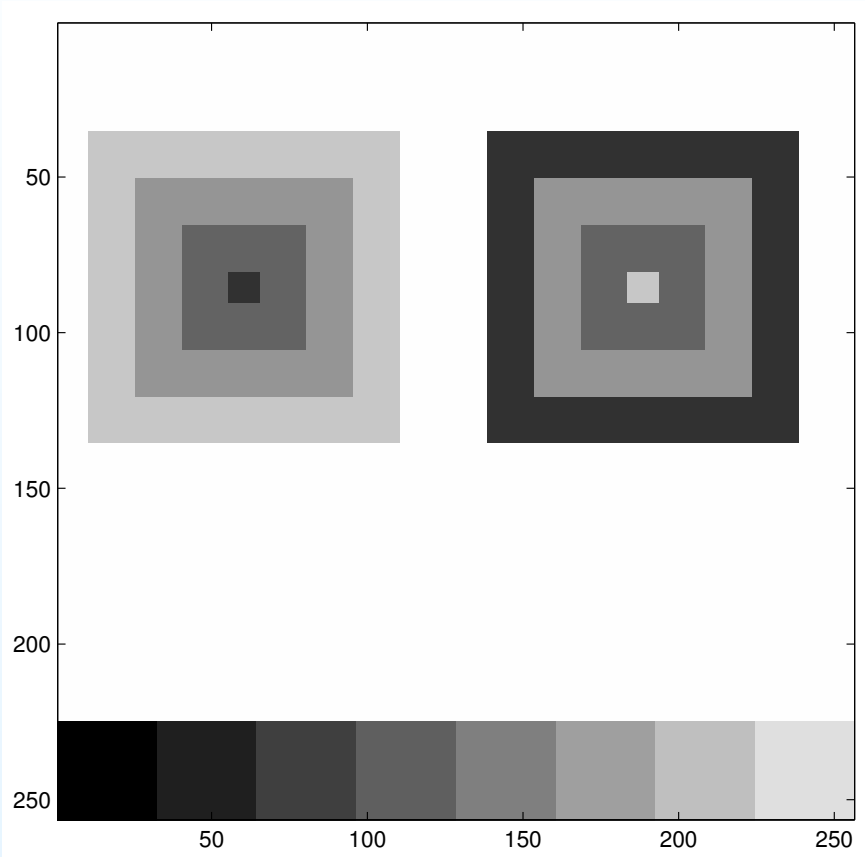
$$\|\nabla u(i,j)\| = \sqrt{(u(i+1,j)-u(i-1,j))^2 + (u(i,j+1)-u(i,j-1))^2}$$

Edge/contour detection (cont.)

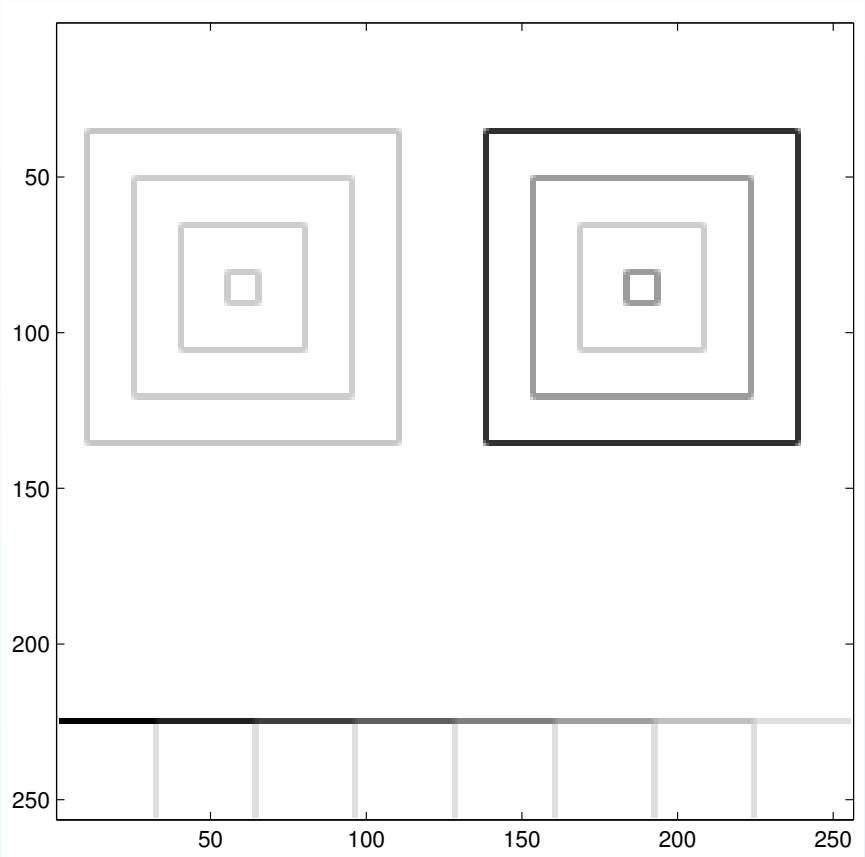


$u(l, c)$

Edge/contour detection (cont.)



$u(l,c)$

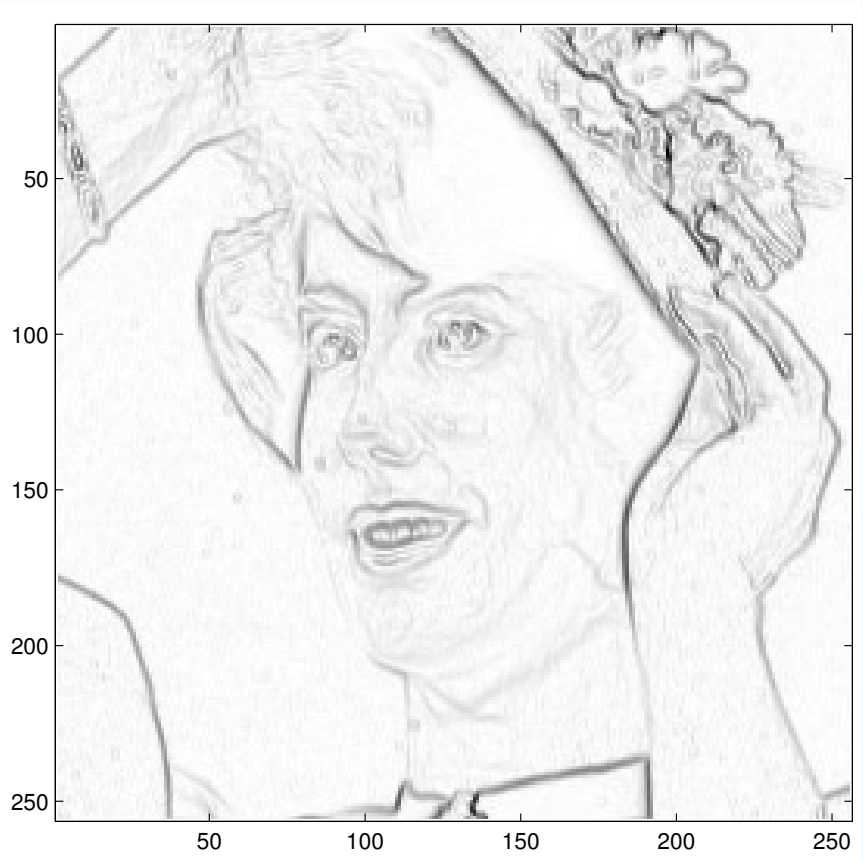


$\|\nabla u\|$

(inverse colormap: 0=white, 255=black)

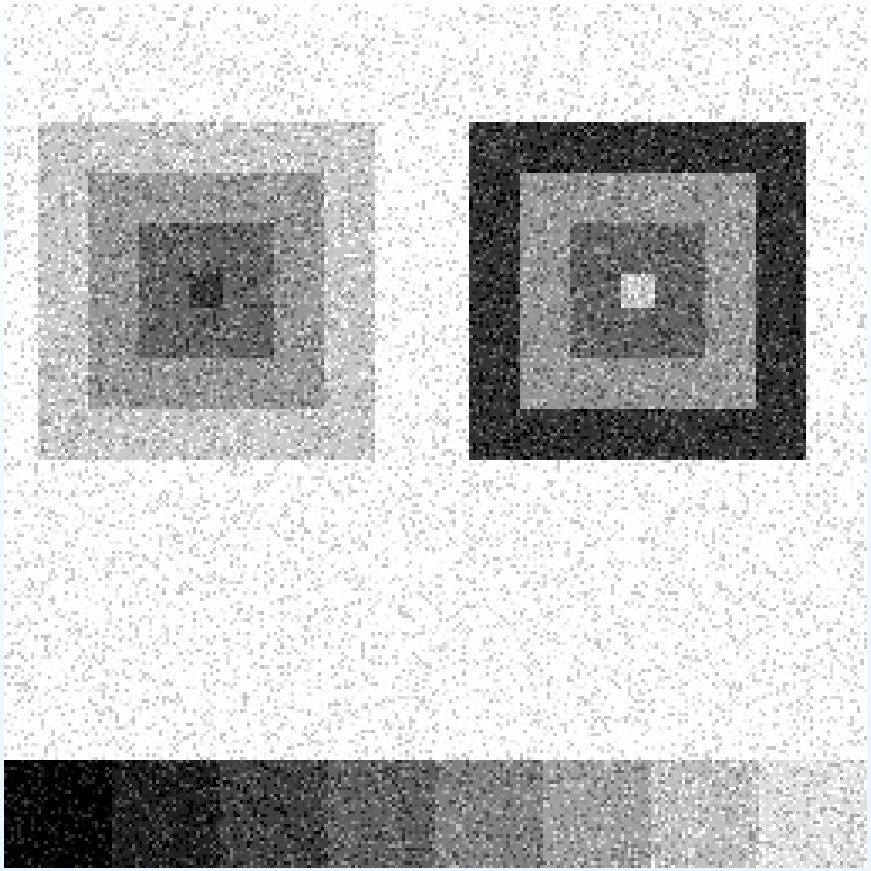
Edge/contour detection (cont.)

Edge/contour detection (cont.)



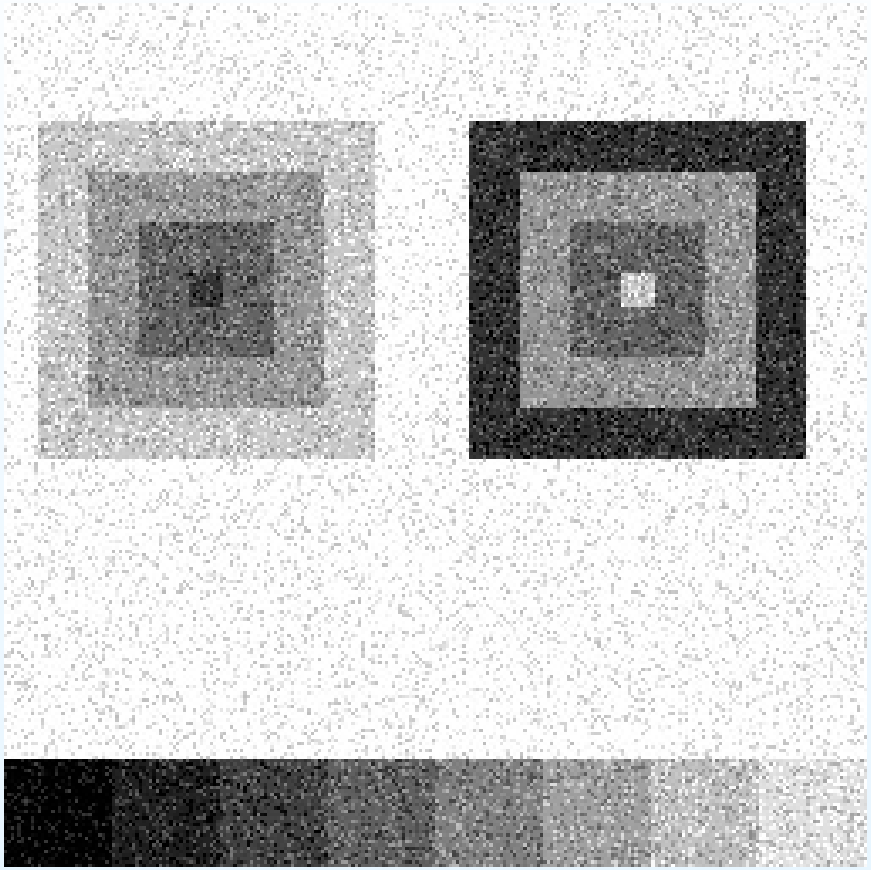
(inverse video)

Edge/contour detection (cont.)

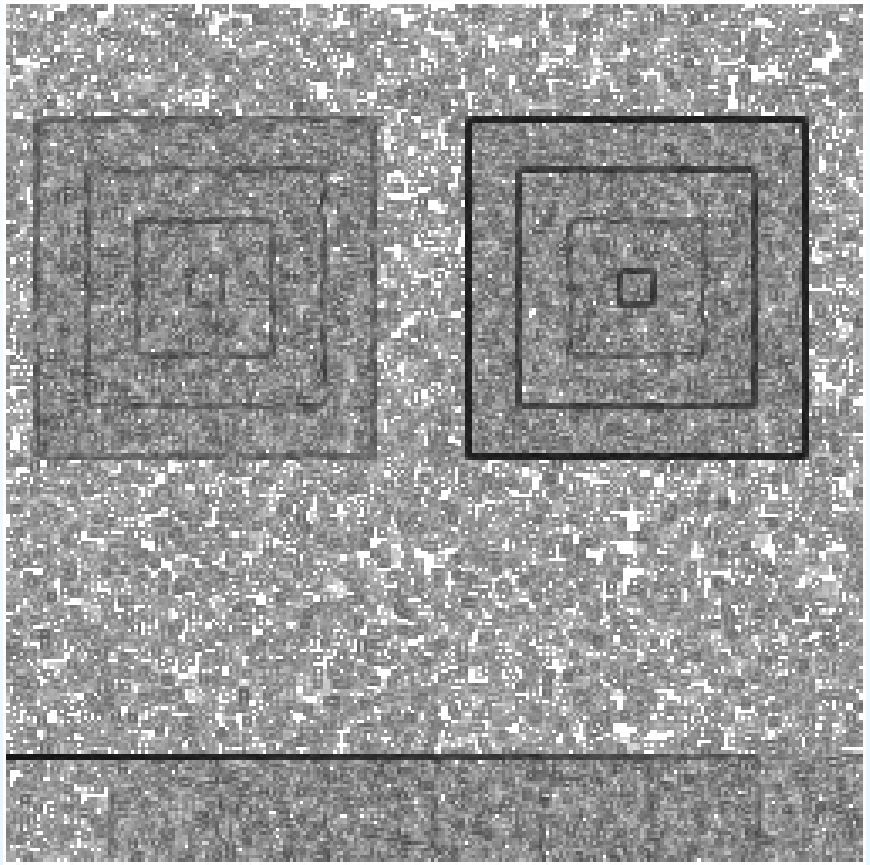


$$u_n(l,c) = u(l,c) + n(l,c)$$

Edge/contour detection (cont.)

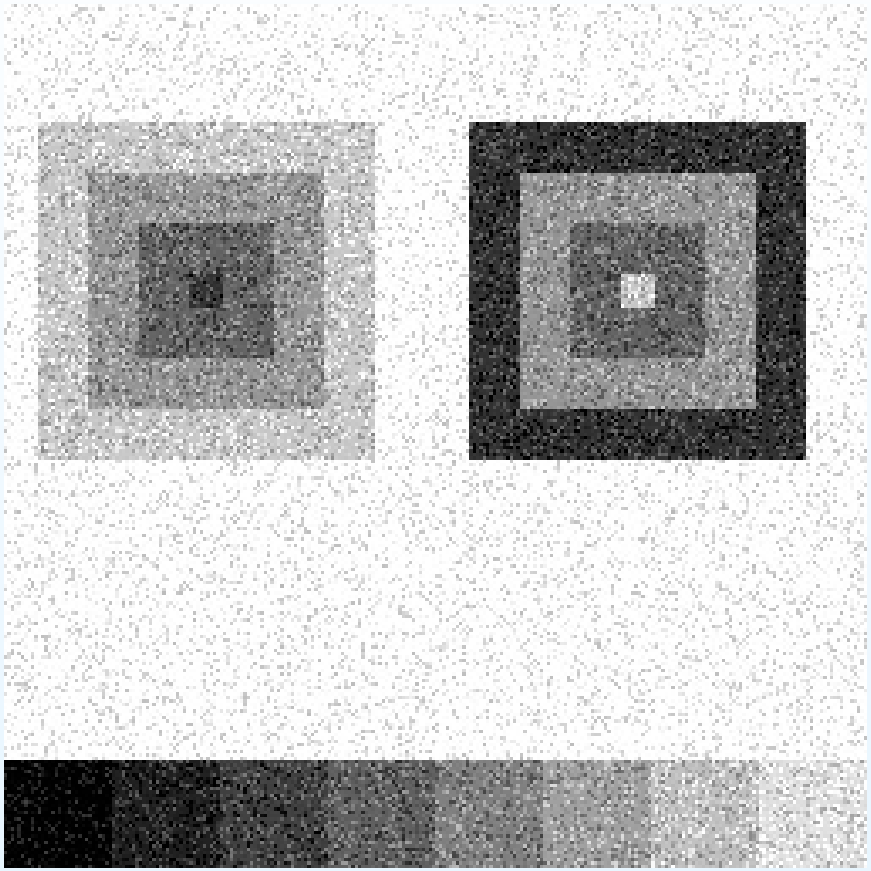


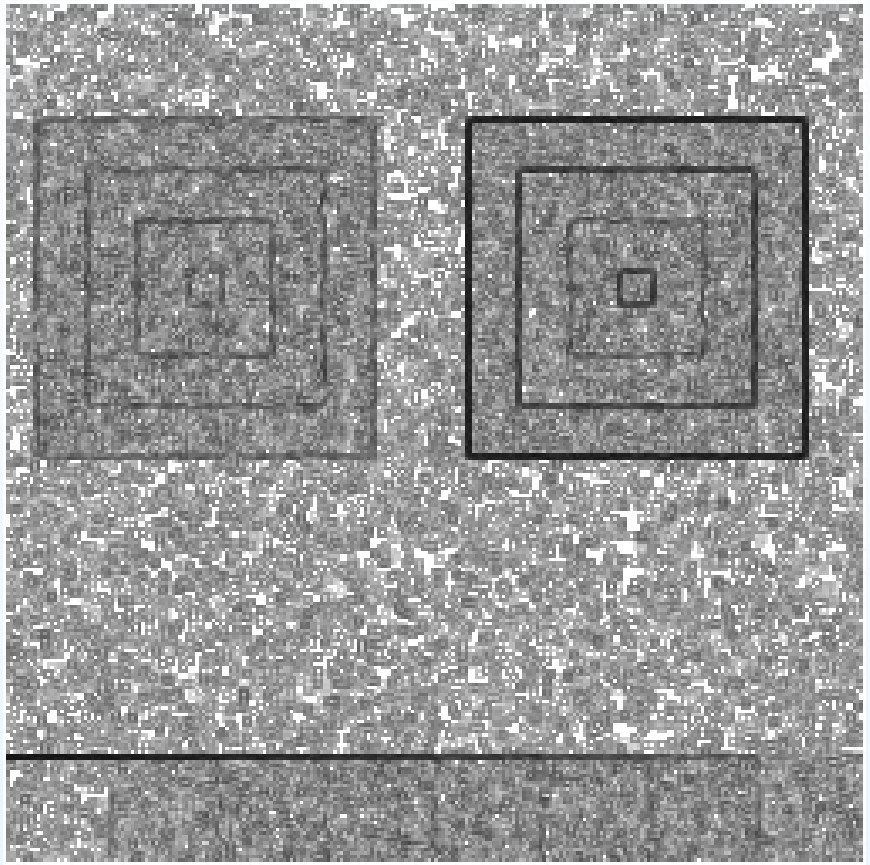
$$u_n(l,c) = u(l,c) + n(l,c)$$



$$\|\nabla u_n\|$$

Edge/contour detection (cont.)





$u_n(l,c) = u(l,c) + n(l,c)$ $\|\nabla u_n\|$

Need to smooth / regularize / denoise the data!

Denoising

Additive noise: a noisy pixel is very different from its neighbors.

Data

0	5	5
7	150	5
1	10	2

Mean Filter

0	5	5
7	20	5
1	10	2

Denoising

Additive noise: a noisy pixel is very different from its neighbors.

Data

0	5	5
7	150	5
1	10	2

Mean Filter

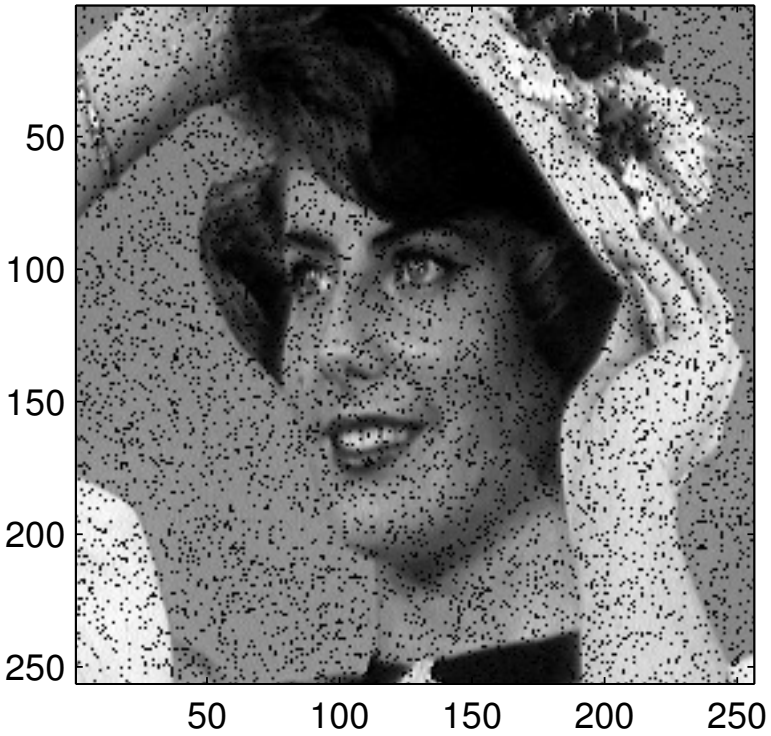
0	5	5
7	20	5
1	10	2

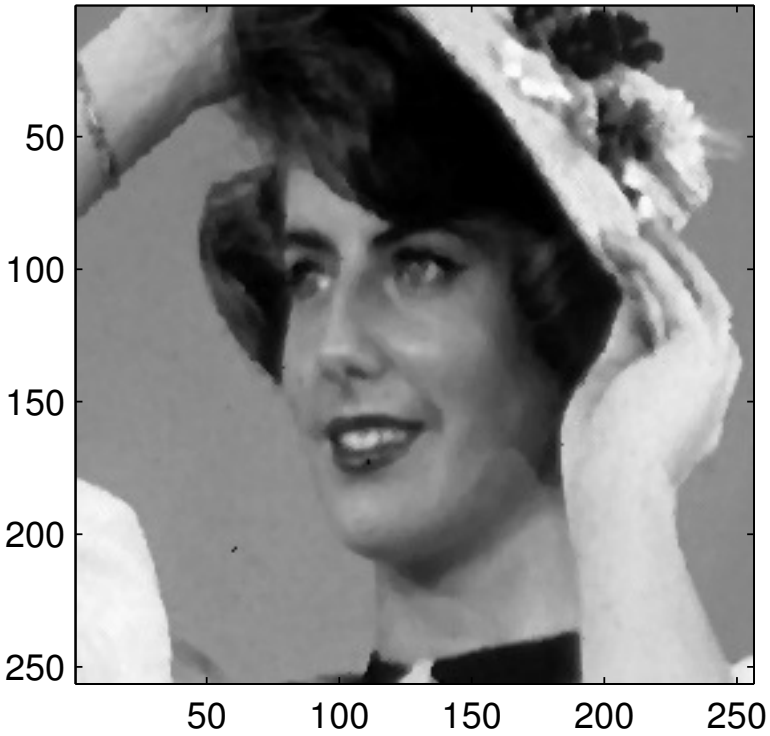
Median Filter

0	5	5
7	5	5
1	10	2

$$0 \leq 1 \leq 2 \leq 5 \leq \mathbf{5} \leq 5 \leq 7 \leq 10 \leq 150 \Longrightarrow \text{median=}\mathbf{5}$$

Median Filter





The Median Filter is non linear, gives quite good results but needs a sorting algorithm.

Denoising: a zoo of equations

- The Mean Filter is replaced by weighted local means, this can be written

$$\frac{\partial u}{\partial t} = \Delta u$$

Denoising: a zoo of equations

- The Mean Filter is replaced by weighted local means, this can be written

$$\frac{\partial u}{\partial t} = \Delta u$$

- Non linear Partial Differential Equations:

$$\frac{\partial u}{\partial t} = |\nabla u| \, \mathrm{div} \left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|} \right)$$

Denoising: a zoo of equations

- The Mean Filter is replaced by weighted local means, this can be written

$$\frac{\partial u}{\partial t} = \Delta u$$

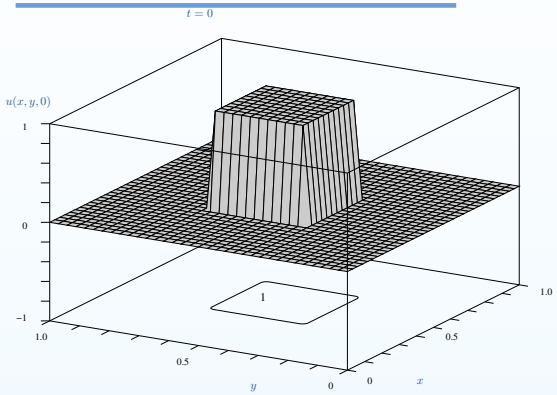
- Non linear Partial Differential Equations:

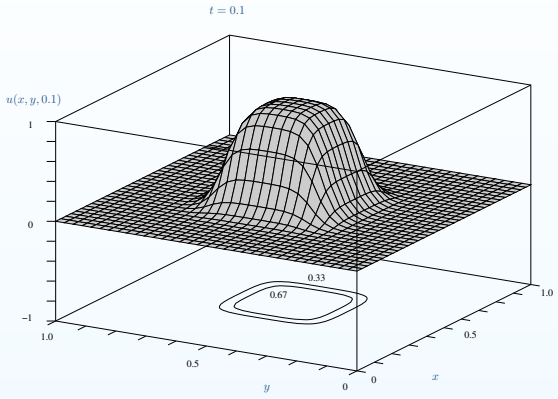
$$\frac{\partial u}{\partial t} = |\nabla u| \, \mathrm{div} \left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|} \right)$$

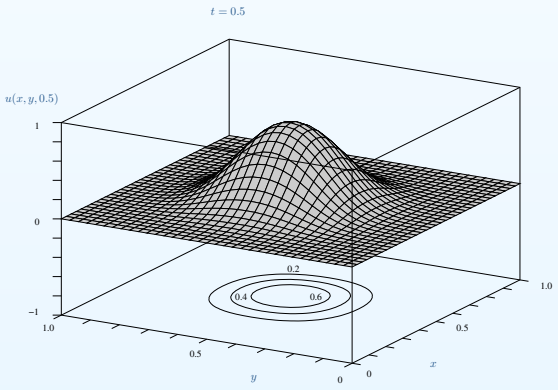
- Total Variation Minimization:

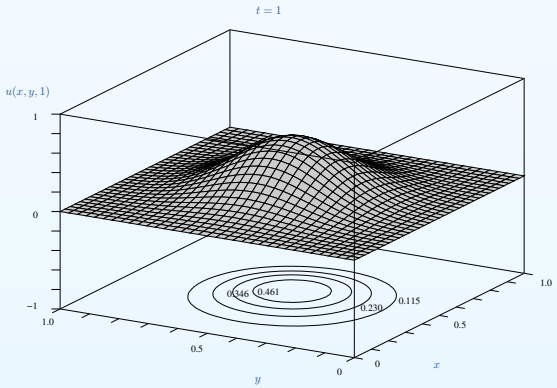
$$\frac{\partial u}{\partial t} = \mathrm{div} \left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|} \right)$$

Heat Equation

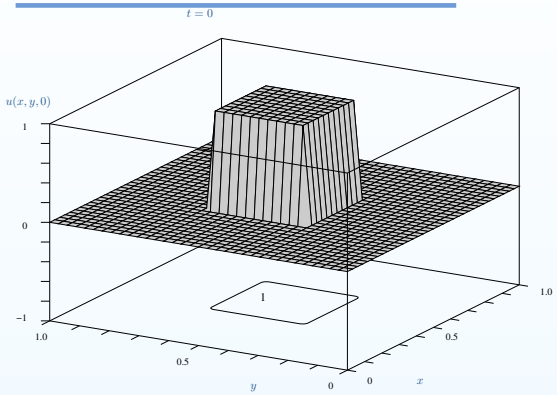


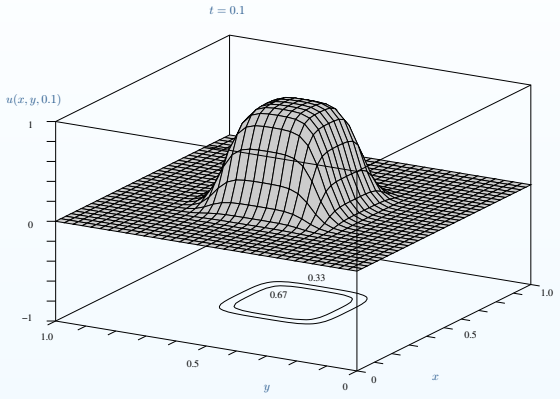


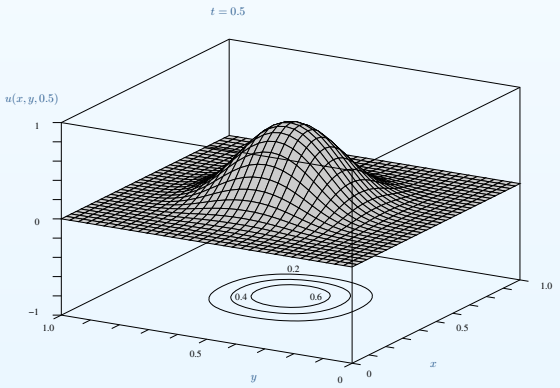


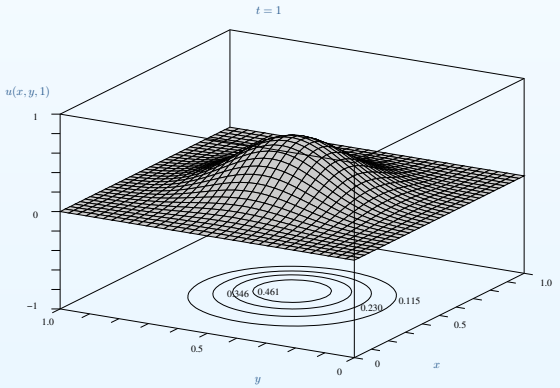


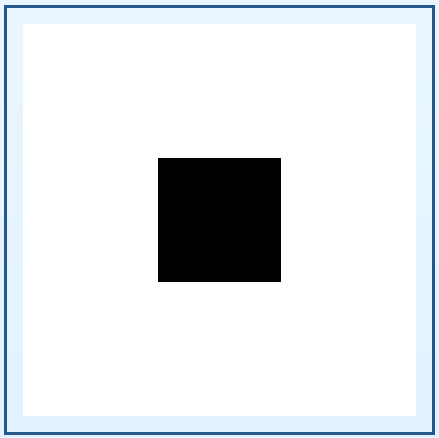
Heat Equation

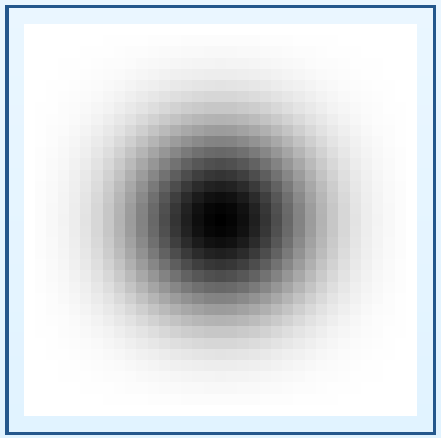




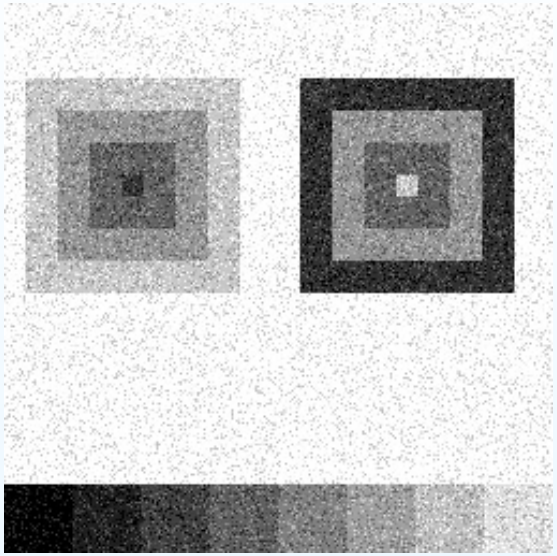




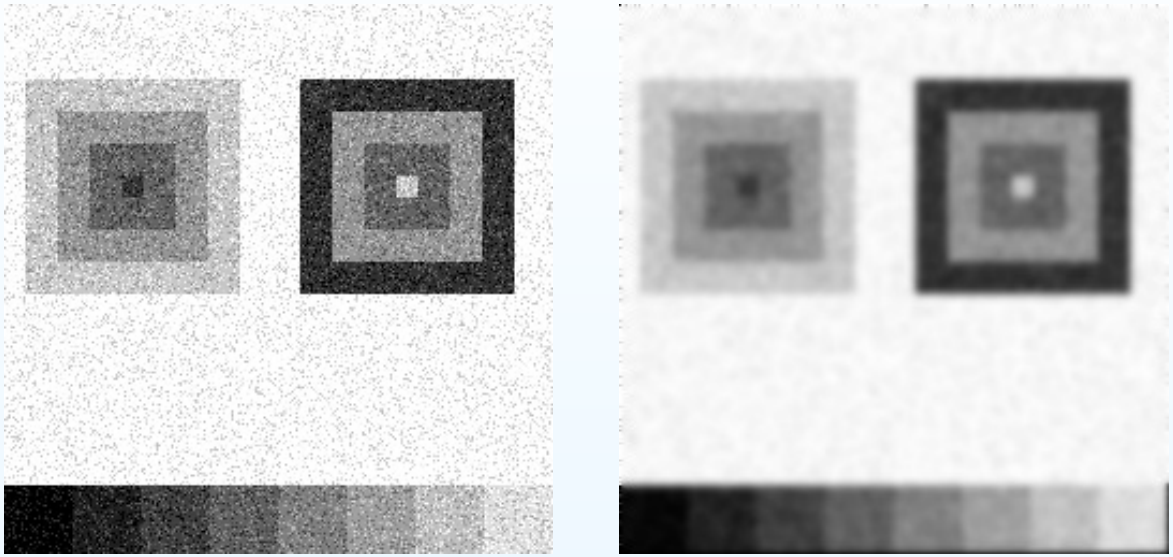




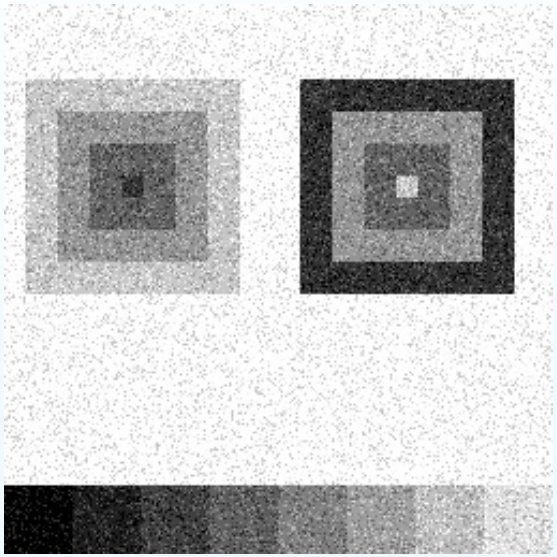
Heat Equation: evolution

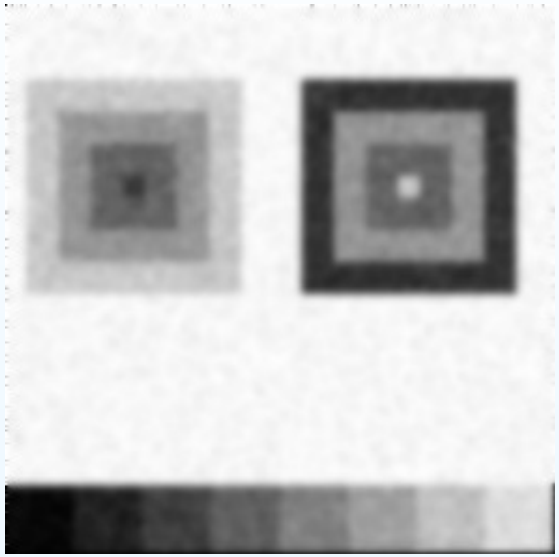


Heat Equation: evolution

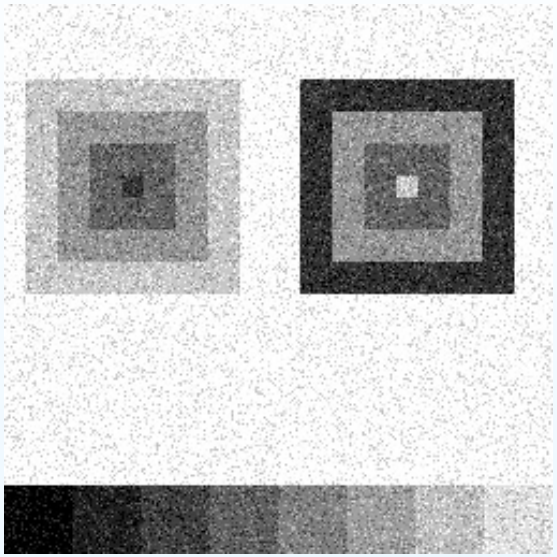


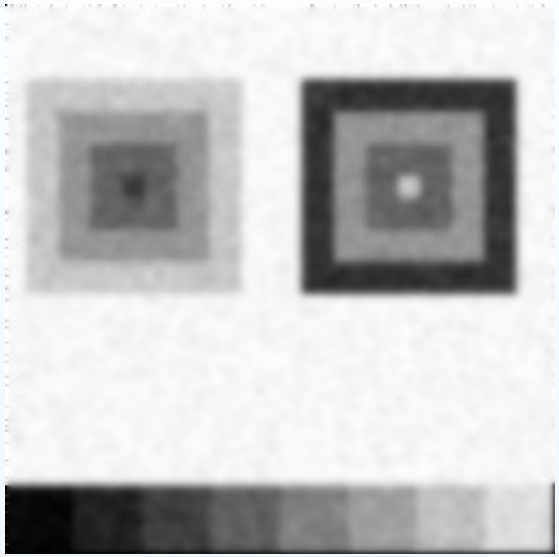
Heat Equation: evolution





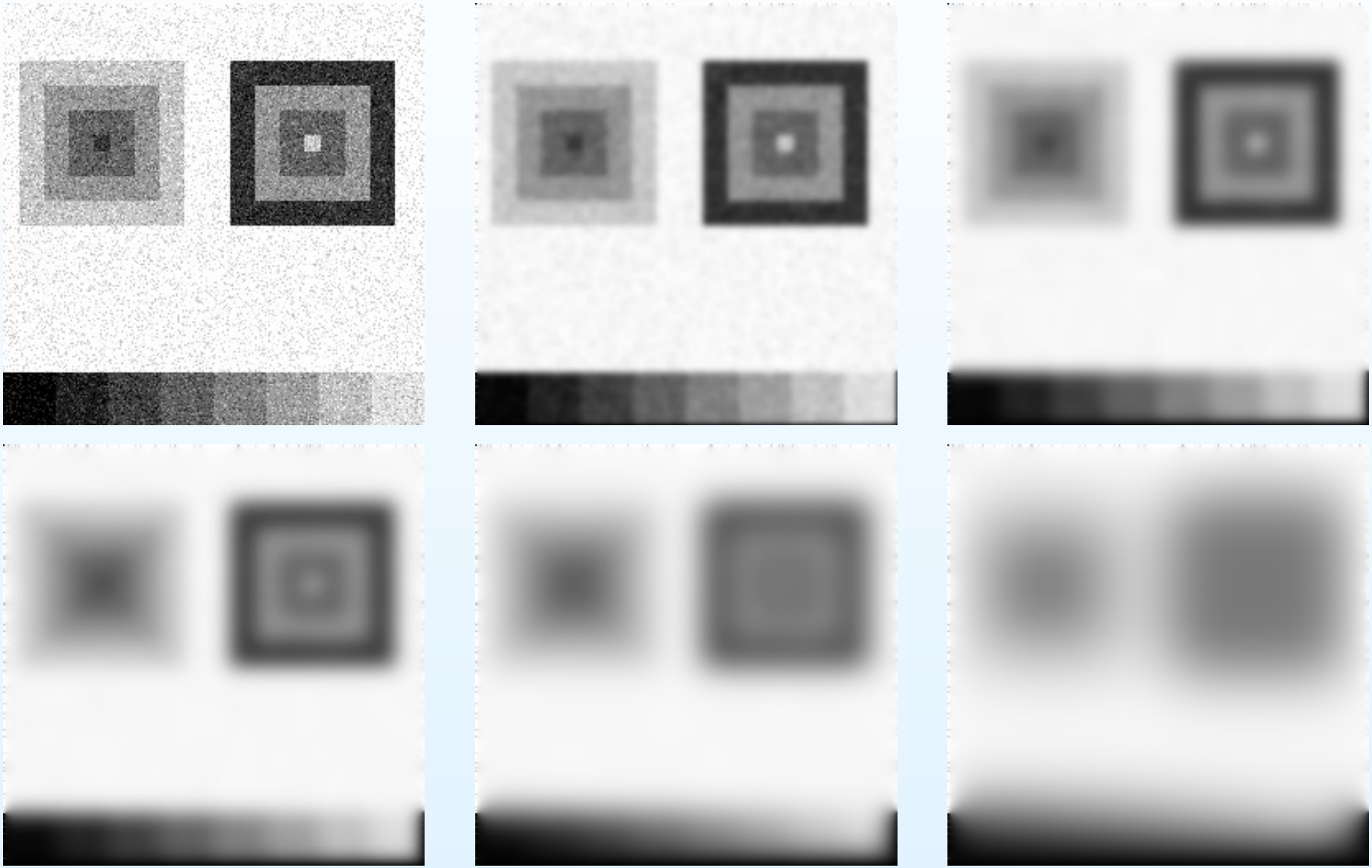
Heat Equation: evolution



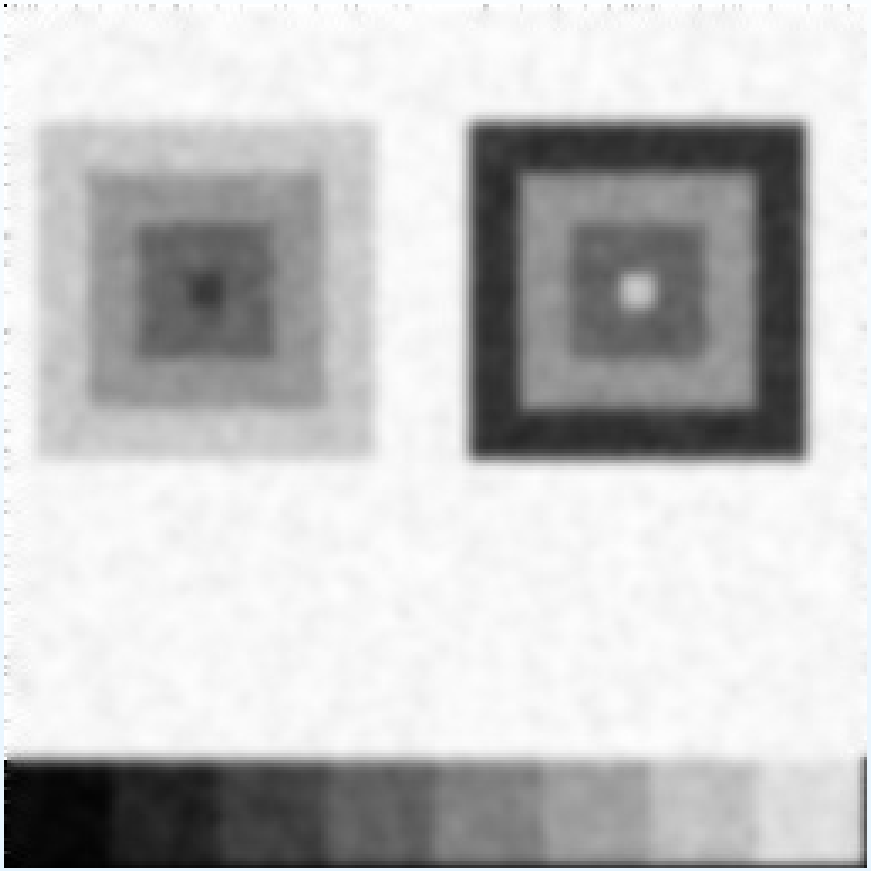


Heat Equation: evolution

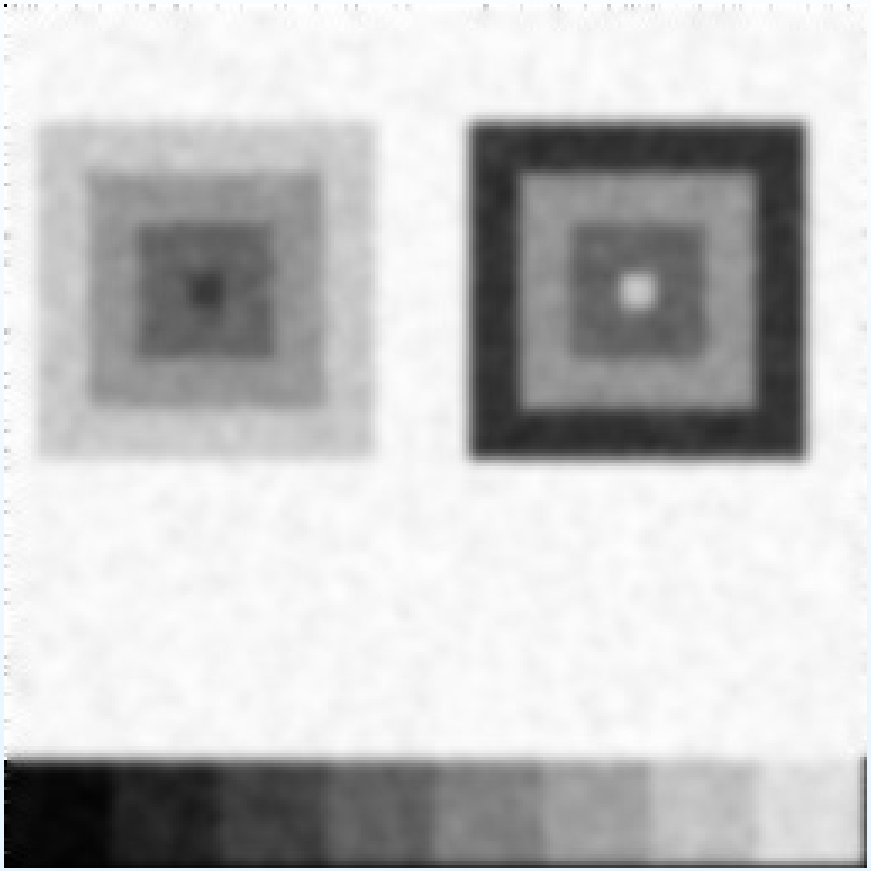
Heat Equation: evolution

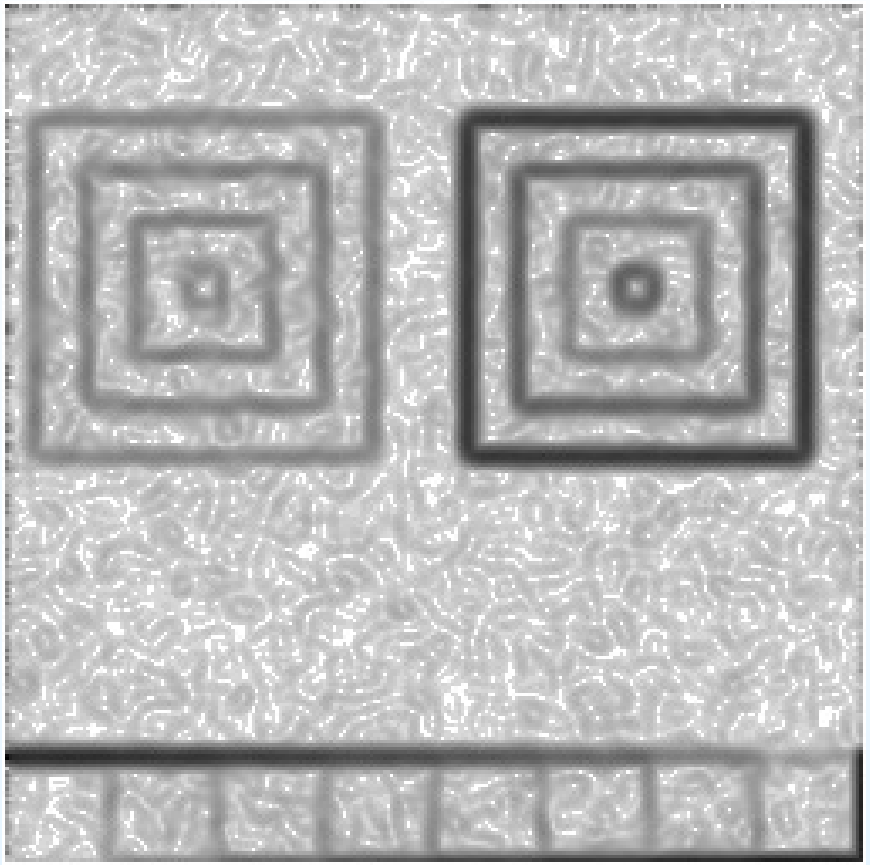


Heat Equation (cont.)



Heat Equation (cont.)

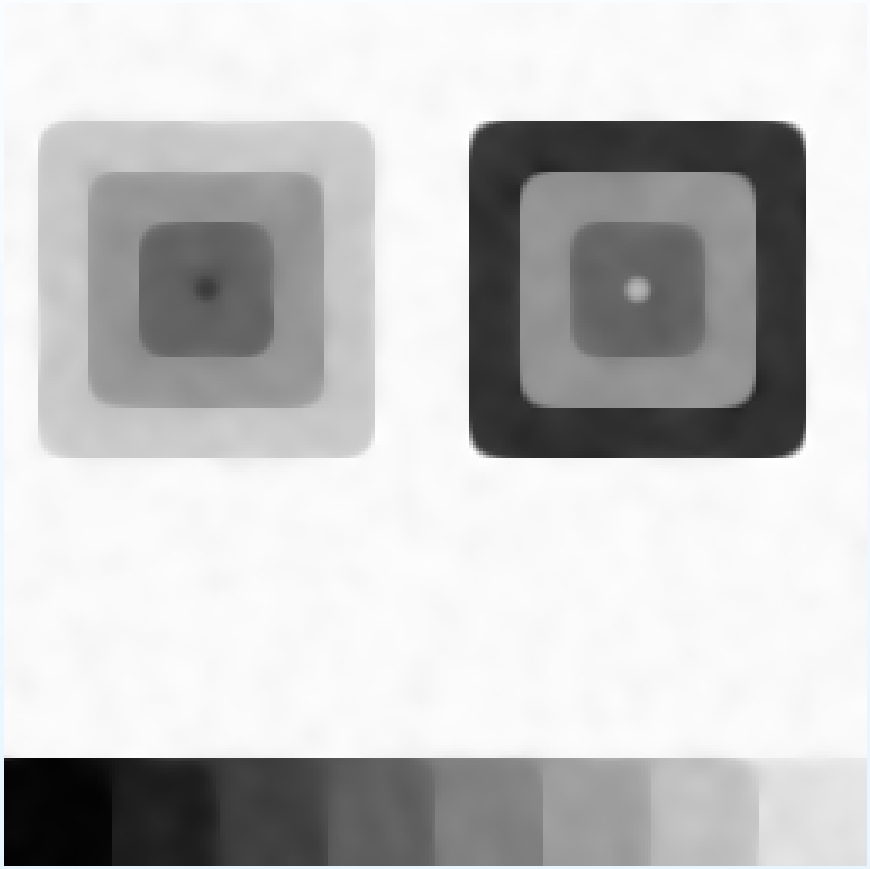


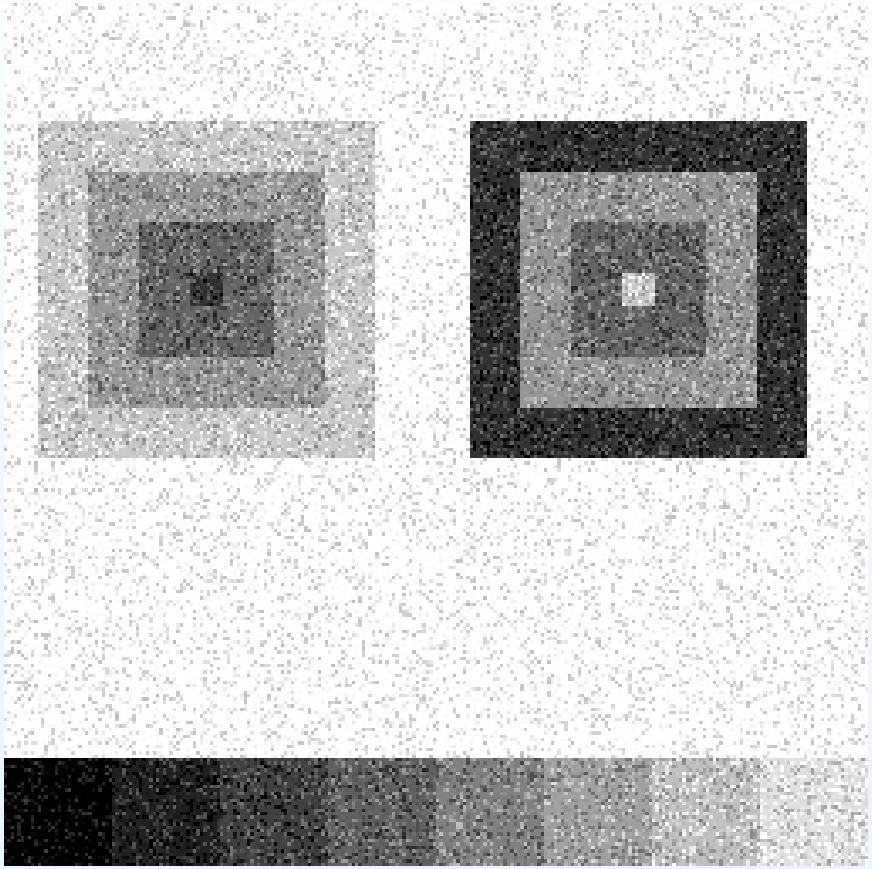


$$G_\sigma \star u_0 = u(.,\sigma^2/2)$$

$$\|\nabla u(.,\sigma^2/2)\|$$

Mean Curvature Motion

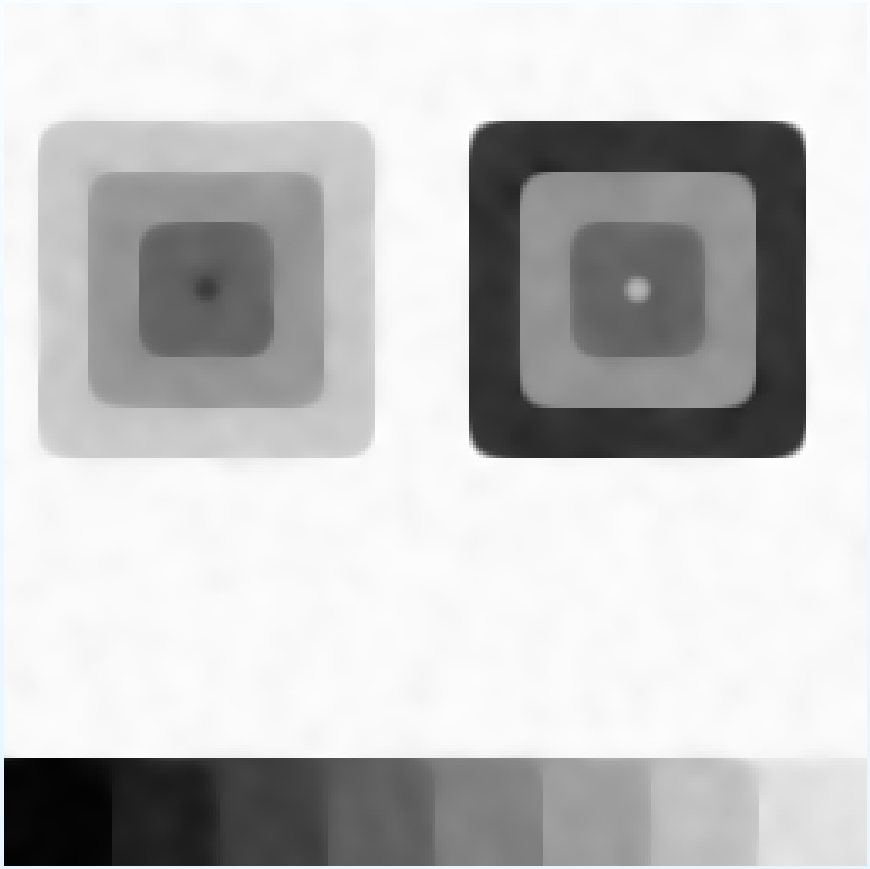


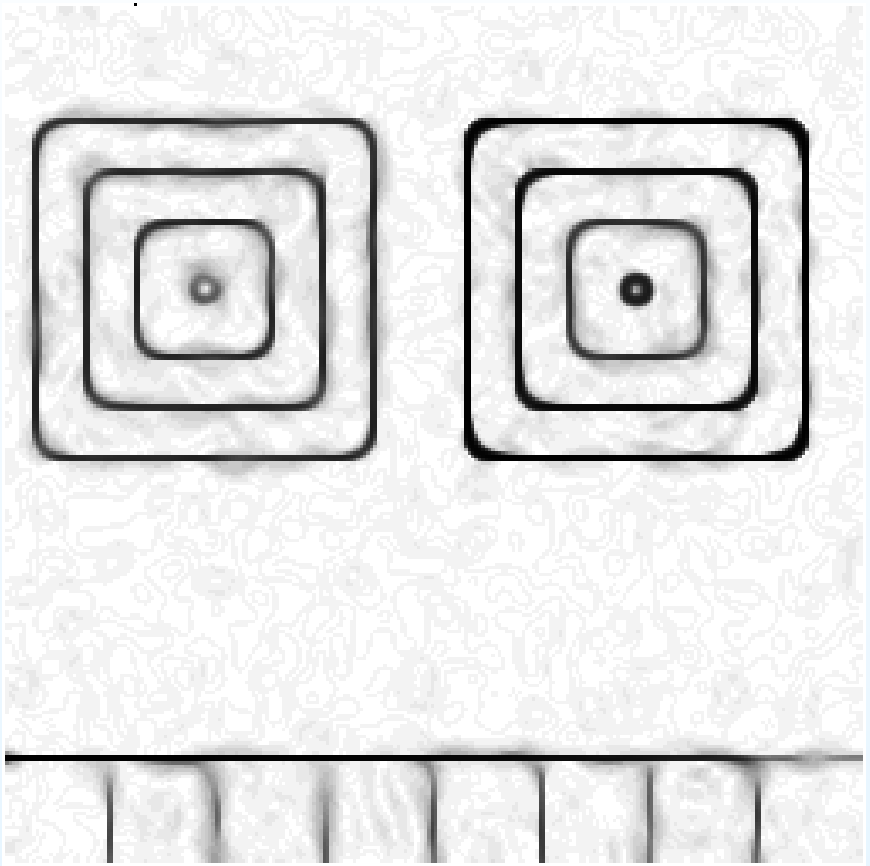


$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

original

Mean Curvature Motion

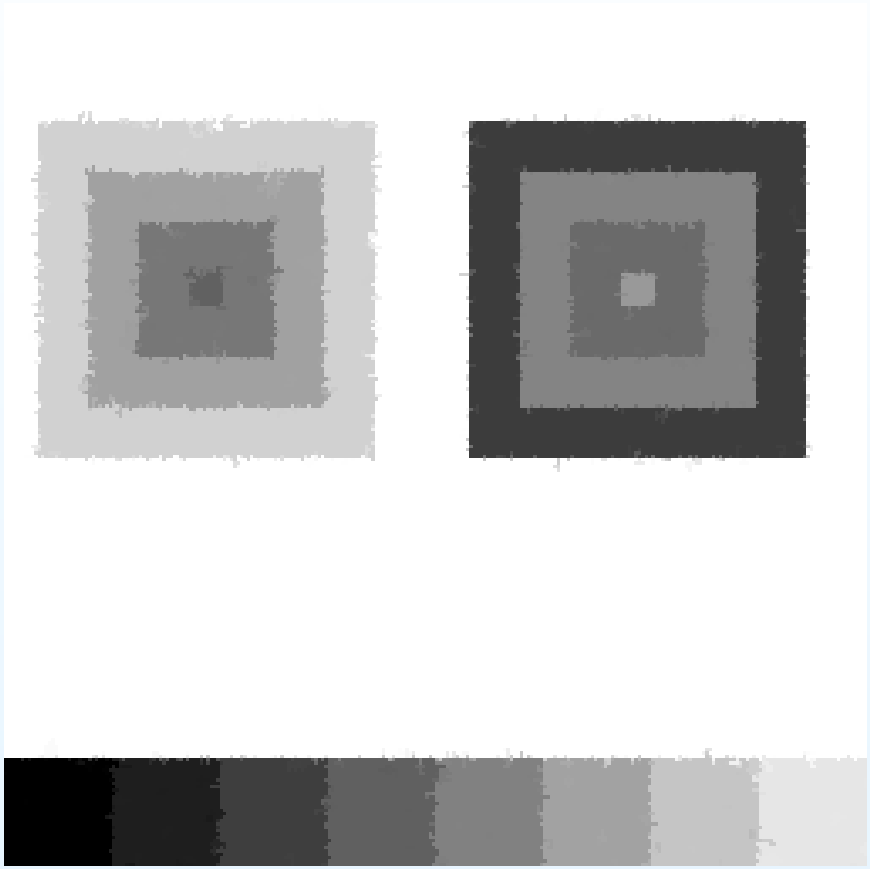


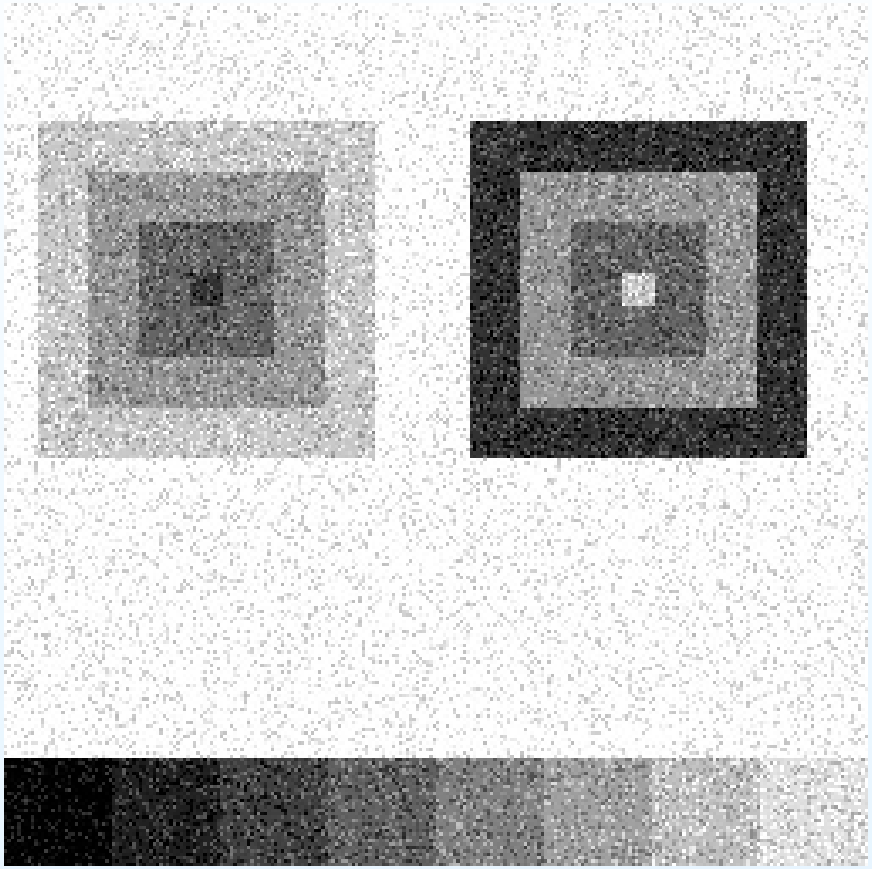


$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

$$\|\nabla u\|$$

Total Variation Minimization

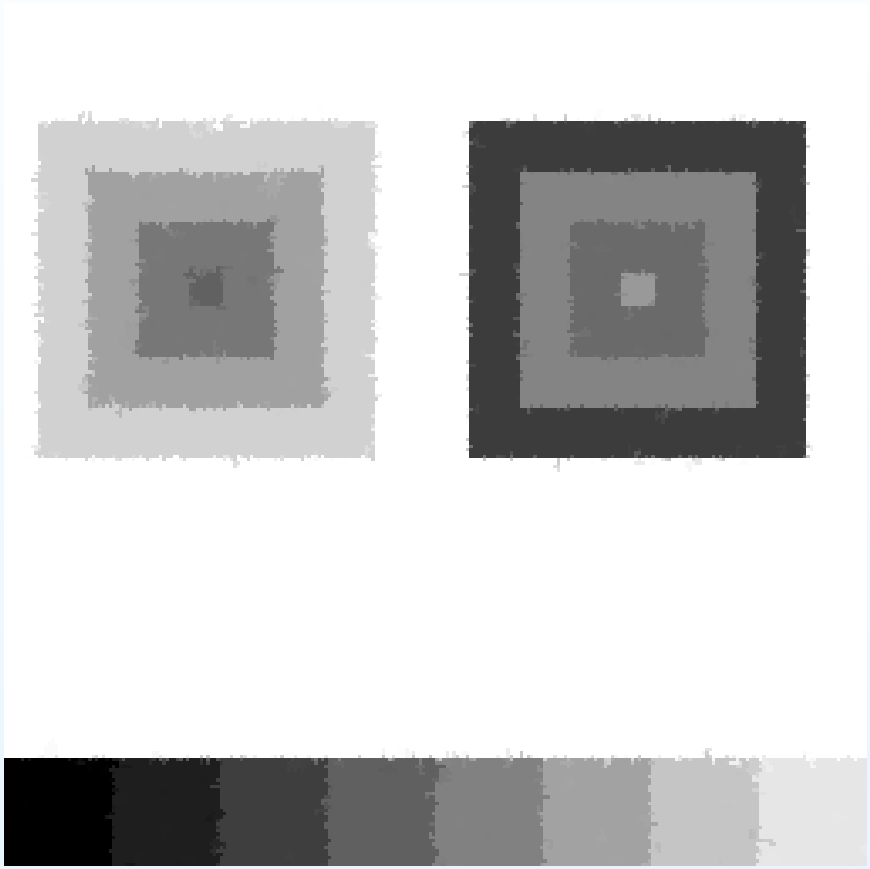


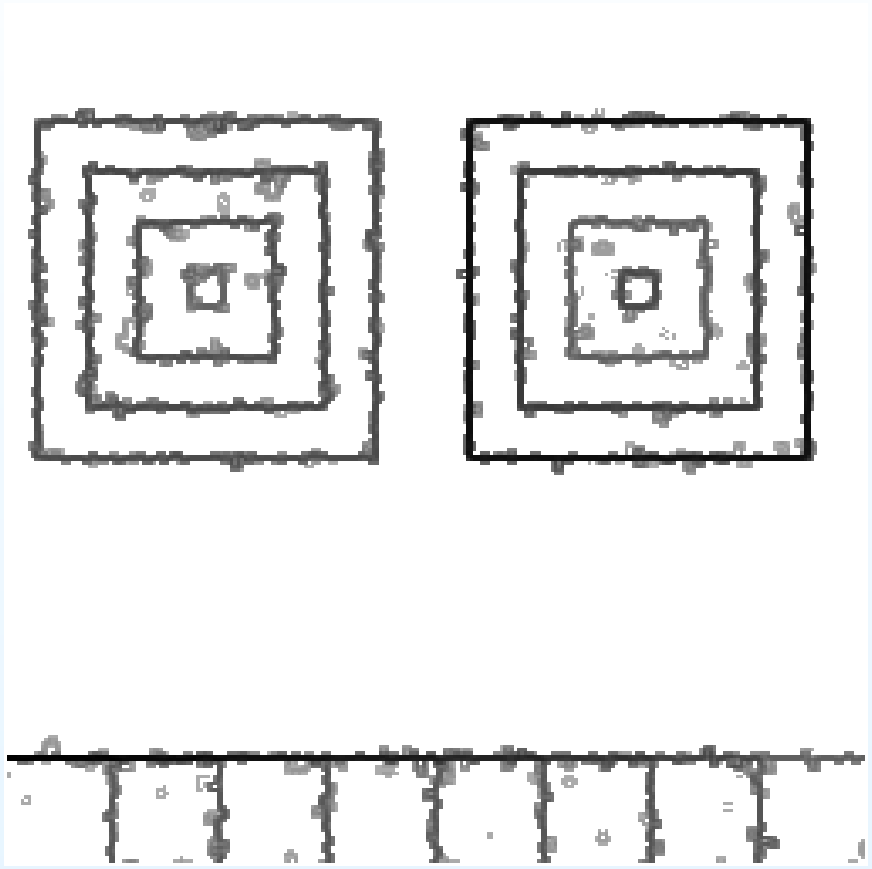


$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

original

Total Variation Minimization





$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

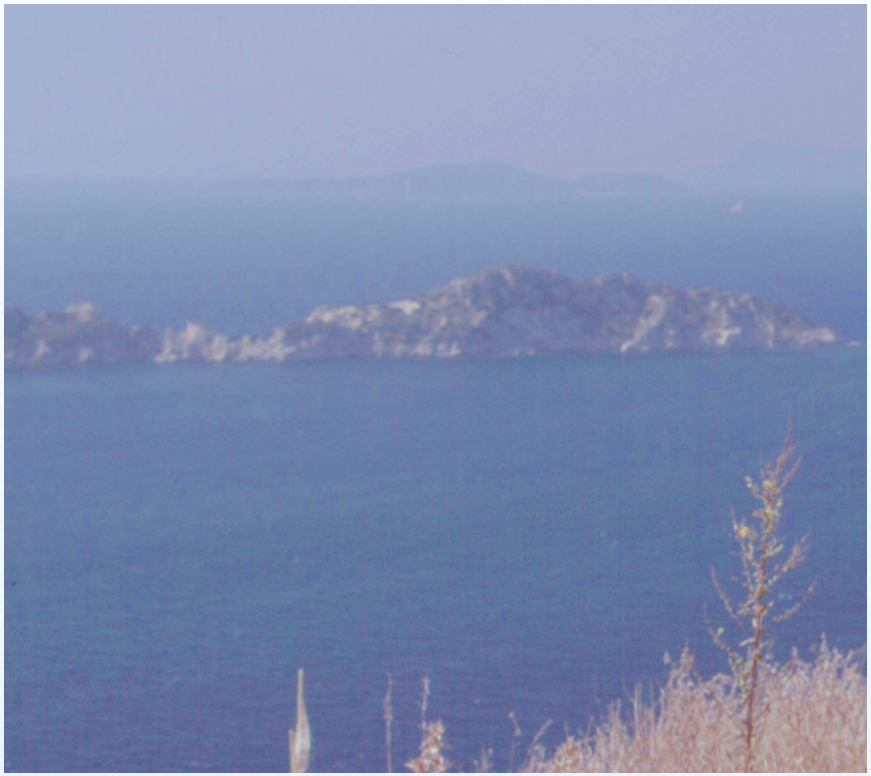
$$\|\nabla u\|$$

Total Variation Minimization (cont.)

noisy image

denoised image

Total Variation Minimization (cont.)

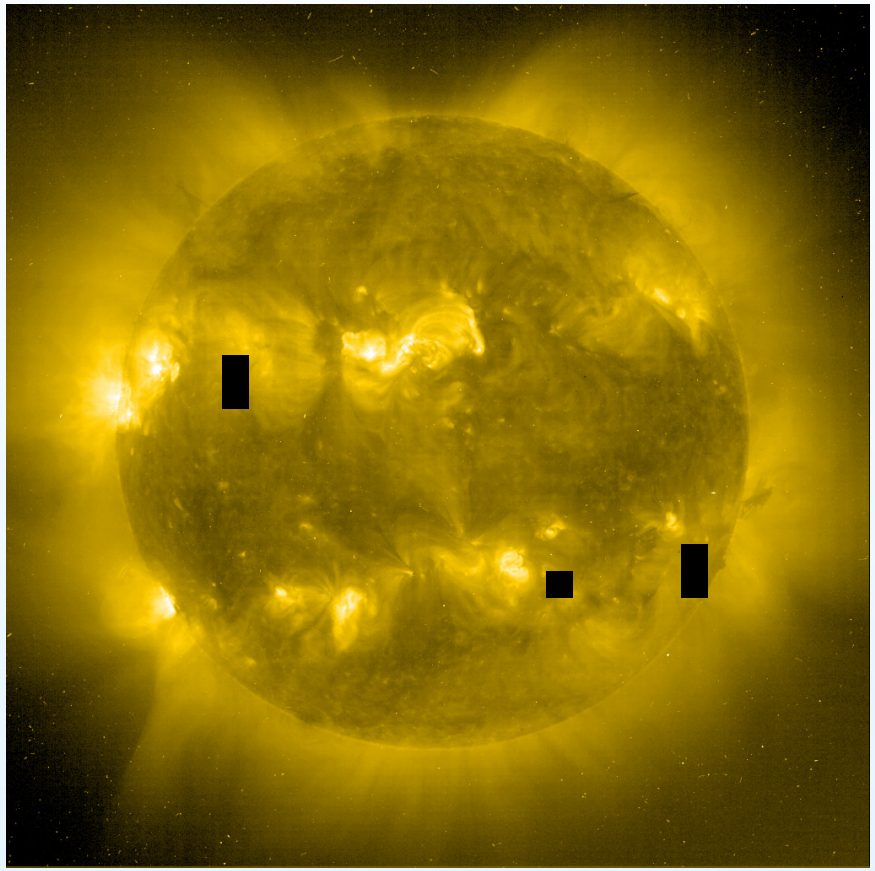


original image

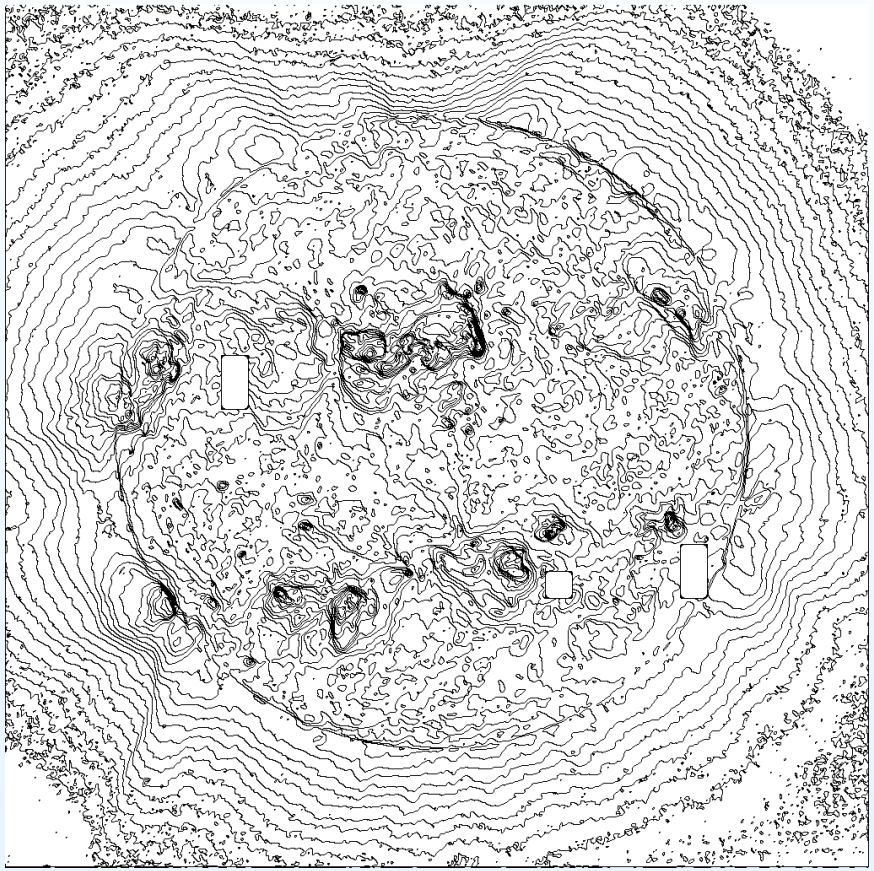
denoised image

Inpainting

Idea: Complete the level lines. . .



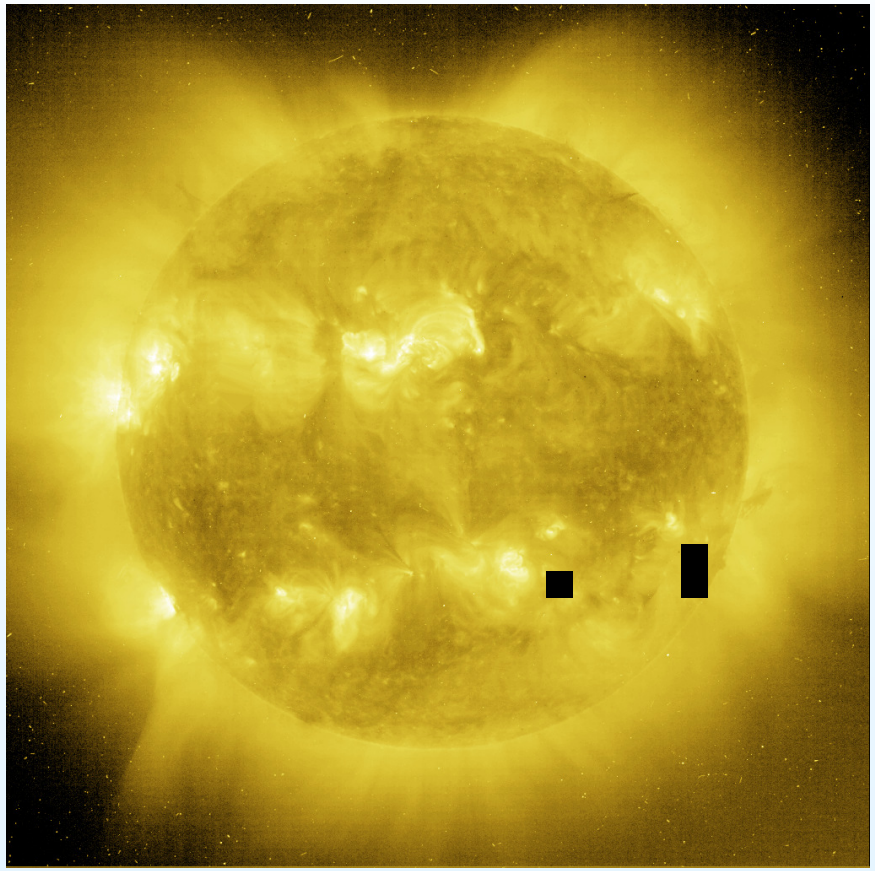
occlusions



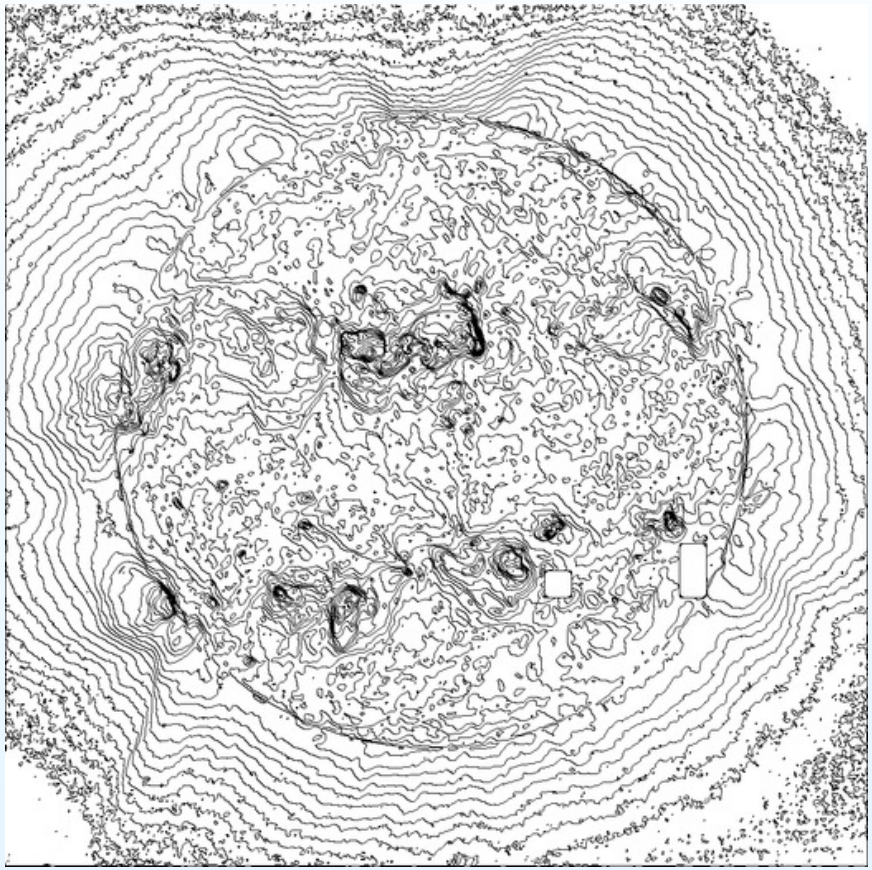
level lines (each 20)

Inpainting

Idea: Complete the level lines. . .



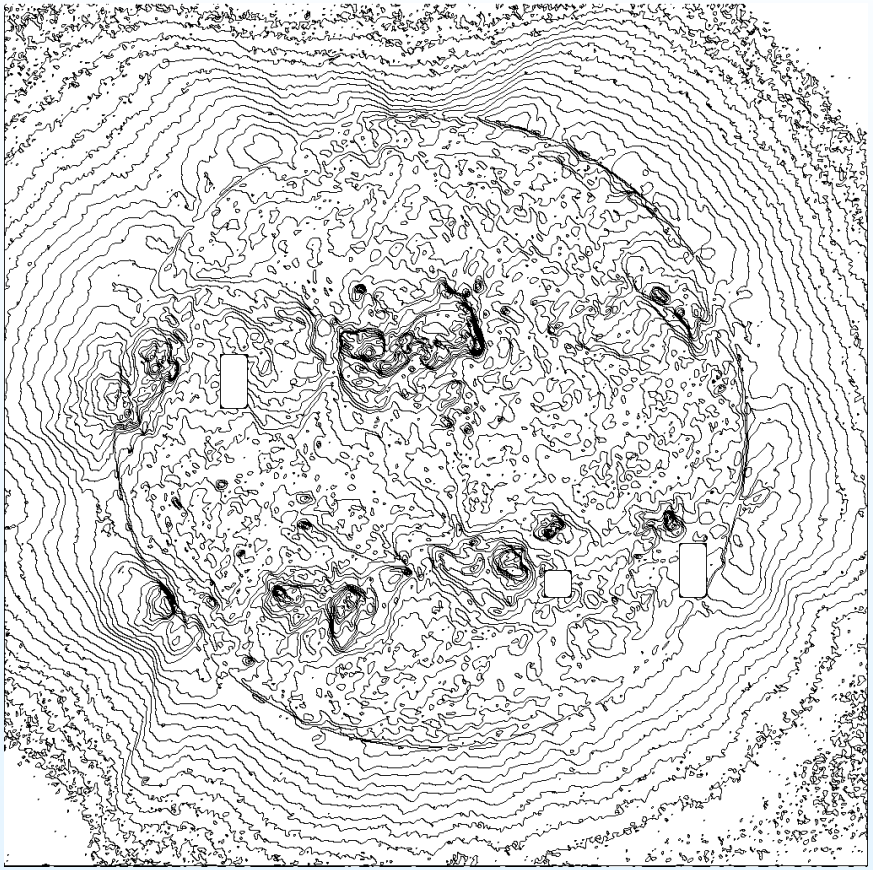
partial restoration



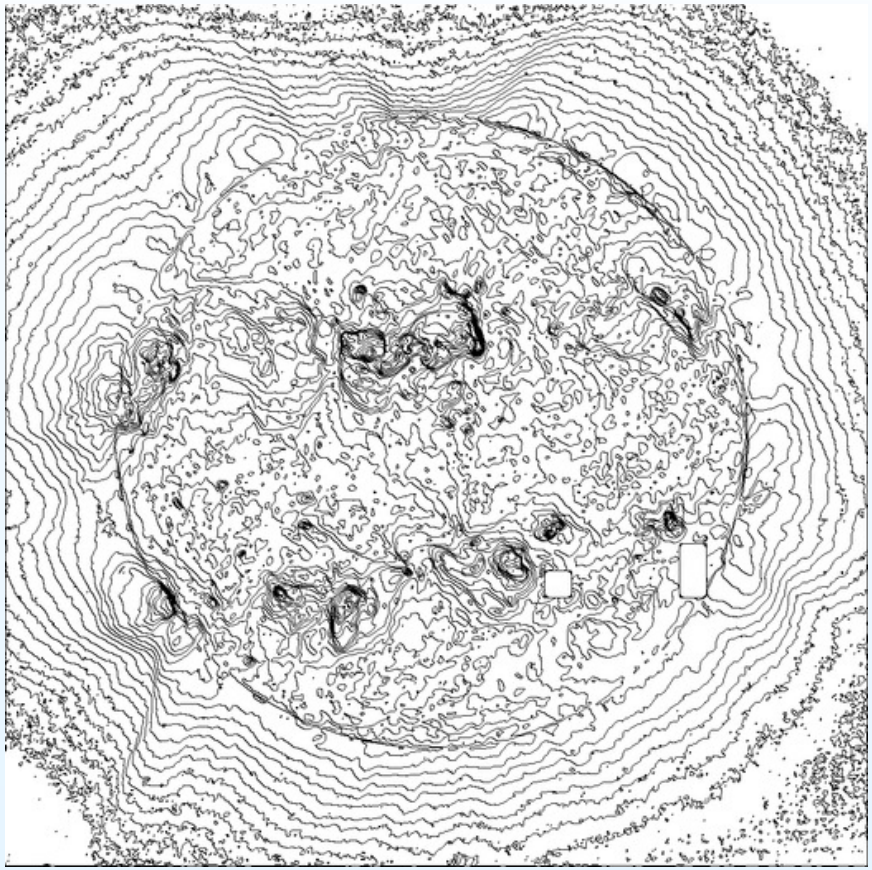
level lines (each 20)

Inpainting

Idea: Complete the level lines. . .



initial level lines



level lines (each 20)

Segmentation

Idea: Analyze original image g and get a simple image u.
Thus to segment an image is:

- replace the original by a cartoon;
- partition the image in homogeneous regions;

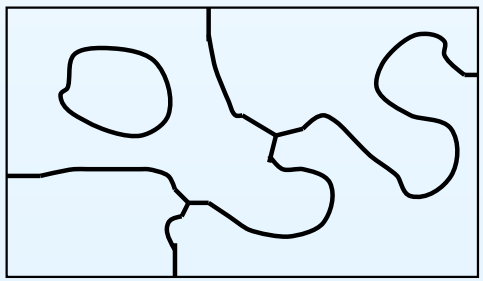
Segmentation

Idea: Analyze original image g and get a simple image u.
Thus to segment an image is:

- replace the original by a cartoon;
- partition the image in homogeneous regions;

MUMFORD-SHAH functional:

$$E(K) = \int_{\Omega \setminus K} (u-g)^2 + \lambda \ell(K)$$



$\Omega = \bigcup O$, O regions s.t. $O \cap O' = \emptyset$
$K = \bigcup \partial O$, set of boundaries

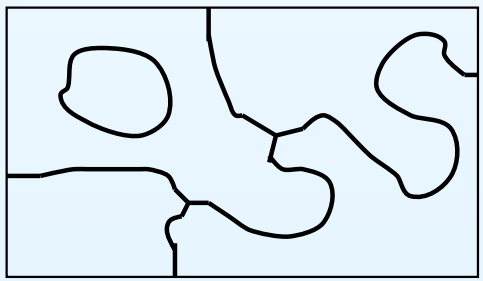
Segmentation

Idea: Analyze original image g and get a simple image u.
Thus to segment an image is:

- replace the original by a cartoon;
- partition the image in homogeneous regions;

MUMFORD-SHAH functional:

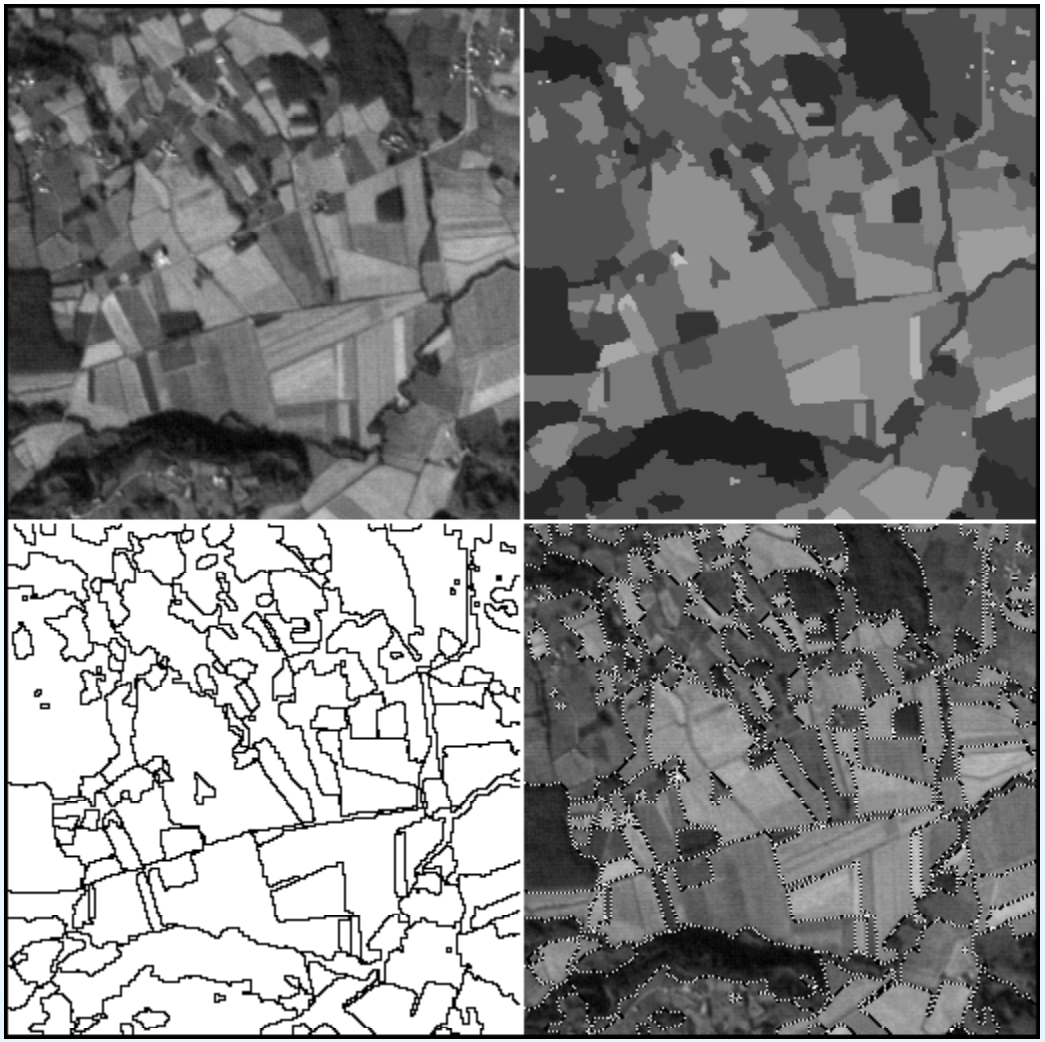
$$E(K) = \int_{\Omega \setminus K} (u-g)^2 + \lambda \ell(K)$$



$\Omega = \bigcup O$, O regions s.t. $O \cap O' = \emptyset$
$K = \bigcup \partial O$, set of boundaries

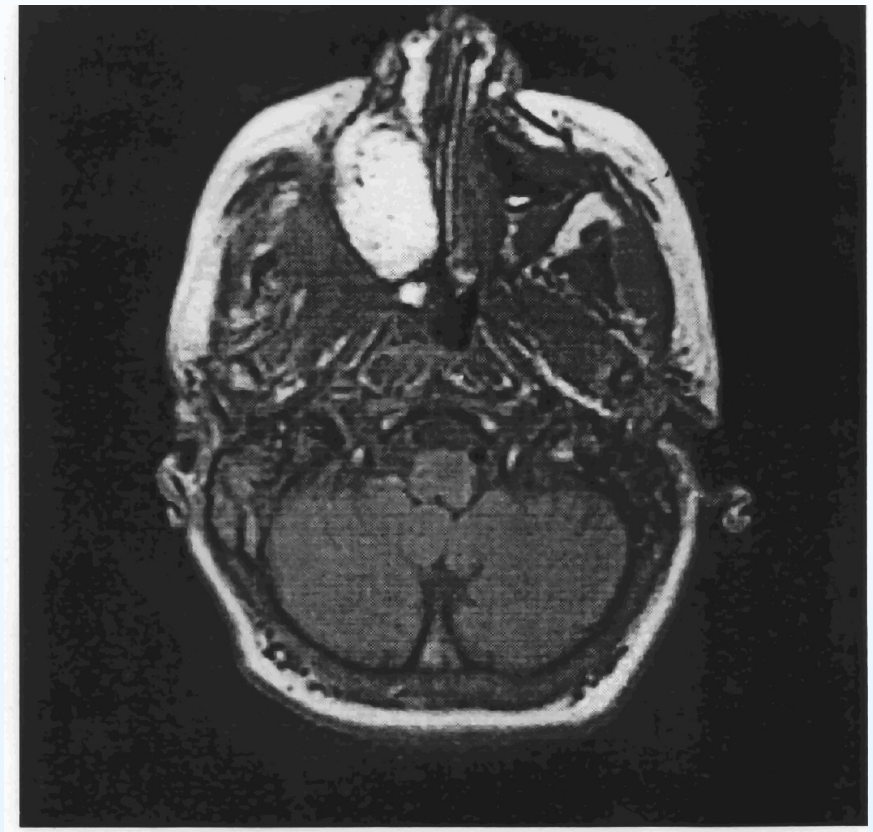
Merge two regions O,O' iff $E(K \setminus (\partial O \cap \partial O')) < E(K)$.

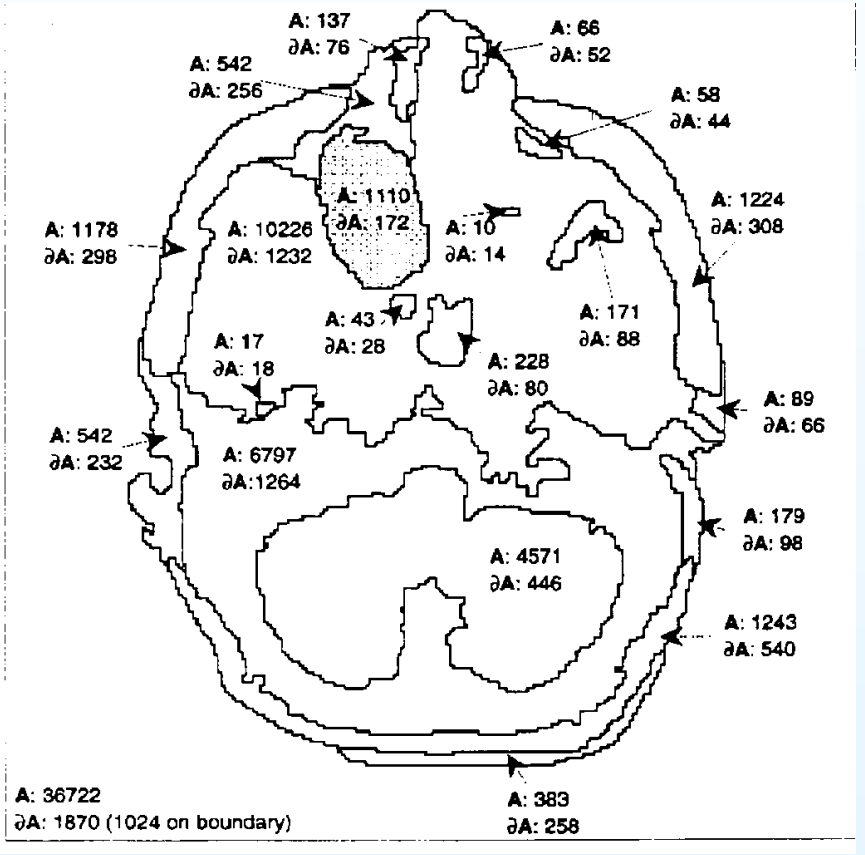
Segmentation



Segmentation: medical application

Segmentation: medical application





Texture Segmentation

Texture Segmentation

Texture Segmentation

Original data

gray level segmentation

wavelet coefficient segmentation

Color Segmentation

Color Segmentation

original

100 regions

Compression

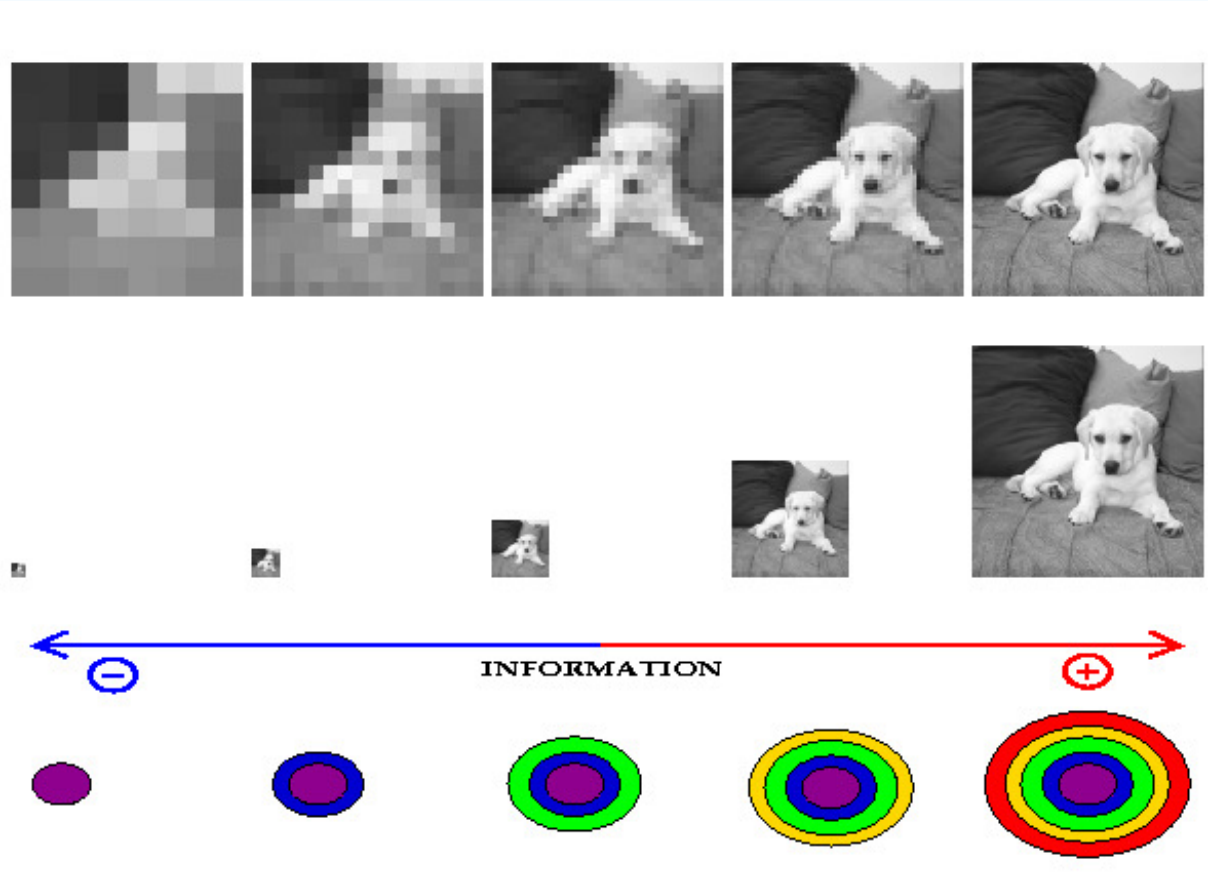
Idea: In an image objects of different scale/size are present.

Get a pyramidal or multiscale representation of an image.

Compression

Idea: In an image objects of different scale/size are present.

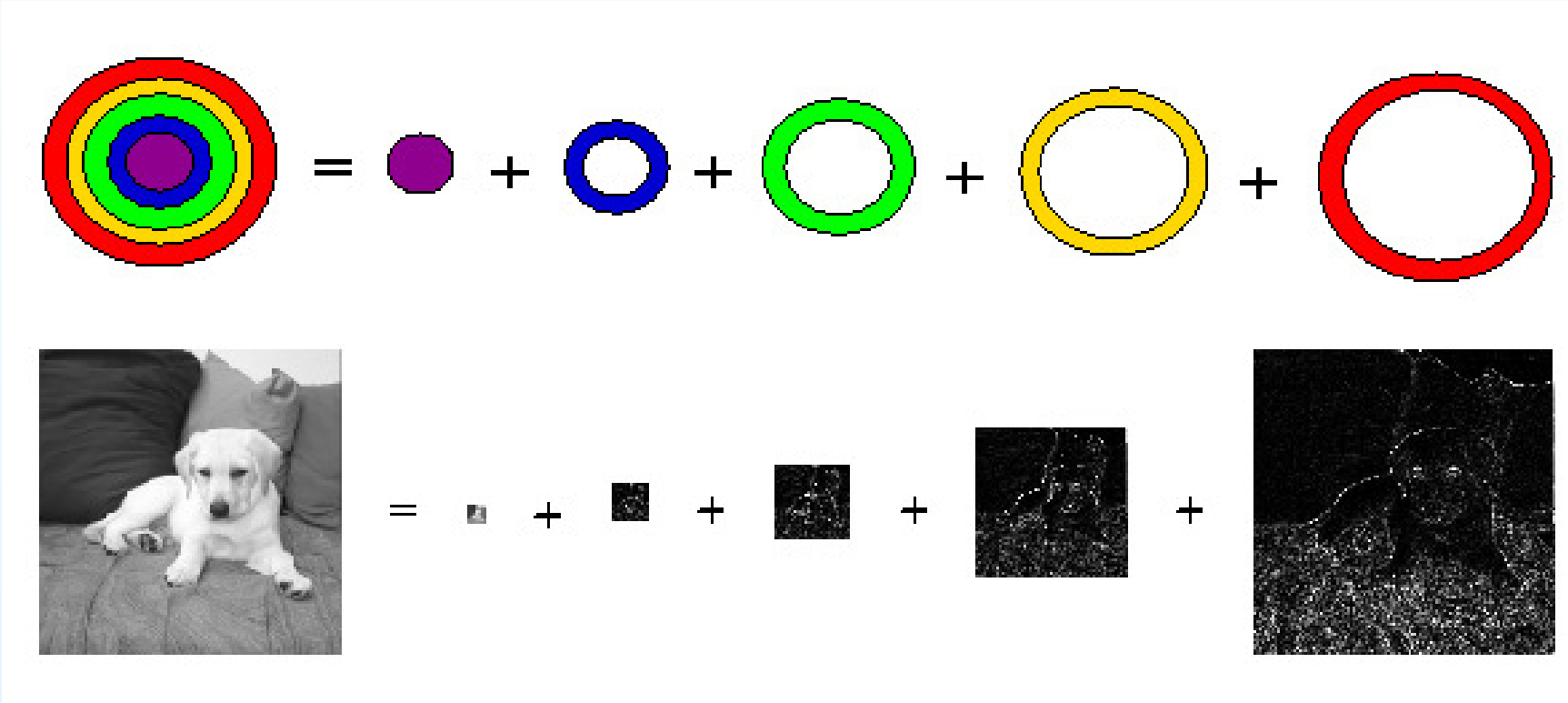
Get a pyramidal or multiscale representation of an image.



(Images & Idea: Basarab MATEI, University Paris 6)

Compression with wavelets

$$u(x,y) = \sum_{\lambda} c_\lambda \Psi_\lambda(x,y)$$



(Images & Idea: Basarab MATEI, University Paris 6)