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Modeling planar shape variation via hamiltonian flows of curves

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Summary. The application of the theory of deformable templates to the study of the action of a group of diffeomorphisms on deformable objects provides a powerful framework to compute dense one to one matchings on a d-dimensional domains. In this paper, we derive the geodesic equations that governs the time evolution of an optimal matching in the case of the action on 2D curves with various driving matching terms, and provide a Hamiltonian formulation in which the initial momentum represented by an L^2 vector field on the boundary of the template.

Key words: Infinite Dimensional Riemannian Manifolds, Hamiltonian System, Shape representation and recognition

1.1 Introduction

This paper focuses on the study of plane curve deformation, and how it can lead to curve evolution, comparison and matching. Our primary interest is in diffeomorphic deformations, in which a template curve is in one-to-one smooth correspondence with a target curve. This correspondence will be expressed as the restriction (to the template curve) of a 2D diffeomorphism, which will control the quality of the matching.

This point of view, which is non standard in the large existing literature on curve matching, emanates from the general theory of “large deformation diffeomorphisms”, introduced in [9, 6, 16], and further developed in [12, 13]. This is a different approach than the one which only considers the restriction of the diffeomorphisms to the curves starting with the introduction of dynamic time warping algorithms in speech recognition [14], and developed in papers like [7, 3, 21, 17, 22, 11, 15, 18].

Like in [21, 11], however, our approach is related to geodesic distances between plane curves. In particular, we will provide a Hamiltonian interpretation of the geodesic equations (which in this case shares interesting properties with a physical phenomenon called solitons [10]), and exhibit the structure of the *momentum*, which is of main importance in this setting.

The deformation will be driven by a data attachment term which measures the quality of the matching. In this paper, we review 3 kinds of attachments. The first one, that we call measure-based, is based on the similarity of the singular measures in $\mathbb{R}^2$ which are supported by the curves. The second, which is adapted to *Jordan curves* corresponds to the measure of the symmetric differences of the domains within the curves (binary shapes). The last one is for data attachement terms based on a potential, as often introduced in the theory of active contours.

The paper is organized as follows. Section 1.2 provides some definition and notation, together with a heuristic motivation of the approach. Section 1.3 develops a first version of the momentum theorem, which relates the momentum of the hamiltonian evolution to the diffierential of the data attachement term. Section 1.4 is an application of this framework to measure-based matching. Section 1.5 deals with binary shapes, and provides a more general version of the momentum theorem, which will also be used in section 1.6 for data attachment terms based on a potential. Finally, section 1.7 proves an existence theorem for the Hamiltonian flow.

1.2 Diffeomorphic curve and shape matching with large variability

In this paper, a shape $S_\gamma \subset \mathbb{R}^2$ is defined as the interior of a sufficiently smooth Jordan curve (i.e. continuous, nonintersecting) $\gamma : \mathbb{T} \to \mathbb{R}^2$ where $\mathbb{T}$ is the 1D torus. (Hence, γ is a parametrization of the boundary of S.)[4].

The emphasis will be on the action of global non-rigid deformation, for which we introduce some notation. Assume that a group G of C^1 diffeomorphisms of $\mathbb{R}^2$ provides a family of admissible non-rigid deformations. The action of a given deformation φ on a shape $S \subset \mathbb{R}^2$ is defined by

$$S_{def} = \varphi(S)\,. \tag{1.1}$$

Selecting one shape as an initial template $S_{temp} = S_{\gamma_{temp}}$, we will look for the best global deformation of the ambient space which transforms S_{temp} into a target shape S_{targ}. The optimal matching of the template on the target will

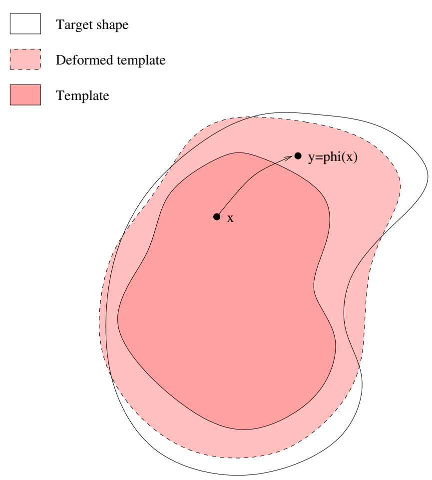

Fig. 1.1. Comparing deformed shapes

be defined as an energy minimization problem

[4] Obviously, the mapping $\gamma \to S_\gamma$ is not one to one since $S_\gamma = S_{\gamma'}$ as soon as $\gamma' = \gamma \circ \zeta$ and ζ is a parameter change.

$$\varphi_* = \text{argmin}_{\varphi \in G} R(\varphi) + g(\varphi(S_{temp}), S_{targ}) \tag{1.2}$$

where R is a regularization term penalizing unlikely deformations and g is the data term penalizing bad approximations of the target S_{targ}. In the framework of large deformations, the group G of admissible deformations is equipped with a right invariant metric distance d_G and the regularization term $R(\varphi)$ is designed as an increasing function of $d_G(\text{Id}, \varphi)$ where Id is the identity $(x \rightarrow x)$ mapping. One of the strengths of this diffeomorphic approach, which introduces a global deformation of the ambient space, is that it allows to model large deformations between shapes while preserving their natural non overlapping constraint. This is very hard to ensure with boundary-based methods, which match the boundaries of the region based on their geometric properties without involving their situation in the ambient space. Then, singularities may occur when, for example, two points which are far from each other for the arc length distance on the boundary are close for the Euclidian distance in the ambient space (cf. fig 1.2).

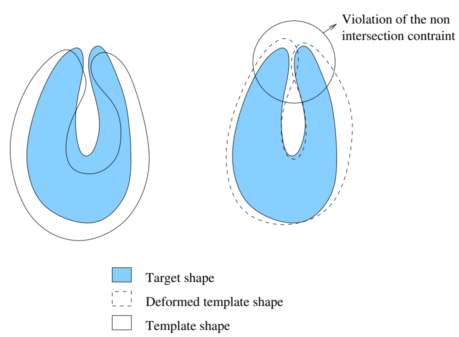

Fig. 1.2. Violation of the non overlapping constraint for usual curve based approaches

Another issue in the context of large deformations is that smoothness constraints acting only on the displacement fields (point displacements from the initial configuration to the deformed one) cannot guarantee the invertibility of the induced mapping, creating, for instance, loops along the boundary. Even if there may be ad hoc solutions to fix this (like penalties on the Jacobian, [5]), we argue that considering the deformation itself φ as the variable instead of linearizing with respect to the displacement field $u = \text{Id} - \varphi$ leads to a more natural geometrical framework. There is a high overhead in such an approach, since such φs live in an infinite dimensional manifold whereas the displacement fields belongs in a more amenable vector space. However, this turns out to be manageable, if one chooses a computational definition of diffeomorphisms in G as the solutions at time 1 of flow equations

$$\frac{\partial \varphi_t}{\partial t} = u_t \circ \varphi_t, \ \varphi_0 = \mathrm{id} \tag{1.3}$$

where at each time t, u_t belongs to a vector space of vector fields on the ambient space. To be slighly more precise, assume that the ambient space is a bounded open domain with smooth boundary $\Omega \subset \mathbb{R}^2$ and that V is a Hilbert space of vector fields continously embedded in $C_0^p(\Omega, \mathbb{R}^2)$ with $p \geq 1$ (the set of C^p vector fields on mR^2 which vanish outside Ω). Then, a unique solution of such flows exists for $t \in [0,1]$ for any time-dependent vector field $u \in L^2([0,1], V)$ ([6]) and we can define for any $t \in [0,1]$, the flow mapping

$$u \to \varphi_t^u, \ u \in L^2([0,1], V). \tag{1.4}$$

We finally define G as

$$G = \{ \ \varphi_1^u \mid u \in L^2([0,1], V) \ \}, \tag{1.5}$$

which is a subgroup of the C^1 diffeomorphisms on Ω (they all coincide with the identity on $\partial\Omega$, because of the boundary condition that has been imposed on V). In the following, we will use the notation $H_1 = L^2([0,1], V)$. This is the basic Hilbert space on which the optimization is performed: any problem involving a diffeomorphism in G as its variable can be formulated as a problem over H_1 through the onto map $u \mapsto \varphi_1^u$. In our setting, the regularization term $R(\varphi)$ is taken as a squared geodesic distance between φ and id on G, this distance being defined by

$$d_G(\varphi, \varphi')^2 = \inf \left\{ \int_0^1 |u_t|_V^2 dt \mid u \in H_1, \ \varphi_1^u \circ \varphi = \varphi' \right\}. \tag{1.6}$$

The variational problem (1.2) becomes

$$u_* = \mathrm{argmin}_{u \in H_1} \left(\int_0^1 |u_t|_V^2 dt + g(\varphi_1^u(S_{temp}), S_{targ}) \right) \tag{1.7}$$

Note that the reformulation of the problem from an infinite dimensional manifold to a Hilbert space comes at the cost of adding a new (time) dimension. One can certainly be concerned by the fact that the initial problem which was essentially matching 1D shape outlines, has become a problem formulated in term of time-dependent vector fields on Ω. However, this expansion from 1D to 3D is only apparent. An optimal solution $u_* \in H_1$ minimizes the kinetic energy $\int_0^1 |u_t|_V^2 dt$ over the set of $\{u \in H_1 : \varphi_1^{u_*} = \varphi_1^u\}$ (for such u, the data term stays unchanged). This means that $t \to \varphi_t^{u_*}$ is a geodesic path from id to $\varphi_1^{u_*}$, so that $t \to u_{*,t}$ satisfies an evolution equation which allows for the whole trajectory and the final $\varphi_* = \varphi_1^{u_*}$ to be reconstructed from inital data $u_{*,0} \in V$. Moreover, the main results in this paper show that this initial data can in turn be put into a form $u_{*,0} = Kp_{*,0}$ where K is a known kernel operator and $p_{*,0}$ is a bounded normal vector field on the boundary of S_{temp}, therefore reducing the problem to its initial dimension.

Let us summarize this discussion: *comparing shapes via a region based approach and global action of non rigid deformations of the ambient space is natural for modelling deformations of non-rigid objects. The estimation of large deformations challenges the usual linearized approaches in terms of dense* displacement *fields. The large deformation approach via the* φ *variable, more natural but apparently more complex has in fact potentially the same coding complexity: a normal vector field* $p_{*,0}$ *on the* $\partial S_{\mathrm{temp}}$ *from which the optimal* φ_* *and thus the deformed template shape* $\varphi_*(S_a)$*, can be reconstructed.*

1.3 Optimal matching and geodesic shooting for shapes

1.3.1 Hypotheses on the compared shapes

The compared shapes S_{temp} and S_{targ} are assumed to correspond to the following class of Jordan shapes. We let $\mathbb{T}$ be the unit 1D torus $\mathbb{T} = [0,1]_{\{0=1\}}$.

Definition 1 (Jordan Shapes). *Let* $k \geq 1$ *be a positive integer.*

1. *We say that* γ *is a non stopping piecewise* C^k *Jordan curve in* Ω *if* $\gamma \in C(\mathbb{T}, \Omega)$*,* γ *has no self-intersections and there exists a subdivision* $0 = s_0 < s_1 < \cdots < s_n = 1$ *of* $\mathbb{T}$ *such that the restriction* $\gamma_{|[s_i,s_{i+1}]}$ *is in* $C^k([s_i, s_{i+1}], \mathbb{R}^2)$ *on each interval and* $\gamma'(s) \neq 0$ *for any* $s_i < s < s_{i+1}$*. Such a subdivision will be called an admissible subdivision for* γ*. We denote* $\mathcal{C}^k_\flat(\Omega)$*, the set of non stopping piecewise* C^k *Jordan curves in* Ω*.*
2. *Let* $\mathcal{S}^k(\Omega)$ *be the set of all subset* S_γ *where* S_γ *is the interior (the unique bounded connected component of* $\mathbb{R}^2 \setminus \gamma(\mathbb{T})$*) of* $\gamma \in \mathcal{C}^k_\flat(\Omega)$*.*

Introducing a parametrization γ of the boundary of a Jordan shape S ($S = S_\gamma$), and considering the action of φ on curves [5] defined by

$$\gamma_{def} = \varphi \circ \gamma \tag{1.8}$$

we get

$$\varphi(S_\gamma) = S_{\varphi\circ\gamma}\,, \tag{1.9}$$

so that we can work as well with the curve representation of the boundary of a shape. A variational problem on Jordan shapes can be translated to a variational problem on Jordan curves thanks to the $\gamma \to S_\gamma$ mapping. Conversly, if $g_c(\gamma)$ is a driving matching term in a variational problem on Jordan curves, this term reduces to a driving matching term in a variational problem on Jordan shapes if

$$g_c(\gamma) = g_c(\gamma \circ \zeta) \tag{1.10}$$

for any C^∞ diffeomorphic change of variable $\zeta : \mathbb{T} \to \mathbb{T}$. Such a driving matching term g_c will be called *a geometric driving matching term.*

[5] We check immediately that $\varphi' \circ (\varphi \circ \gamma) = (\varphi' \circ \varphi) \circ \gamma$ so that we have an action.

1.3.2 Momentum theorem for differentiable driving matching term

We first study the case of a differentiable g_c, in the following sense:

Definition 2. *1. Let $(\gamma_n)_{n\geq 0}$ be a sequence in $\mathcal{C}^k_\flat(\Omega)$. We say that $\gamma_n \overset{\mathcal{C}^k_\flat(\Omega)}{\longrightarrow} \gamma_\infty$ if there exists a common admissible subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ of $\mathbb{T}$ for all the γ_n, $n \in \mathbb{N}\cup\{+\infty\}$ such that for any $j \leq k$*

$$\sup_{i,s\in[s_i,s_{i+1}]} |\frac{d^j}{ds^j}(\gamma_n - \gamma_\infty)| \to 0\,.$$

2. We say that $\Gamma : \mathbb{T}\times]-\eta,\eta[$ is a smooth perturbation of γ in $\mathcal{C}^k_\flat(\Omega)$ if
   a) $\Gamma(s,0) = \gamma(s)$, for any $s \in \mathbb{T}$,
   b) $\Gamma(.,\epsilon) \in \mathcal{C}^k_\flat(\Omega)$, for any $|\epsilon| < \eta$,
   c) there exists an admissible subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ of γ such that for any $0 \leq i < n$, $\Gamma_{|[s_i,s_{i+1}]\times]-\eta,\eta[} \in C^{k,1}([s_i,s_{i+1}]\times]-\eta,\eta[,\mathbb{R}^2)$.

3. Let $g_c : \mathcal{C}^k_\flat(\Omega) \to \mathbb{R}$ and $\gamma \in \mathcal{C}^k_\flat(\Omega)$. We say that g_c is Γ-differentiable (in $L^2(\mathbb{T},\mathbb{R}^2)$) at γ if there exists $\partial g_c(\gamma) \in L^2(\mathbb{T},\mathbb{R}^2)$ such that for any smooth perturbation Γ in $\mathcal{C}^k_\flat(\Omega)$ of γ, $q(\epsilon) \doteq g_c(\Gamma(.,\epsilon))$ has a derivative at $\epsilon = 0$ defined by $q'(0) = \int_\mathbb{T}\langle \partial g_c(\gamma)(s), (\partial\Gamma/\partial\epsilon)(s,0)\rangle ds$.

Our goal in this section is to prove the following:

Theorem 1. *Let $p \geq k \geq 0$ and assume that V is compactly embedded in $C_0^{p+1}(\Omega,\mathbb{R})$ and let $g_c : \mathcal{C}^k_\flat(\Omega) \to \mathbb{R}$ be lower semi-continuous on $\mathcal{C}^k_\flat(\Omega)$ ie $\liminf g_c(\gamma_n) \geq g_c(\gamma)$ for any sequence $\gamma_n \overset{\mathcal{C}^k_\flat(\Omega)}{\longrightarrow} \gamma$.*

1. Let $H_1 = L^2([0,1],V)$. There exists $u_* \in H_1$ such that $J(u_*) = \min_{u\in H_1} J(u)$ where

$$J(u) = \int_0^1 |u_t|_V^2 dt + \lambda g_c(\varphi_1^u \circ \gamma_{\text{temp}})\,.$$

2. Assume that g_c is Γ-differentiable in $\mathcal{C}^k_\flat(\Omega)$ at $\gamma_* = \varphi_1^{u_*}\circ\gamma_{\text{temp}}$. Then, the solution u_* is in fact in $C^1([0,1],V)$ and there exists $(\gamma_t, p_t) \in \mathcal{C}^k_\flat(\Omega)\times L^2(\mathbb{T},\mathbb{R}^2)$ such that
   a) $\gamma_0 = \gamma_{\text{temp}}$, $p_1 = -\lambda\partial g_c(\gamma_*)$ and for any $t \in [0,1]$

$$u_{*,t}(m) = \int_\mathbb{T} K(m,\gamma_t(s))p_t(s)ds,\ \gamma_t = \varphi_t^{u_*}\circ\gamma_{\text{temp}}$$
$$\text{and } p_t = (d\varphi_{t,1}^{u_*}(\gamma_t))^*(p_1)$$

   where $\varphi_{t',t}^u = \varphi_t^u\circ(\varphi_{t'}^u)^{-1}$ and K is the reproducing kernel [6] [7] associated with V.

[6] We have use the notation $d\varphi_{t',t}^u(x)$ for the differential at x and $(d\varphi_{t',t}^u(x))^*$ for the adjoint of $d\varphi_{t',t}^u(x)$.

[7] $K : \Omega\times\Omega \to \mathcal{M}_2(\mathbb{R})$ (the set of 2 by 2 matrices) is defined by $\langle K(.,x)a, v\rangle_V = \langle a, v(x)\rangle_{\mathbb{R}^2}$ for $(a,v) \in \mathbb{R}^2\times V$ and its existence and uniqueness are guaranteed by Riesz's theorem on continuous linear forms in a Hilbert space.

b) γ_t *and* p_t *are solutions in* $C^1([0,1], L^2(\mathbb{T}, \mathbb{R}^2))$ *of*

$$\begin{cases} \dfrac{\partial \gamma}{\partial t} = \dfrac{\partial}{\partial p} H(\gamma, p) \\ \\ \dfrac{\partial p}{\partial t} = -\dfrac{\partial}{\partial \gamma} H(\gamma, p) \end{cases} \tag{1.11}$$

where $H(\gamma, p) = \dfrac{1}{2} \int {}^t p(y) K(\gamma(y), \gamma(x)) p(x) dx dy.$

Moreover, if $k \geq 1$ *and* g_c *is geometric, then for any* $t \in [0,1]$*, the momentum* p_t *is normal to* γ_t *i.e* $\langle p_t(s), (\partial \gamma_1 / \partial s)(s) \rangle = 0$ *a.e.*

Remark 1. Not surprisingly, H can be interpreted as the reduced hamiltonian associated with the following control problem on $L^2(\mathbb{T}, \mathbb{R})$, with control variable $u \in V$:

$$\begin{cases} \dot{\gamma} = f(\gamma, u) \\ \\ \dot{\gamma}^0 = f^0(\gamma, u) \end{cases}$$

where $f(\gamma, u) = u(\gamma(.))$ and $f^0(\gamma, u) = \frac{1}{2} |u|_V^2$.

1.3.3 Proof

We give in this section a proof of Theorem 1. Let us recall a regularity result we borrow from [18] (lemma 11). If V is compactly embedded in $C_0^{p+1}(\Omega, \mathbb{R}^2)$, then for any $u, h \in H_1$, $\Phi : \Omega \times [-\eta, \eta] \to \mathbb{R}^2$ defined by $\Phi(x, \epsilon) = \varphi_1^{u+\epsilon h}(x)$ is a map in $C^{p,1}(\overline{\Omega} \times [-\eta, \eta], \mathbb{R}^2)$. From it, we deduce easily for $u = u_*$ and $h \in H_1$ that $\Gamma(s, \epsilon) \doteq \Phi(\gamma_{temp}(s), \epsilon)$ is a smooth perturbation of γ_{temp} in $\mathcal{C}_\flat^k(\Omega)$.

Let us denote $\gamma_0 = \gamma_{temp}$. The first step is the decomposition of J as $G \circ F$ where $F : H_1 \to M$ with $H_1 = L^2([0,1], V)$, $M = \mathbb{R} \times \mathcal{C}_\flat^k(\Omega)$,

$$F(u) = \left(\frac{1}{2} \int_0^1 |u_t|_V^2 dt, \gamma_1^u \right) \text{ where } \gamma_t^u = \varphi_t^u \circ \gamma_0 \tag{1.12}$$

and $G : M \to \mathbb{R}$ is given by

$$G(x, \gamma) = x + \lambda g_c(\gamma) \tag{1.13}$$

so that

$$J(u) = G \circ F(u) = \frac{1}{2} \int_0^1 |u_t|_V^2 dt + \lambda g_c(\gamma)\,. \tag{1.14}$$

With this decomposition, we emphasize with F that we have an underlying curve evolution structure and G appears has a terminal cost in a optimal control point of view [20].

Point (1) of Theorem 1 follows from the strong continuity of the mapping $u \to \varphi_1^u$ for the weak convergence in H_1 [18] (Theorem 9): if $u_n \rightharpoonup u$ in H_1, then $\varphi_1^{u_n} \to \varphi_1^u$ in $C^p(\overline{\Omega}, \mathbb{R}^2)$ so that $\gamma_n \overset{\mathcal{C}_\flat^k(\Omega)}{\longrightarrow} \gamma$ where $\gamma_n = \varphi_1^{u_n} \circ \gamma_0$ and $\gamma = \varphi_1^u \circ \gamma_0$. Using the lower semicontinuity property of g_c for the convergence in $\mathcal{C}_\flat^k(\Omega)$ and the lower semi-continuity of $\frac{1}{2}\int_0^1 |u_t|_V^2 dt$ for weak convergence in H_1, we deduce that J is lower semi-continuous for the weak convergence in H_1. Thus, the existence of u_* comes then from standard compactness argument of the strong balls in H_1 for the weak topology.

Point (2) of Theorem 1: For any $h \in H_1$, F admits a Gâteaux derivative in $H_2 = \mathbb{R} \times L^2(\mathbb{T}, \mathbb{R}^2)$ in the direction h , denoted $\partial F(u)(h)$, and given by (cf. [18], lemma 10)

$$\partial F(u)(h) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(u+\epsilon h) - F(u)) = \left(\int_0^1 \langle u_t, h_t \rangle dt, v^h \circ \gamma_1^u \right) \qquad (1.15)$$

where $\gamma_1^u = \varphi_1^u \circ \gamma_0$ and

$$v^h = \int_0^1 d\varphi_{t,1}^u(\varphi_{1,t}^u) h_t \circ \varphi_{1,t}^u dt \,. \qquad (1.16)$$

Considering $u = u_*$, $\eta > 0$, $|\epsilon| < \eta$ and $\Gamma(s,\epsilon) = \gamma_1^{u_*+\epsilon h}(s)$, Γ is a smooth perturbation of $\gamma_* = \gamma_1^{u_*}$ so that if $Q(\epsilon) = J(u_* + \epsilon h) = \frac{1}{2}\int_0^1 |u_{*,t} + \epsilon h_t|_V^2 dt + \lambda q(\epsilon)$, we get

$$\begin{aligned} Q'(0) &= \int_0^1 \langle u_{*,t}, h_t \rangle_V dt + \lambda \int_{\mathbb{T}} \left\langle \partial g_c(\gamma_*)(s), \frac{\partial \Gamma}{\partial \epsilon}(s,0) \right\rangle_{\mathbb{R}^2} ds \\ &= \int_0^1 \langle u_{*,t}, h_t \rangle_V dt + \lambda \int_{\mathbb{T}} \langle \partial g_c(\gamma_*)(s), v^h \circ \gamma_1^{u_*} \rangle_{\mathbb{R}^2} ds \end{aligned}$$

Using (1.16), we deduce that

$$\int_{\mathbb{T}} \langle \partial g_c(\gamma_*)(s), v^h \circ \gamma_1^{u_*} \rangle ds = \int_0^1 \int_{\mathbb{T}} \langle (d\varphi_{t,1}^{u_*}(\gamma_t))^*(\partial g_c(\gamma_*)(s)), h_t(\gamma_t^{u_*}(s) \rangle ds dt \,.$$

Hence, introducing $p_t(s) = -\lambda (d\varphi_{t,1}^{u_*}(\gamma_t))^*(\partial g_c(\gamma_*)(s))$, we get

$$Q'(0) = \int_0^1 \left\langle u_{*,t} - \int_{\mathbb{T}} K(.,\gamma_t^{u_*}(s)) p_t(s) ds \,,\, h_t \right\rangle_V dt \,.$$

Since $J(u_*)$ is the minimum of J, $Q'(0) = 0$ for any $h \in H_1$ and we have

$$u_{*,t}(m) = \int_{\mathbb{T}} K(m, \gamma_t^{u_*}(s)) p_t(s) ds \,.$$

Since, $t \to \varphi_t^{u_*}$ (resp. $t \to d\varphi_t^{u_*}$) is a continuous path in $C^1(\overline{\Omega}, \overline{\Omega})$ (resp. in $C(\overline{\Omega}, \mathcal{M}_2(\mathbb{R}))$), as soon as V is compactly embedded in $C_0^1(\Omega, \mathbb{R}^2)$ [18], we deduce that $t \to \gamma_t^{u_*}$ is continuous in $C(\mathbb{T}, \Omega)$, $t \to p_t$ in $L^2(\mathbb{T}, \mathbb{R}^2)$ and $t \to u_{*,t}$ in V. Thus, (2a) is proved.

The part (2b) is straightforward: Let us denote $\gamma_t = \gamma_t^{u_*}$. We first check that

$$u_{*,t}(\gamma_t) = (\partial/\partial p)H(\gamma_t, p_t)$$

so that $\partial\gamma_t/\partial t(s) = u_{*,t}(\gamma_t(s)) = (\partial/\partial p)H(\gamma_t, p_t)$. Now, from

$$p_t(s) = -(d\varphi_{t,1}^{u_*}(\gamma_t))^*(\partial g_c(\gamma_*)(s)) = (d\varphi_{t,1}^{u_*}(\gamma_t))^*(p_1(s))$$

we get

$$\frac{\partial p_t}{\partial t}(s) = \frac{\partial}{\partial t}(d\varphi_{t,1}^{u_*}(\gamma_t))^*(p_1(s)) = -(du_t(\gamma_t(s)))^*(p_t(s)). \tag{1.17}$$

Since V is continuously embedded in $C_0^1(\Omega, \mathbb{R}^2)$, the kernel K is in $C_0^1(\Omega \times \Omega, M_2(\mathbb{R}))$ and

$$du_{*,t}(m) = \int_{\mathbb{T}} \partial_1 K(m, \gamma_t(s'))p_t(s')ds', m \in \Omega$$

so that[8]

$$(du_t(\gamma_t(s)))^*(p_t(s)) = \int_{\mathbb{T}} {}^t p_t(s)\partial_1 K(\gamma_t(s), \gamma_t(s'))p_t(s')ds' = \frac{\partial}{\partial \gamma}H(\gamma_t, p_t)$$

and this combined with (1.17) provides the required evolution of p.

Now, from the previous expression of $\partial\gamma_t/\partial t$ and $\partial p_t/\partial t$, one deduces easily that $t \to \gamma_t$ and $t \to p_t$ belongs to $C^1([0,1], L^2([0,1], \mathbb{R}^2))$.

The last thing to be proved is the normality of the momentum for geometric driving matching terms. Indeed, let $\alpha \in C^\infty(\mathbb{T}, \mathbb{R})$ such that $\alpha(s_i) = 0$ for any $0 \le i < n$ where $0 = s_0 < \cdots < s_n = 1$ is an admissible subdivision for γ_*. Let $\zeta(s, \epsilon)$ be the flow defined for any $s \in \mathbb{T}$ by $\zeta(s, 0) = s$ and

$$\frac{\partial}{\partial \epsilon}\zeta(s, \epsilon) = \alpha(\zeta(s, \epsilon))\,.$$

Obviously the flow is defined for $\epsilon \in \mathbb{R}$ and $\zeta \in C^\infty(\mathbb{T} \times \mathbb{R}, \mathbb{T})$ and satisfies $\zeta(s_i, \epsilon) = s_i$ for any $0 \le i \le n$ so that $\Gamma(s, \epsilon) = \gamma_*(\zeta(s, \epsilon))$ is a smooth perturbation in $\mathcal{C}_\flat^k(\Omega)$ of γ_*. Since g_c is geometric, $g_c(\Gamma(., \epsilon)) \equiv g_c(\gamma_*)$ so that

$$\int_{\mathbb{T}} \left\langle \partial g_c(\gamma_*)(s), \frac{\partial \Gamma}{\partial \epsilon}(s, 0) \right\rangle ds = \int_{\mathbb{T}} \langle \partial g_c(\gamma_*)(s), \frac{\partial}{\partial s}\gamma_*(s)\alpha(s)\rangle ds = 0$$

[8] Here and in the following, when α is a function of several variables, the notation $\partial_1\alpha$ refers to the partial derivative or differential with respect to the first variable. We will use this notation in particular when the variables in α are not identified with a specific letter, which makes notation like $\partial/\partial x$ ambiguous.

Considering all the possible choice for α, we deduce that $\langle \partial g_c(\gamma_*)(s), (\partial\gamma_*/\partial s)\rangle = 0$ a.e. so that $\langle p_1(s), \frac{\partial}{\partial s}\gamma_*(s)\rangle = 0$ a.e. Since $p_t(s) = (d\varphi^{u_*}_{t,1}(\gamma_t))^*(p_1(s))$, we get

$$\langle p_t(s), \frac{\partial}{\partial s}\gamma_t(s)\rangle = \langle p_1(s), d\varphi^{u_*}_{t,1}(\gamma_t)(\frac{\partial}{\partial s}\gamma_t(s))\rangle = \langle p_1(s), \frac{\partial}{\partial s}\gamma_*(s)\rangle$$

so that $\langle p_t(s), \frac{\partial}{\partial s}\gamma_t(s)\rangle = 0$ a.e.

1.4 Application to measure based matching

1.4.1 Measure matching

We present here a first application of Theorem (1) for shape matching. This is a particular case of a more general framework introduced in [8] for measure matching.

Let $\mathcal{M}_s(\Omega)$ be the set of signed measures on Ω and consider I, a Hilbert space of functions on Ω, such that I is continously embedded in $C_b(\Omega, \mathbb{R})$, the set of bounded continuous functions. Since $\mathcal{M}_s(\Omega)$ is the dual of $C_b(\Omega, \mathbb{R})$ and $I \overset{\text{cont.}}{\hookrightarrow} C_b(\Omega, \mathbb{R})$, we have $\mathcal{M}_s(\Omega) \overset{\text{cont.}}{\hookrightarrow} I^*$ where I^* is the dual of I. Define the action of diffeomorphisms on I^*, $(\varphi, \mu) \to \varphi.\mu$, by $(\varphi.\mu, f) = (\mu, f \circ \varphi)$, which, in the case when μ is a measure, yields

$$(\varphi.\mu, f) \doteq \int f d(\varphi.\mu) = \int f \circ \varphi \, d\mu, \ \forall f \in I \subset C_b(\Omega, \mathbb{R}).$$

The dual norm on I^* provides a nice way to compare two signed measures μ and ν:

$$|\mu|_{I^*} = \sup_{f \in I, |f|_I \leq 1} \int_\Omega f d\mu.$$

Introduce the reproducing kernel $(x, y) \mapsto k_I(x, y)$ on I, which is such that, for $f \in I$ and $x \in \Omega$,

$$f(x) = \langle f, k_I(x)\rangle_I$$

with $k_I(x) : y \mapsto k_I(x, y)$. We have

$$\langle \mu, \nu\rangle_{I^*} = \int_{\Omega\times\Omega} k_I(x, y) d\mu(x) d\nu(y). \tag{1.18}$$

Indeed,

$$\int_\Omega f(x) d\mu(x) = \int_\Omega \langle f, k_I(x)\rangle_I \, d\mu(x) = \left\langle f, \int_\Omega k_I(x, .) d\mu(x) \right\rangle_I$$

which is maximized for $f(y) = \frac{1}{C}\int_\Omega k_I(x, y) d\mu(x)$ with

$$C = \left| \int_\Omega k_I(., x) d\mu(x) \right|_I$$

so that $|\mu|_{I^*} = C$. Now, we have

$$C^2 = \int_\Omega \int_\Omega \langle k_I(x), k_I(y) \rangle_I d\mu(x) d\mu(y) = \int_\Omega \int_\Omega k_I(x, y) d\mu(x) d\mu(y)$$

from the properties of a reproducing kernel. This proves (1.18).

Coming back to the shape matching problem, for any curve $\gamma : \mathbb{T} \to \mathbb{R}^2$, we define $\mu_\gamma \in \mathcal{M}_s(\Omega)$ by

$$\int_\Omega f d\mu_\gamma = \int_\mathbb{T} f \circ \gamma(s) ds \,.$$

For example, when S is a Jordan shape and γ is a parametrization with constant speed, μ_γ is a uniform measure on ∂S (a probability measure if properly normalized). More generally, given a compact submanifold M of dimension k, one can associate with M the uniform probability measure denoted μ_M. This measure framework is useful also to represent finite union of submanifolds of different dimensions or more irregular structures (see [8]). Moreover, this allows various approximation schemes since for any reasonable sampling process over the manifold M, $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \to \mu_M$. We focus on the simple case of $2D$ shape modeling but instead of working with the approximation scheme $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ (by uniform sampling on the curve) we will work with a continuous representation as a 1D measure μ_γ where $S = S_\gamma$. We introduce as in [8] the following energy:

$$\begin{aligned} J(u) \doteq \frac{1}{2} \int_0^1 |u_t|_V^2 dt + \frac{\lambda}{2} |\varphi_1^u.\mu_{\partial S_{temp}} - \mu_{\partial S_{targ}}|_{I^*}^2 \\ = \frac{1}{2} \int_0^1 |u_t|_V^2 dt + \frac{\lambda}{2} |\varphi_1^u.\mu_{\gamma_{temp}} - \mu_{\gamma_{targ}}|_{I^*}^2 \end{aligned}$$

where γ_{temp} (resp. γ_{targ}) is a constant speed parametrization of S_{temp} (resp. S_{targ}).

Note that for any $f \in I$,

$$\int f d(\varphi.\mu_\gamma) = \int f \circ \varphi d\mu_\gamma = \int f \circ \varphi \circ \gamma ds = \int f d(\mu_{\varphi \circ \gamma})$$

so that, with

$$g_c(\gamma) = \frac{1}{2} |\mu_\gamma - \mu_{\gamma_{targ}}|_{I^*}^2 ,$$

minimizing J is a variational problem which is covered by Theorem 1. It is clear that g_c is not geometric since, in general, $\mu_{\gamma \circ \zeta} \neq \mu_\gamma$ for a change of variable $\zeta : \mathbb{T} \to \mathbb{T}$. However, this approach provides a powerful matching algorithm between unlabelled sets of points and submanifolds.

Let $p \geq k \geq 0$ and consider Γ a smooth perturbation of a curve $\gamma \in \mathcal{C}^k_\flat(\Omega)$. Then if $v(s) = (\partial\Gamma/\partial\epsilon)(s,0)$ and $q(\epsilon) = g_c(\Gamma(.,\epsilon))$ we get immediately

$$q'(0) = \int_{\mathbb{T}\times\mathbb{T}} \langle \partial_1 k_I(\gamma(s),\gamma(s')) - \partial_1 k_I(\gamma(s),\gamma_{targ}(s')), v(s)\rangle\, dsds'$$

giving

$$\partial g_c(\gamma)(s) = \int_{\mathbb{T}} (\partial_1 k_I(\gamma(s),\gamma(s')) - \partial_1 k_I(\gamma(s),\gamma_{targ}(s')))\, ds'$$

Theorem 1 can therefore be directly applied, yielding

Theorem 2. *Let $p \geq k \geq 0$ and assume that V is compactly embedded in $C_0^{p+1}(\Omega,\mathbb{R})$. Let I be a Hilbert space of real valued functions on Ω and assume that I is continuously embedded in $C^k(\overline{\Omega},\mathbb{R})$. Let $S_{\rm temp}$ and $S_{\rm targ}$ be two Jordan shapes in $\mathcal{S}^k(\Omega)$. Then the conclusions of Theorem 1 are true, with*

$$p_1(s) = -\lambda\partial g_c(\gamma_1)(s) = \int_{\mathbb{T}} (\partial_1 k_I(\gamma_1(s),\gamma_{\rm targ}(s')) - \partial_1 k_I(\gamma_1(s),\gamma_1(s')))\, ds'$$

From Theorem 1, we have $p_t = (d\varphi^{u_*}_{t,1}(\gamma_t))^*(p_1)$, and since $p \geq k$, inherits the smoothness properties of p_1. Now, if $0 \leq s_0 < \cdots < s_n = 1$ is an admissible partition of γ_{temp} (ie S_{temp} has a C^k boundary except at a finite number $\gamma_{temp}(s_0),\cdots,\gamma_{temp}(s_n)$ of possible “corners”) then p_1 is continuous and p_1 restricted to $[s_i,s_{i+1}]$ is C^k, and this conclusion is true also for all p_t.

1.4.2 Geometric measure-based matching

As said before, the previous formulation is not geometric and in particular, μ_{γ_*} is not generally the uniform measure on $S_* = \varphi_1^{u_*}(S_{temp})$ ie $\mu_{\gamma_*} \neq \mu_{S_*}$. If we want to consider a geometric action, we can propose a new data term, derived from the previous one, which is now fully geometric

$$g_c(\gamma) = \frac{1}{2}|\mu_{\partial S_\gamma} - \mu_{\partial S_{targ}}|^2_{I^*}$$

or equivalently

$$\begin{aligned} g_c(\gamma) = \frac{1}{2}\int_{\mathbb{T}\times\mathbb{T}} & k_I(\gamma(s),\gamma(r))\, |\gamma'(s)|\, |\gamma'(r)|\, dsdr \\ & + \frac{1}{2}\int_{\mathbb{T}\times\mathbb{T}} k_I(\gamma_{targ}(s),\gamma_{targ}(r))\, |\gamma'_{targ}(s)|\, |\gamma'_{targ}(r)|\, dsdr \\ & \qquad - \int_{\mathbb{T}\times\mathbb{T}} k_I(\gamma(s),\gamma_{targ}(r))\, |\gamma'(s)|\, |\gamma'_{targ}(r)|\, dsdr \end{aligned} \quad (1.19)$$

The main difference from the previous non geometric matching term is the introduction of speed of γ and γ_{targ} in the integrals (with the notation $\gamma'(s) = \partial\gamma(s)/\partial s$).

The derivative of $g_c(\gamma)$ under a smooth perturbation Γ of γ in $\mathcal{C}^k_\flat(\Omega)$ for $k \geq 2$ can be computed. Note first that for $\gamma \in \mathcal{C}^k_\flat(\Omega)$ and $k \geq 2$, we can define for any $s \in \mathbb{T} \setminus \{s_0, \cdots, s_n\}$ (where $0 = s_0 < \cdots < s_n = 1$ is an admissible subdivision of γ), the Frenet frame (τ_s, n_s) along the curve, and the curvature κ_s. In the following we will use the relations $\gamma'(s) = |\gamma'(s)|\tau_s$ and $\partial\tau_s/\partial s = \kappa_s|\gamma'(s)|n_s$. Let Γ be a smooth perturbation of γ in $\mathcal{C}^k_\flat(\Omega)$ for $k \geq 2$. As previously, we will denote $v(s) = (\partial\Gamma/\partial\varepsilon)(s, 0)$. Since Γ is C^1, we have $(\partial v/\partial s) = (\partial\gamma'/\partial\varepsilon)(s, \epsilon)|_{\epsilon=0}$. Then, if $q(\epsilon) = g_c(\Gamma(., \epsilon))$, assuming that $k_I \in C^1(\Omega \times \Omega, \mathbb{R})$,

$$
\begin{aligned}
q'(0) = \quad & \int_{\mathbb{T}\times\mathbb{T}} [\langle \partial_1 k_I(\gamma(s), \gamma(r)), v(s)\rangle \ |\gamma'(s)| \\
& \qquad + k_I(\gamma(s), \gamma(r)) \ \langle \tau_s, \partial v/\partial s\rangle] \ |\gamma'(r)| \, dsdr \\
- \quad & \int_{\mathbb{T}\times\mathbb{T}} [\langle \partial_1 k_I(\gamma(s), \gamma_{targ}(r)), v(s)\rangle \ |\gamma'(s)| \\
& \qquad + k_I(\gamma(s), \gamma_{targ}(r)) \ \langle \tau_s, \partial v/\partial s\rangle] \ |\gamma'_{targ}(r)| \, dsdr.
\end{aligned}
$$

Consider the term $\int_{\mathbb{T}} k_I(\gamma(s), \gamma(r)) \ \langle \tau_s, \partial v/\partial s\rangle \ ds$. Integrating by parts on each $[s_i, s_{i+1}]$ yields

$$
\begin{aligned}
&\int_{\mathbb{T}} k_I(\gamma(s), \gamma(r)) \ \langle \tau_s, \partial v/\partial s\rangle \ ds = \sum_{i=0}^{n} k_I(\gamma(s_i), \gamma(r)) \ \langle -\delta\tau_i, v(s_i)\rangle \\
&- \int_{\mathbb{T}} \langle [\langle \partial_1 k_I(\gamma(s), \gamma(r)), \tau(s)\rangle \ \tau_s + k_I(\gamma(s), \gamma(r)) \ \kappa_s n_s] \, , v(s)\rangle \ |\gamma'(s)| \, ds,
\end{aligned}
\tag{1.20}
$$

where $\delta\tau_i = \lim_{r\to 0} \tau_{s_i+r} - \lim_{r\to 0} \tau_{s_i-r}$ (note that v is always continuous). Since we have allowed corners in our model of shapes, the boundary terms of the integration do not vanish, and consequently g_c is not Γ-differentiable, unless we allow *singular terms* (Dirac measures) in the gradient, which is possible but will not be addressed here. In the case of smooth curves, the singular terms cancel and we have

Theorem 3. *Let $p \geq k \geq 2$ and assume $V \overset{comp.}{\hookrightarrow} C_0^{p+1}(\Omega, \mathbb{R})$ and $I \overset{cont.}{\hookrightarrow} C^k(\overline{\Omega}, \mathbb{R})$. Let S_{temp} and S_{targ} be two C^k Jordan shapes. Then, the conclusions of Theorem 1 are valid for*

$$
J(u) = \frac{1}{2}\int_0^1 |u_t|_V^2 dt + \frac{\lambda}{2}|\mu_{\partial S_\gamma} - \mu_{\partial S_{\mathrm{targ}}}|_{I^*}^2
$$

with

$$
\begin{aligned}
p_1(s) = -\lambda \Bigg[&\int_{\mathbb{T}} [\langle \partial_1 k_I(\gamma_1(s), \gamma_1(r)), n_s\rangle - k_I(\gamma_1(s), \gamma_1(r)) \ \kappa_s] \ |\gamma_1'(r)| \, dr \\
- &\int_{\mathbb{T}} [\langle \partial_1 k_I(\gamma_1(s), \gamma_{\mathrm{targ}}(r)), n_s\rangle - k_I(\gamma_1(s), \gamma_{\mathrm{targ}}(r)) \ \kappa_s] \ |\gamma'_{\mathrm{targ}}(r)| \, dr \Bigg] \ |\gamma_1'(s)| \, n_s.
\end{aligned}
\tag{1.21}
$$

More over, p_t is at all times normal to the boundary of γ_t.

The normality of p_t at all times is a consequence of Theorem 1, but can be seen directly from the fact that p_1 is normal to γ_1 and from the equations $p_t = (d\varphi^{u_*}_{t,1}(\gamma_t))^*(p_1)$ and $\gamma_t = \varphi_{1,t}(\gamma_1)$.

1.4.3 Geometric measure-based matching, second formulation

The following version of the driving term has a non singular gradient, at the difference of the previous one. Define

$$\begin{aligned} g_c(\gamma) = \frac{1}{2}\int_{\mathbb{T}\times\mathbb{T}} & k_I(\gamma(s),\gamma(r))\ \langle \gamma'(s),\gamma'(r)\rangle\ dsdr \\ &+ \frac{1}{2}\int_{\mathbb{T}\times\mathbb{T}} k_I(\gamma_{targ}(s),\gamma_{targ}(r))\ \left\langle \gamma'_{targ}(s),\gamma'_{targ}(r)\right\rangle\ dsdr \\ &\qquad - \int_{\mathbb{T}\times\mathbb{T}} k_I(\gamma(s),\gamma_{targ}(r))\ \left\langle \gamma'(s),\gamma'_{targ}(r)\right\rangle\ dsdr, \end{aligned} \tag{1.22}$$

i.e. we replace products of scalar velocities by dot products of vector velocities. This expression may be interpreted as follows: given a curve γ, one may define the vector-valued Borel measure $\overrightarrow{\mu}_\gamma$ such that for any continuous vector field $v : \Omega \to \mathbb{R}^2$,

$$\overrightarrow{\mu}_\gamma(v) = \int_{\mathbb{T}} \langle v(\gamma(s)),\gamma'(s)\rangle\ ds.$$

Now extend the $|\cdot|_I$ norm introduced in the preceding section to vector-valued maps $v = (v_x, v_y) : \Omega \to \mathbb{R}^2$ by defining $|v|_I = \sqrt{|v_x|_I^2 + |v_y|_I^2}$. One may check that the corresponding matrix-valued kernel is the scalar kernel $k_I(x,y)$ times the identity matrix. Consequently, formula (1.22) corresponds in this setting to the dual norm squared error $|\overrightarrow{\mu}_\gamma - \overrightarrow{\mu}_{\gamma_{targ}}|^2_{I^*}$.

Let Γ be a smooth perturbation of γ in $\mathcal{C}^k_\flat(\Omega)$ for $k \geq 1$, and denote $v(s) = (\partial\Gamma/\partial\varepsilon)(s,0)$ and $q(\epsilon) = g_c(\Gamma(.,\epsilon)$ as before. We have

$$\begin{aligned} q'(0) = & \int_{\mathbb{T}\times\mathbb{T}} [\langle \partial_1 k_I(\gamma(s),\gamma(r)), v(s)\rangle\ \langle \gamma'(s),\gamma'(r)\rangle \\ &\qquad + k_I(\gamma(s),\gamma(r))\ \langle \partial v/\partial s, \gamma'(r)\rangle]\ dsdr \\ - & \int_{\mathbb{T}\times\mathbb{T}} [\langle \partial_1 k_I(\gamma(s),\gamma_{targ}(r)), v(s)\rangle\ \left\langle \gamma'(s),\gamma'_{targ}(r)\right\rangle \\ &\qquad + k_I(\gamma(s),\gamma_{targ}(r))\ \left\langle \partial v/\partial s, \gamma'_{targ}(r)\right\rangle]\ dsdr \end{aligned}$$

Integrating by parts on each $[s_i, s_{i+1}]$ the second part of each integral,

$$\begin{aligned} q'(0) = & \int_{\mathbb{T}\times\mathbb{T}} [\langle \partial_1 k_I(\gamma(s),\gamma(r)), v(s)\rangle \ \langle \gamma'(s),\gamma'(r)\rangle \\ & \quad - \langle \partial_1 k_I(\gamma(s),\gamma(r)), \gamma'(s)\rangle \ \langle v(s), \gamma'(r)\rangle]\, dsdr \\ - & \int_{\mathbb{T}\times\mathbb{T}} [\langle \partial_1 k_I(\gamma(s),\gamma_{targ}(r)), v(s)\rangle \ \left\langle \gamma'(s),\gamma'_{targ}(r)\right\rangle \\ & \quad - \langle \partial_1 k_I(\gamma(s),\gamma_{targ}(r)), \gamma'(s)\rangle \ \left\langle v(s), \gamma'_{targ}(r)\right\rangle]\, dsdr. \end{aligned}$$

Hence in this case we get a Γ-derivative

$$\begin{aligned} \partial g_c(\gamma)(s) = & \int_{\mathbb{T}} [\langle \gamma'(s),\gamma'(r)\rangle \ \partial_1 k_I(\gamma(s),\gamma(r)) \\ & \quad - \langle \partial_1 k_I(\gamma(s),\gamma(r)), \gamma'(s)\rangle \ \gamma'(r)]\, dr \\ - & \int_{\mathbb{T}} [\left\langle \gamma'(s),\gamma'_{targ}(r)\right\rangle \ \partial_1 k_I(\gamma(s),\gamma_{targ}(r)) \\ & \quad - \langle \partial_1 k_I(\gamma(s),\gamma_{targ}(r)), \gamma'(s)\rangle \ \gamma'_{targ}(r)]\, dr. \end{aligned}$$

As expected, this can be rewritten to get an expression which is purely normal to the curve γ. Indeed,

$$\begin{aligned} \partial g_c(\gamma)(s) = & \left[\int_{\mathbb{T}} \langle n_r, \partial_1 k_I(\gamma(s),\gamma(r))\rangle \ |\gamma'(r)|\, dr \right. \\ & \left. - \int_{\mathbb{T}} \left\langle n_r^{targ}, \partial_1 k_I(\gamma(s),\gamma_{targ}(r))\right\rangle \ |\gamma'_{targ}(r)|\, dr \right] \ |\gamma'(s)|\, n_s. \end{aligned}$$

This implies

Theorem 4. *Let $p \geq k \geq 1$ and assume $V \overset{comp.}{\hookrightarrow} C_0^{p+1}(\Omega,\mathbb{R})$ and $I \overset{cont.}{\hookrightarrow} C^k(\overline{\Omega},\mathbb{R})$. Let S_{temp} and S_{targ} be two Jordan shapes in $\mathcal{S}^k(\Omega)$. Then the conclusions of Theorem 1 hold for*

$$J(u) = \frac{1}{2}\int_0^1 |u_t|_V^2 dt + \frac{\lambda}{2} |\overrightarrow{\mu}_{\varphi_1^u \circ \gamma_{\mathrm{temp}}} - \overrightarrow{\mu}_{\gamma_{\mathrm{targ}}}|_{I^*}^2$$

with

$$\begin{aligned} p_1(s) = -\lambda & \left[\int_{\mathbb{T}} \langle n_r, \partial_1 k_I(\gamma_1(s),\gamma_1(r))\rangle \ |\gamma_1'(r)|\, dr \right. \\ & \left. - \int_{\mathbb{T}} \left\langle n_r^{targ}, \partial_1 k_I(\gamma_1(s),\gamma_{\mathrm{targ}}(r))\right\rangle \ |\gamma'_{\mathrm{targ}}(r)|\, dr \right] \ |\gamma_1'(s)|\, n_s. \quad (1.23) \end{aligned}$$

More over, p_t is at all times normal to the boundary of γ_t, continuous and C^{k-1} on any interval on which γ_{temp} is C^k.

1.5 Application to shape matching via binary images

1.5.1 Shape matching via binary images

Another natural way to build a geometric driving matching term is to consider, for any shape S, the binary image χ_S such that $\chi_S(m) = 1$ if $m \in S$ and 0 otherwise. Then the usual L^2 matching term between images ($\int_\Omega (I_{temp} \circ \varphi^{-1} - I_{targ})^2 dm$) leads to the area of the set symmetric difference $\int_\Omega |\chi_{\varphi(S_{temp})} - \chi_{S_{targ}}| dm$. Introducing

$$g_c(\gamma) = \int_\Omega |\chi_{S_\gamma} - \chi_{S_{targ}}| dm$$

we get an obviously geometric driving matching term leading to the definition of

$$J(u) = \int_0^1 |u_t|_V^2 dt + \lambda \int_\Omega |\chi_{S_{\gamma_1^u}} - \chi_{S_{targ}}| dm \, .$$

where $\gamma_1^u = \varphi_1^u \circ \gamma_{temp}$. The problem of diffeomorphic image matching has been quite studied in the case of sufficiently smooth images in ([12], [18], [1]). It has been proved that the momentum, p_0, is a function defined on Ω of the form $p_0 = \alpha \nabla I_{temp}$, where $\alpha = |d\varphi_{0,1}^{u_*}|(I_{temp} - I_{targ} \circ \varphi_{0,1}^{u_*}) \in L^2(\Omega, \mathbb{R})$. This particular expression $\alpha \nabla I_{temp}$ shows that the momentum is normal to the level sets of the template image, and vanishes on regions over which I_{temp} is constant. (This property is conserved over time for the deformed images I_t. This is what we called the normal momentum constraint [13].) In the case of binary images, we lose the smoothness property since ∇I_{temp} is singular and much less was known except that the momentum is a distribution whose support is concentrated on the boundary of S_{temp}. We show in this section that this distribution is as simple as it can be, and is essentially an L^2 function on the boundary of the template, or using a parametrization (and with a slight abuse of notation), an element of $p_0 \in L^2(\mathbb{T}, \mathbb{R}^2)$ which is everywhere normal to the boundary.

The main idea is to proceed like in Theorem 1, but we here have to deal with the fact that g_c *is not Γ-differentiable* in $\mathcal{C}_\flat^k(\Omega)$ (it is still lower semi-continuous for $k \geq 1$). We need to introduce for this the weaker notion of Γ-semi-differentiability and a proper extension of Theorem 1.

1.5.2 Momentum Theorem for semi-differentiable driving matching term

We start with the definition of the Γ-semi-differentiability.

Definition 3. *Let $g_c : \mathcal{C}_\flat^k(\Omega) \to \mathbb{R}$ and $\gamma \in \mathcal{C}_\flat^k(\Omega)$. We say that g_c is Γ-semi differentiable at γ if for any smooth perturbation Γ in $\mathcal{C}_\flat^k(\Omega)$ of γ, $q(\epsilon) \doteq g_c(\Gamma(.,\epsilon))$ has left and right derivatives at $\epsilon = 0$. We say that g_c has Γ-semi-derivatives upper bounded by B if B is a bounded subset of $L^2(\mathbb{T}, \mathbb{R}^2)$ such* *that for any smooth perturbation Γ in $\mathcal{C}^k_\flat(\Omega)$ of γ, there exists $b \in B$ such that*

$$\partial^+ q(0) \le \int_{\mathbb{T}} \langle b(s), (\partial\Gamma/\partial\epsilon)(s,0)\rangle ds$$

where $\partial^+ q(0)$ denotes the right derivative of q at 0.

Under this weaker condition, we can prove the following extension of Theorem 1:

Theorem 5. *Let $p \ge k \ge 0$ and assume that V is compactly embedded in $C^{p+1}_0(\Omega,\mathbb{R})$ and let $g_c : \mathcal{C}^k_\flat(\Omega) \to \mathbb{R}$ be lower semi-continuous on $\mathcal{C}^k_\flat(\Omega)$ ie $\liminf g_c(\gamma_n) \ge g_c(\gamma)$ for any sequence $\gamma_n \xrightarrow{\mathcal{C}^k_\flat(\Omega)} \gamma$.*

1. *Let $H_1 = L^2([0,1],V)$. There exists $u_* \in H_1$ such that $J(u_*) = \min_{u\in H_1} J(u)$ where*
$$J(u) = \int_0^1 |u_t|^2_V dt + \lambda g_c(\varphi^u_1 \circ \gamma_{\mathrm{temp}})\,.$$
2. *Assume that g_c is Γ-semi-differentiable in $\mathcal{C}^k_\flat(\Omega)$ at $\gamma_* = \varphi^{u_*}_1 \circ \gamma_{\mathrm{temp}}$ with Γ-semi-derivative upper bounded by $B \subset L^2(\mathbb{T},\mathbb{R}^2)$. Then, the solution u_* is in fact in $C^1([0,1],V)$ and there exist $(\gamma_t, p_t) \in \mathcal{C}^k_\flat(\Omega) \times L^2(\mathbb{T},\mathbb{R}^2)$ such that*
   a) *$\gamma_0 = \gamma_{\mathrm{temp}}$, $p_1 = -\lambda b$ with $b \in \overline{conv(B)}$ and for any $t \in [0,1]$*
   $$u_{*,t}(m) = \int_{\mathbb{T}} K(m,\gamma_t(s))p_t(s)ds,\ \gamma_t = \varphi^{u_*}_t \circ \gamma_{\mathrm{temp}} \text{ and } p_t = (d\varphi^{u_*}_{t,1}(\gamma_t))^*(p_1)$$
   where $\varphi^u_{s,t} = \varphi^u_t \circ (\varphi^u_s)^{-1}$ and K is the reproducing kernel associated with V.
   b) *γ_t and p_t are solutions in $C^1([0,1],L^2(\mathbb{T},\mathbb{R}^2))$ of*
   $$\begin{cases} \dfrac{\partial\gamma}{\partial t} = \dfrac{\partial}{\partial p} H(\gamma,p) \\ \dfrac{\partial p}{\partial t} = -\dfrac{\partial}{\partial \gamma} H(\gamma,p) \end{cases} \tag{1.24}$$
   where $H(\gamma,p) = \frac{1}{2}\int {}^t p(y)K(\gamma(y),\gamma(x))p(x)dxdy$.

Proof. The proof of the Theorem 5 follows closely the lines of the proof of Theorem 1. In particular, introduce F and G as in equations (1.12) and (1.13), and for $u \in H_1$, consider $\partial F(u)$ defined by (1.15) and (1.16). We focus on the proof of point (2), since point (1) does not differ from Theorem 1. Let $h \in H_1$, $\eta > 0$, $|\epsilon| < \eta$ and $\Gamma(s,\epsilon) = \gamma^{u_*+\epsilon h}_1(s)$ where $\gamma^u_t = \varphi^u_t \circ \gamma_{temp}$. The mapping Γ is a smooth perturbation of $\gamma_* = \gamma^{u_*}_1$ in $\mathcal{C}^k_\flat(\Omega)$ and if $Q(\epsilon) = J(u_* + \epsilon h) = \frac{1}{2}\int_0^1 |u_{*,t} + \epsilon h_t|^2_V dt + \lambda q(\epsilon)$ where $q(\epsilon) \doteq g_c(\Gamma(.,\epsilon))$, we deduce from the hypothesis that there exists $b \in B$ such that

$$\partial^+ Q(0) \leq \int_0^1 \langle u_t, h_t \rangle dt + \int_{\mathbb{T}} \langle b(s), (\partial \Gamma/\partial \epsilon)(s,0)\rangle ds = \langle (\partial F(u_*)h, \overline{b}\rangle_{H_2}$$

where $H_2 = \mathbb{R} \times L^2(\mathbb{T}, \mathbb{R}^2)$ and $\overline{b} = (1, b)$. We need now the following lemma:

Lemma 1. *Let $F : H_1 \to M$ and $G : M \to \mathbb{R} \cup \{+\infty\}$ be two mappings where H_1 is a separable Hilbert space. Let us assume the following:*

(H1) There exists $u_* \in H_1$ such that

$$G \circ F(u_*) = \inf_{u \in H_1} G \circ F(u) < +\infty.$$

(H2) For any $h \in H_1$, the function $\rho_h(\varepsilon) = G \circ F(u_* + \varepsilon h)$ has left and right derivatives at 0 and the following holds for a separable Hilbert space H_2 and a bounded subset D of H_2: there exists a linear mapping $\partial F(u_*) : H_1 \to H_2$ such that, for any $h \in H_1$, there exists $\overline{b} \in D$ with

$$\partial^+ \rho_h(0) \leq \langle \overline{b}, \partial F(u_*)h \rangle. \tag{1.25}$$

Then, there exists $\overline{b}_* \in \overline{conv(D)}$, the closure in H_2 of the convex hull of D, such that for any $h \in H_1$

$$\langle \overline{b}_*, \partial F(u_*)h \rangle = 0\,. \tag{1.26}$$

Proof. Let $\tilde{E}$ be the closure in H_2 of the linear space $\partial F(u_*)(H_1)$ and π the orthogonal projection on $\tilde{E}$. Now, let $C = \overline{\mathrm{conv}(D)}$. From (H2), we get that C is a non-empty bounded closed convex subset of H_2 so that we deduce from corollary III.19 in [2] that C is weakly compact. Now, π is continuous for the weak topology so that $\tilde{C} = \pi(C)$ is weakly compact and thus strongly closed. From the projection Theorem on closed non-empty convex subsets of an Hilbert space (Theorem V2 in [2]), we deduce that there exist $\tilde{b}_* \in \tilde{C}$ such that $|\tilde{b}_*| = \inf_{\tilde{b} \in \tilde{C}} |\tilde{b}|$ and $\langle \tilde{b}_*, \tilde{b} - \tilde{b}_* \rangle \geq 0$ for any $\tilde{b} \in \tilde{C}$. Considering $\overline{b}_* \in C$ such that $\pi(\overline{b}_*) = \tilde{b}_*$ we deduce eventually that for any $\overline{b} \in C$,

$$|\tilde{b}_*|^2 = \langle \tilde{b}_*, \overline{b}_* \rangle \leq \langle \tilde{b}_*, \overline{b} \rangle\,. \tag{1.27}$$

Assume that $\tilde{b}_* \neq 0$, and let $h \in H_1$ such that $|\tilde{b}_* + \partial F(u_*)h| \leq |\tilde{b}_*|^2/2M$ where $\sup_{\overline{b} \in C} |\overline{b}| \leq M < \infty$. From $(H2)$, there exists $\overline{b} \in C$ such that

$$\partial_0^+ \rho_h \leq \langle \overline{b}, \partial F(u_*)h \rangle \leq (|\tilde{b}_*|^2/2 - \langle \overline{b}, \tilde{b}_* \rangle)$$

so that using (1.27), we get

$$\partial_0^+ \rho_h \leq -|\tilde{b}_*|^2/2 < 0$$

which is in contradiction with (H1).

Hence $\tilde{b}_* = 0$ and $\overline{b}_*$ is orthogonal to $\tilde{E}$ which gives the result.

Using the lemma, we deduce that there exists $b \in B$ such that for any $h \in H_1$,

$$\int_0^1 \langle u_{*,t}, h_t \rangle_V dt + \lambda \int_{\mathbb{T}} \langle b(s), v^h(\gamma(s)) \rangle_{\mathbb{R}^2} ds = 0$$

where

$$v^h = \int d\varphi^{u_*}_{t,1}(\varphi^{u_*}_{1,t}) h_t \circ \varphi^{u_*}_{1,t} dt \,.$$

Denoting $p_t(s) = -\lambda (d\varphi^{u_*}_{t,1}(\gamma_t(s)))^*(b(s))$, we get eventually for any $h \in H_1$

$$\int_0^1 \langle u_{*,t} - \int_{\mathbb{T}} K(., \gamma^{u_*}_t(s)) p_t(s) ds, h_t \rangle_V dt = 0 \,.$$

so that

$$u_{*,t}(m) = \int_{\mathbb{T}} K(m, \gamma^{u_*}_t(s)) p_t(s) ds \,. \tag{1.28}$$

Given this representation of $u_{*,t}$ the remaining of the proof of Theorem 5 is identical to Theorem 1.

1.5.3 Momentum description for shape matching via binary images

Coming back to the case of the driving matching term g_c defined by

$$g_c(\gamma) = \int_{\Omega} |\chi_{S_\gamma} - \chi_{S_{targ}}| dm \,,$$

the Γ-semi-differentiability is given in the following proposition. For a shape S in $\mathcal{S}^k(\Omega)$, denote by d_S the function equal to -1 within S and to 1 outside.

Proposition 1. *Let $p \geq k \geq 1$ and assume that V is compactly embedded in $C^{p+1}_0(\Omega, \mathbb{R})$. Let S_{targ} be a Jordan shape in $\mathcal{S}^k(\Omega)$ and $g_c : \mathcal{C}^k_\flat(\Omega) \to \mathbb{R}$ such that*

$$g_c(\gamma) = \int_{\Omega} |\chi_{S_\gamma} - \chi_{S_{\mathrm{targ}}}| dm \,.$$

Let $\gamma_1 \in \mathcal{C}^k_\flat(\Omega)$ be positively oriented. Denote $\mathbb{T}_0 = \{ \ s \in \mathbb{T} \mid \gamma_1(s) \notin \partial S_{\mathrm{targ}} \ \}$ and

$$\mathbb{T}_+ = \{ \ s \in \mathbb{T} \backslash \mathbb{T}_0 |\ n_{\mathrm{targ}}(\gamma_1(s)) \text{ and } n_1(\gamma_1(s)) \text{ exist and } n_1(\gamma_1(s)) = n_{\mathrm{targ}}(\gamma_1(s)) \ \},$$

n^1 and n_{targ} being the outward normals to the boundaries of S_{γ_1} and S_{targ} (which are well-defined except at a finite number of locations).

Then, g_c is Γ-semi-differentiable at γ_1 and for any smooth perturbation Γ of γ_1 in $\mathcal{C}^k_\flat(\Omega)$, if $q(\epsilon) = g_c(\Gamma(., \epsilon))$, we have

$$\partial^+ q(0) \leq \int_{\mathbb{T}_0} d_{S_{\mathrm{targ}}}(\gamma_1(s)) \langle (\partial \Gamma / \partial \epsilon)(s, 0), n_1(\gamma_1(s)) \rangle |\partial \gamma_1 / \partial s| ds$$
$$+ \int_{\mathbb{T}_+} |\langle (\partial \Gamma / \partial \epsilon)(s, 0), n_1(\gamma_1(s)) \rangle| |\partial \gamma_1 / \partial s| ds.$$

Moreover, if

$$B = \{\ b \in L^2(\mathbb{T},\mathbb{R}^2)\ |$$
$$b(s) = d_{S_{\rm targ}}(\gamma_1(s))n_1(\gamma_1(s)) \text{ when } \gamma_1(s) \notin \partial S_{\rm targ} \text{ and } |b(s)| \leq 1 \text{ otherwise } \},$$

then the Γ-semi-derivatives of g_c at γ_1 are upper bounded by B.

Proof. Let Γ be a smooth perturbation of γ_1 in $\mathcal{C}^k_\flat(\Omega)$ and let $v(s) = (\partial\Gamma/\partial\varepsilon)(s,0)$. Denote for any $\epsilon \in]-\eta,\eta[$, $S_\epsilon = S_{\Gamma(.,\epsilon)}$, $S'_\epsilon = \Omega \setminus \overline{S_\epsilon}$ so that $S_0 = S_{\gamma_1}$ and

$$\int_\Omega |\chi_{S_\epsilon} - \chi_{S_{targ}}|dm = \int_{S_\epsilon} |1-\chi_{S_{targ}}|dm + \int_{S'_\epsilon} |0-\chi_{S_{targ}}|dm = \int_{S_\epsilon} (1-2\chi_{S_{targ}})dm + \text{Cst}$$

The proof relies on the following remark: for any bounded measurable function f on Ω, we have:

$$\int_{S_\epsilon} f(m)dm - \int_{S_0} f(m)dm = \int_0^\epsilon \int_{\mathbb{T}} f \circ \Gamma(s,\alpha)|(\partial\Gamma/\partial\alpha),(\partial\Gamma/\partial s)|(s,\alpha)dsd\alpha$$

where $|a,b|$ denotes $\det(a,b)$ for $a,b \in \mathbb{R}^2$. If Γ is C^1 and f is smooth, one can assume that there exists a diffeomorphism φ_ϵ such that $\varphi_0 = \text{id}$ and for $\Gamma(s,\varepsilon) = \varphi_\varepsilon(\Gamma(s,0))$ in which case the result is a consequence of the divergence Theorem [4]. The general case can be derived by density arguments that we skip to avoid technicalities.

Denote, for any $a, m \in \mathbb{R}^2$,

$$\chi^a_{S_{targ}}(m) = \limsup_{t\to 0, t>0} \chi_{S_{targ}}(m+ta)$$

Since $S_{targ} \in \mathcal{S}^k(\Omega)$, we can define n_m, the outwards normal to the boundary of S_{targ} everywhere except in a finite number of locations and we get immediately that $\chi^a_{S_{targ}}(m) = \chi_{S_{targ}}(m)$ for $m \notin \partial S_{targ}$ and $\chi^a_{S_{targ}}(m) = (1 - \text{sgn}(\langle a, n_{targ}(m)\rangle))/2$ for $\langle a, n_m\rangle \neq 0$.

Let $\mathbb{T}' = \{s \in \mathbb{T} \mid \gamma_1(s) \in \partial S_{targ},\ \langle v(s), n_{targ}(\gamma_1(s))\rangle = 0\ \}$. There can be at most a finite number of points $s \in \mathbb{T}'$ such that $\langle(\partial\gamma_1/\partial s), n_{targ}(\gamma_1(s))\rangle \neq 0$, since this implies that s is isolated in $\mathbb{T}'$. For all other $s \in \mathbb{T}'$, we have $\langle(\partial\gamma_1/\partial s), n_{targ}(\gamma_1(s))\rangle = 0$ and $|v(s),(\partial\gamma_1/\partial s)| = 0$ so that

$$\begin{aligned} 0 &= \lim_{\alpha\to 0,\alpha>0} (1 - 2\chi_{S_{targ}} \circ \Gamma(s,\alpha))|(\partial\Gamma/\partial\alpha),(\partial\Gamma/\partial s)|(s,\alpha) \qquad (1.29) \\ &= (1 - 2\chi^{v(s)}_{S_{targ}} \circ \gamma_1(s))|v(s),(\partial\gamma_1/\partial s)| \end{aligned}$$

We check easily that if $s \notin \mathbb{T}'$, then $\gamma_1(s) \notin \partial S_{targ}$ or $\gamma_1(s) \in \partial S_{targ}$ and $\langle v(s), n_{targ}(\gamma_1(s))\rangle \neq 0$, so that

$$\begin{aligned} &\lim_{\alpha\to 0,\alpha>0} (1 - 2\chi_{S_{targ}} \circ \Gamma(s,\alpha))|(\partial\Gamma/\partial\alpha),(\partial\Gamma/\partial s)|(s,\alpha) \\ &\qquad = (1 - 2\chi^{v(s)}_{S_{targ}} \circ \gamma_1(s))|v(s),(\partial\gamma_1\partial s)| \qquad (1.30) \end{aligned}$$

Using the dominated convergence Theorem and equations (1.29) and (1.30), we deduce

$$\lim_{\epsilon\to 0,\epsilon>0}\frac{1}{\epsilon}\left(\int_{S_\epsilon} d_{S_{targ}}dm - \int_{S_0} d_{S_{targ}}dm\right) = \int_{\mathbb{T}}(1-2\chi^{v(s)}_{S_{targ}}\circ\gamma_1(s))|v(s),(\partial\gamma_1/\partial s)|ds \quad (1.31)$$

(We have $d_{S_{targ}} = 1-2\chi_{S_{targ}}$.) Considering $\mathbb{T}_0$, $\mathbb{T}_+$ and $\mathbb{T}_- = \mathbb{T}\setminus(\mathbb{T}_0\cup\mathbb{T}_+)$ as introduced in Theorem 1 we get

$$\partial^+ q(0) = \int_{\mathbb{T}_0} d_{S_{targ}}(\gamma_1(s))\langle v(s), n_1(\gamma_1(s))\rangle|\partial\gamma_1/\partial s|ds + \int_{\mathbb{T}_+}|\langle v(s), n_1(\gamma_1(s))\rangle||\partial\gamma_1/\partial s|ds - \int_{\mathbb{T}_-}|\langle v(s), n_1(\gamma_1(s))\rangle||\partial\gamma_1/\partial s|ds \quad (1.32)$$

which ends the proof of Proposition 1.

Given Proposition 1, we can apply immediatly Theorem 5 and get a precise description of the initial momentum.

Theorem 6. *Let $p\geq k\geq 1$ and assume that V is compactly embedded in $C_0^{p+1}(\Omega,\mathbb{R})$. Let S_{temp} and S_{targ} be two Jordan shapes in $\mathcal{S}^k(\Omega)$. Then the conclusions of theorem 5 hold for*

$$J(u) = \frac{1}{2}\int_0^1 |u_t|_V^2 dt + \lambda\int_\Omega |\chi_{S_{\gamma_1^u}} - \chi_{S_{\mathrm{targ}}}|dm\,.$$

with

$$p_1(s) = \lambda\beta_1(s)|\partial\gamma_1/\partial s|n_1(s)$$

where

$$\beta_1(s) = (2\chi_{S_{\mathrm{targ}}} - 1)\circ\gamma_1(s) \text{ if } \gamma_1(s)\in\Omega\setminus\partial S_{\mathrm{targ}} \quad (1.33)$$

and $|\beta_1(s)|\leq 1$ for all s. Here n_1 is the outwards normal to the boundary ∂S_{γ_1} (which is defined everywhere except on a finite number of points).

Proof. This is a direct consequence of Proposition 1 and Theorem 5.

Using the fact that $p_t(s) = (d\varphi^{u_*}_{t,1}(\gamma_t(s)))^*(p_1(s))$ a straightforward computation gives

$$p_0(s) = \lambda\beta_1(s)|d\varphi^{u_*}_{0,1}(\gamma_0(s))||\partial\gamma_0/\partial s|n^0_{\gamma_0(s)},$$

where n^0 is the outwards normal to ∂S_{temp}. In particular, assuming an arc-length parametrization of the boundary of S_{temp}, we get that the norm of the inital momentum is exactly equal to the value of the Jacobian of the optimal matching at any location $s\in\mathbb{T}_0$ (see Proposition 1) along the boundary.

1.6 Application to driving terms based on a potential

In this section, we consider the case

$$g_c(\gamma) = \int_\gamma U_{targ}(x)dx = \int_{\mathbb{T}} U_{targ}(\gamma(s))|\gamma'(s)|ds$$

where $U_{targ} \geq 0$ is a function, depending on the target shape, which vanishes only for $x \in \partial S_{targ}$, the main example being the distance function $U_{targ}(x) = \mathrm{dist}(\partial S_{targ}, x)$.[9] However, before dealing specifically with the distance function, we first address the simpler case of smooth U_{targ}. We moreover restrict to smooth templates (without corners) to avoid the introduction of additional singularities. Then, an easy consequence of Theorem 1 is

Theorem 7. *Let $p \geq k \geq 2$ and assume that V is compactly embedded in $C_0^{p+1}(\Omega, \mathbb{R})$. Let S_{temp} be a C^2 Jordan shape and U_{targ} be a C^1 function in $\mathbb{R}^2$. Then the conclusions of Theorem 1 hold for*

$$J(u) = \frac{1}{2}\int_0^1 |u_t|_V^2 dt + \lambda \int_{\mathbb{T}} U_{targ}(\gamma_1(s))|\partial\gamma_1|/\partial s ds$$

with

$$p_1 = -\lambda|\gamma_1'(s)|(\nabla^\perp_{\gamma_1(s)} U_{targ} - U_{targ}(\gamma_1(s))\kappa_1(s)n_1(s))$$

where n_1 is the normal to γ_1, κ_1 is the curvature on γ_1 and $\nabla^\perp_{\gamma_1(s)} U_{targ}$ is the normal component of the gradient of U_{targ} to γ_1.

Proof. The hypothesis on U_{targ} obviously implies the continuity of g_c. Let γ be a C^2 curve and Γ a smooth perturbation of γ. The derivative at 0 of the function $q(\varepsilon) = g_c(\Gamma(., \varepsilon))$ is (letting $v(s) = (\partial\Gamma/\partial\varepsilon)(s, 0)$):

$$\begin{aligned} q'(0) = & \int_{\mathbb{T}} \left(\langle \nabla_{\gamma(s)} U_{targ} \, , \, v(s)\rangle |\gamma_1'(s)| + U_{targ}(\gamma(s))\langle \tau_s \, , \, \partial v/\partial s\rangle\right) ds \\ = & \int_{\mathbb{T}} \left(\langle \nabla_{\gamma(s)} U_{targ} - \langle \nabla_{\gamma(s)} U_{targ} \, , \, \tau_s\rangle \tau_s \, , \, v(s)\rangle \right. \\ & \quad \left. - \langle U_{targ}(\gamma(s))\kappa_s n_s \, , \, v(s)\rangle\right) |\gamma_1'(s)| ds \\ = & \int_{\mathbb{T}} \left(\left\langle \nabla^\perp_{\gamma(s)} U_{targ} \, , \, v(s)\right\rangle - \langle U_{targ}(\gamma(s))\kappa_s n_s \, , \, v(s)\rangle\right) |\gamma_1'(s)| ds \end{aligned}$$

where the second equation comes from an integration by parts. This proves Theorem 7.

[9] This can be seen as a form of *diffeomorphic active contours* since the potential U_{targ} can obviously arise from other contexts, for example from the locations of discontinuities within an image.

Now, consider the case $U_{targ} = \mathrm{dist}(\partial S_{targ}, .)$. This function has singularities on ∂S_{targ} and on the medial axis, denoted $\hat{\Sigma}_{targ}$, which consists in points $m \in \mathbb{R}^2$ which have at least two closest points in ∂S_{targ}. Denote

$$\partial_m^+ U_{targ}(h) \doteq \lim_{\varepsilon \to 0, \varepsilon > 0} (U_{targ}(m + \varepsilon h) - U_{targ}(m))/\varepsilon$$

when the limit exists. We assume that there is a subset $\Sigma_{targ} \subset \hat{\Sigma}_{targ}$ such that

- $\hat{\Sigma}_{targ} \setminus \Sigma_{targ}$ has a finite or number of points.
- Σ_{targ} is a union of smooth disjoint curves in $\mathbb{R}^2$.
- The directional derivatives

$$\partial_m^+ U_{targ}(h) \doteq \lim_{\varepsilon \to 0, \varepsilon > 0} (U_{targ}(m + \varepsilon h) - U_{targ}(m))/\varepsilon = |\langle h \, , \, n_{targ}(m) \rangle| \, .$$

  exist for $m \in \Sigma_{targ}$ and $h \in \mathbb{R}^2$, and are negative if h is not tangent to Σ_{targ}. If h is tangent to Σ_{targ}, the function $U(m + \varepsilon h)$ is differentiable at $\varepsilon = 0$, with derivative denoted $\partial_m U_{targ}.h$.

Let $R_{targ} = \mathbb{R}^2 \setminus (\partial S_{targ} \cup \Sigma_{targ})$. The gradient of U_{targ} on this set is well-defined and has norm 1. On ∂S_{targ}, we have $U_{targ} = 0$ and

$$\partial^+ U_{targ}(m)(h) = |\langle h \, , \, n_{targ}(m) \rangle| \, .$$

We have:

$$q'(0) = \int_{\mathbb{T}} \partial^+ U_{targ}(\gamma(s))(v(s)) |\gamma'(s)| ds + \int_{\mathbb{T}} U_{targ}(\gamma(s)) \langle \tau_s \, , \, \partial v / \partial s \rangle ds$$

Denote $\mathbb{T}_0 = \gamma^{-1}(R_{targ})$, $\mathbb{T}_+ = \gamma^{-1}(\partial S_{targ})$ and

$$\mathbb{T}_* = \{ s \in \mathbb{T}, \gamma(s) \in \Sigma_{targ}, v(s) \text{ tangent to } \Sigma_{targ} \}$$

with the convention that 0 is always tangent to Σ_{targ}. For the remaining points in $\mathbb{T}$ (up to a finite number), $\partial^+ U_{targ}(m)(v(s)) \leq 0$ so that the first integral is bounded by

$$\int_{\mathbb{T}_0} \langle \nabla_{\gamma_s} U_{targ} \, , \, v(s) \rangle |\gamma'(s)| ds + \int_{\mathbb{T}_+} |\langle n(s) \, , \, v(s) \rangle| \, |\gamma'(s)| ds \\ + \int_{\mathbb{T}_*} \partial_{\gamma(s)} U_{targ}(v(s)) |\gamma'(s)| ds$$

We now address the integration by parts needed for the second integral. This leads to compute the derivative, with respect to s, of $U_{targ}(\gamma(s))$. Consider the three cases: (i) $\gamma(s) \in R_{targ}$; (ii) $\gamma(s) \in \partial S_{targ}$ and $\gamma'(s)$ is tangent to ∂S_{targ} ; (iii) $\gamma(s) \in \Sigma_{targ}$ and $\gamma'(s)$ is tangent to Σ_{targ}. Points which are in none of these categories are isolated in $\mathbb{T}$ and therefore do not contribute to the integral. In all these cases, the function $s \mapsto U_{targ}(\gamma(s))$ is differentiable. Moreover, in case (ii), the differential is 0, and in case (iii), the resulting term cancels with the integral over T_* above. All this together implies that

$$\partial^+ q(0) \leq \int_{\mathbb{T}_0} \left\langle \nabla^\perp_{\gamma_s} U_{targ} \, , \, v(s) \right\rangle |\gamma'(s)| ds + \int_{\mathbb{T}_+} |\langle n(s) \, , \, v(s) \rangle| \, |\gamma'(s)| ds \\ - \int_{\mathbb{T}} U_{targ}(\gamma(s)) \kappa_s n_s ds.$$

This finally implies

Theorem 8. *Let $p \geq k \geq 2$ and assume that V is compactly embedded in $C_0^{p+1}(\Omega, \mathbb{R})$. Let S_{temp} and S_{targ} be two C^k Jordan shapes. Then the conclusions of Theorem 5 hold for*

$$J(u) = \frac{1}{2} \int_0^1 |u_t|_V^2 dt + \lambda \int_{\mathbb{T}} U_{targ}(\gamma_1^u(s)) |\partial \gamma_1^u / \partial s| ds.$$

with $U_{targ} = dist(\partial S_{targ}, .)$ and

$$p_1(s) = -\lambda |\gamma_1'(s)| (\beta_1(s) - U_{targ}(\gamma_1(s)) \kappa_1(s)) n_1(s)$$

with $\beta_1(s) = \left\langle \nabla^\perp_{\gamma_s} U_{targ} \, , \, n_1(s) \right\rangle$ if $\gamma_1(s) \in R_{targ}$, $\beta_1(s) = 0$ if $\gamma_1(s) \in \Sigma_{targ}$ and $|\beta_1(s)| \leq 1$ if $\gamma_1(s) \in \partial S_{targ}$.

1.7 Existence and uniqueness of the hamiltonian flow

In this short section, we show that the hamiltonian flow exists globally in time for any inital data in the phase space.

Theorem 9 (Flow Theorem). *Assume that V is continuously embedded in $C_0^1(\Omega, \mathbb{R}^2)$ with a C^2 kernel K having bounded second order derivative. Let $H : L^2(\mathbb{T}, \mathbb{R}^2) \times L^2(\mathbb{T}, \mathbb{R}^2) \to \mathbb{R}$ be defined by*

$$H(\gamma, p) = \frac{1}{2} \int {}^t p(y) K(\gamma(y), \gamma(x)) p(x) dx dy$$

Then for any initial data (γ_0, p_0) there exists a unique solution $(\gamma, p) \in C^1(\mathbb{R}, L^2(\mathbb{T}, \mathbb{R}^2) \times L^2(\mathbb{T}, \mathbb{R}^2))$ of the ODE

$$\begin{cases} \dot{\gamma} = \dfrac{\partial}{\partial p} H(\gamma, p) \\ \\ \dot{p} = -\dfrac{\partial}{\partial \gamma} H(\gamma, p) \end{cases} \tag{1.34}$$

where $\partial H(\gamma, p)/\partial p = \int K(\gamma(.), \gamma(y)) \gamma(y) dy$ and

$$\partial H(\gamma, p)/\partial \gamma = \int {}^t p(.)\partial_1 K(\gamma(.), \gamma(y))p(y)dy.$$

Here, the notation ${}^t u\partial_1 K(\alpha_0, \beta)v$ *refers to the gradient at* α_0 *of the function* $\alpha \mapsto {}^t uK(\alpha, be)v$.

Proof. The existence of a solution in small time is straightforward since the smoothness conditions on the kernel imply that there exists $M > 0$ such that $|\partial H(\gamma, p)/\partial p - \partial H(\gamma', p')/\partial p|_2 \leq M(|p - p'|_2 + |p|_2|\gamma - \gamma'|_2)$ and $|(\partial/\partial\gamma)H(\gamma, p) - (\partial/\partial\gamma)H(\gamma', p')|_2 \leq M(|p|_2^2|\gamma - \gamma'|_2 + |p|_2|p - p'|_2)$. Thus $\partial H/\partial\gamma$ and $\partial H/\partial p$ is uniformly Lipschitz on any ball in $L^2(\mathbb{T}, \mathbb{R}^2) \times L^2(\mathbb{T}, \mathbb{R}^2)$. This implies obviously the local existence and uniqueness of the solution for any inital data but also that for any maximal solution defined on $[0, T[$ with $T > \infty$, then

$$\lim_{t \to T}(|\gamma_t|_2 + |p_t|_2) = +\infty \tag{1.35}$$

The global existence in time follows from standard arguments: Assume that (γ_t, p_t) is a maximal solution defined on $[0, T[$ with $T < \infty$. Since V is continuously embedded in $C_0^1(\overline{\Omega}, \mathbb{R}^2)$, we deduce that $m \to v(m) = \int K(m, \gamma_t(s'))p_t(s')ds$ defines an element $v \in V$ with continuous differential and such that $|dv|_\infty \leq M|v|_2$ with M independent of v. Hence $|\partial H(\gamma_t, p_t)/\partial\gamma|_2 = |dv(\gamma_t)(p_t)|_2 \leq M|v|_V = MH(\gamma_t, p_t)^{1/2}$. Since H is constant along the solution, we get $|\gamma_t - p_0|_2 \leq MT\sqrt{H(\gamma_0, p_0)}$ so that $|\dot\gamma_t|_2 \leq |K|_\infty(|p_0|_2 + MT\sqrt{H(\gamma_0, p_0)})$ and $|\gamma_t - \gamma_0|_2 \leq |K|_\infty T(|p_0|_2 + MT\sqrt{H(\gamma_0, p_0)})$. This is in contradiction with (1.35).

1.8 Conclusion

We have spent some time, in this paper, in order to provide, for specific examples of interest, the Hamiltonian structure of large deformation curve matching. The central element in this structure, is the momentum $p_t, t \in [0, 1]$, and the fact that the deformation can be reconstructed exactly from the template and the knowledge of the initial momentum p_0.

This implies that p_0 can be considered as a *relative signature* for the deformed shape with respect to the template. In all cases, it was a vector-valued function defined on the unit circle, characterized in fact by a scalar when the data attachement term is geometric. Because the initial momentum is always supported by the template, it is possible to add them, or average them without any issue of registering the data, since the work is already done. This facts lead to simple procedures for statistical shape analysis, when they are based on the momentum, and some developments have already been provided in [19] in the case of landmark-based matching.

This paper therefore provides the theoretical basis for the computation of this representation. Future works will include the refinement and development of numerical algorithms for its computation. Such algorithms already exist, for example, in the case of measure-based matching, but still need to be developed in the other cases.

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