

On the functional central limit theorem for stationary processes

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Abstract

We give a sufficient condition for a stationary sequence of square-integrable and real-valued random variables to satisfy a Donsker-type invariance principle. This condition is similar to the \mathbb{L}^1 -criterion of Gordin for the usual central limit theorem and provides invariance principles for α -mixing or β -mixing sequences as well as stationary Markov chains. In the latter case, we present an example of a non irreducible and non α -mixing chain to which our result applies.

Résumé

Nous donnons une condition suffisante pour qu'une suite stationnaire de variables aléatoires réelles de carré intégrable satisfasse le principe d'invariance de Donsker. Cette condition est comparable au critère \mathbb{L}^1 de Gordin pour le théorème limite central usuel. Nous en déduisons des principes d'invariance pour les suites α -mélangeantes ou β -mélangeantes, ainsi que pour les chaînes de Markov stationnaires. Dans ce dernier cas, nous exhibons une chaîne de Markov ni irréductible ni α -mélangeante à laquelle notre résultat s'applique.

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Short Title: Functional CLT for stationary sequences

1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability \mathbb{P} . In this paper, we shall study the invariance principle for the strictly stationary process $(X_0 \circ T^i)$, where X_0 is some real-valued, square-integrable and centered random variable. To be precise, write $X_i = X_0 \circ T^i$,

$$S_n = X_1 + \cdots + X_n \quad \text{and} \quad S_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}.$$

We say that the sequence $(X_0 \circ T^i)$ satisfies the invariance principle if the process $\{n^{-1/2}S_n(t) : t \in [0, 1]\}$ converges in distribution to a mixture of Wiener processes in the space $C([0, 1])$ equipped with the metric of uniform convergence.

One of the possible approaches to study the asymptotic behaviour of the normalized partial sum process is to approximate S_n by a related martingale with stationary differences. Then, under some additional conditions, the central limit theorem can be deduced from the martingale case. This approach was first explored by Gordin (1969), who obtained a sufficient condition for the asymptotic normality of the normalized partial sums. One of the most interesting cases arises when the sequence $(X_0 \circ T^i)$ admits a coboundary decomposition. This means that $(X_0 \circ T^i)$ differs from the approximating martingale differences sequence $(M_0 \circ T^i)$ in a coboundary, i.e.

$$X_0 - M_0 = Z - Z \circ T \tag{1.1}$$

where Z is some real-valued random variable. In this case, the invariance principle and the functional law of the iterated logarithm hold as soon as M_0 and Z are square integrable variables. As shown by Heyde (1975), this condition is equivalent to the convergence in \mathbb{L}^2 of some sequences of random variables derived from the stationary process $(X_0 \circ T^i)$. To say more on this subject, we need the following definition.

Definition 1. Let \mathcal{M}_0 be a σ -algebra of \mathcal{A} satisfying $\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$, and define the nondecreasing filtration $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ by $\mathcal{M}_i = T^{-i}(\mathcal{M}_0)$. For any integrable random variable Y , we denote by $\mathbb{E}_i(Y)$ the conditional expectation of Y with respect to the σ -algebra \mathcal{M}_i .

From Heyde (1975) and Volný (1993), we know that the stationary sequence $(X_i) = (X_0 \circ T^i)$ admits the coboundary decomposition (1.1) with M_0 in

$\mathbb{L}^2(\mathcal{M}_0)$ and Z in $\mathbb{L}^2(\mathcal{A})$ if and only if

$$\sum_{n=0}^{\infty} \mathbb{E}_0(X_n) \text{ and } \sum_{n=0}^{\infty} (X_{-n} - \mathbb{E}_0(X_{-n})) \text{ converge in } \mathbb{L}^2. \quad (1.2)$$

Consequently, the invariance principle holds as soon as (1.2) is satisfied. However, criterion (1.2) may be suboptimal when applied to Markov chains or to strongly mixing sequences (cf. Section 2, Remark 2).

To improve on condition (1.2) it seems quite natural to weaken the convergence assumption. For instance, if we replace the convergence in \mathbb{L}^2 by the convergence in \mathbb{L}^1 in (1.2), then (1.1) holds with both M_0 and Z in \mathbb{L}^1 . Under this assumption, it follows from Gordin (1973) that a sufficient condition for M_0 to belong to \mathbb{L}^2 is: $\liminf_{n \rightarrow +\infty} n^{-1/2} \mathbb{E}|S_n| < +\infty$. In that case, $n^{-1/2}S_n$ converges in distribution to a normal law. Nevertheless, this is not sufficient to ensure that Z belongs to \mathbb{L}^2 , and therefore the invariance principle may fail to hold (see Volný (1993), Remark 3).

The proofs of these criteria are mainly based on the martingale convergence theorem. Another way to obtain central limit theorems is to adapt Lindeberg's method, as done by Ibragimov (1963) in the case of stationary and ergodic martingale differences sequences. This approach has been used by Dedecker (1998) who gives a projective criterion for strictly stationary random fields. In the case of bounded random variables, this criterion is an extension of the \mathbb{L}^1 -criterion of Gordin (1973). In the present work, we aim at proving the invariance principle for the stationary sequence $(X_i)_{i \in \mathbb{Z}}$ under this new condition. To establish the functional central limit theorem, the usual way is first to prove the weak convergence of the finite dimensional distributions of the normalized partial sums process, and second to prove tightness of this process (see Billingsley (1968), Theorem 8.1). Let

$$\bar{S}_n = \max\{|S_1|, |S_2|, \dots, |S_n|\}.$$

In the stationary case the tightness follows from the uniform integrability of the sequence $(n^{-1}\bar{S}_n^2)_{n>0}$ via Theorem 8.4 in Billingsley (1968). In the adapted case (i.e. X_i is \mathcal{M}_i -measurable) we proceed as follows: first we prove the uniform integrability of the sequence $(n^{-1}\bar{S}_n^2)_{n>0}$ under the condition

$$\sum_{n=0}^{\infty} X_n \mathbb{E}_0(X_n) \text{ converges in } \mathbb{L}^1. \quad (1.3)$$

In order to achieve this, we adapt Garsia's method (1965), as done in Rio (1995) for strongly mixing sequences. Second, we use both the uniform integrability of $(n^{-1}\bar{S}_n^2)_{n>0}$ and Lindeberg's decomposition to obtain the weak convergence of the finite dimensional distributions. The invariance principle follows then straightforwardly. In the adapted case, criterion (1.3) is weaker than (1.2) and its application to strongly mixing sequences leads to the invariance principle of Doukhan *et al.* (1994). Furthermore, condition (1.3) provides new criteria for stationary Markov chains, which cannot be deduced from (1.2) or from mixing assumptions either.

In the general case we apply (1.3) to the adapted sequences $(\mathbb{E}_i(X_{i-k}))_{i \in \mathbb{Z}}$, for arbitrary large values of k . In order to obtain the uniform integrability of the initial sequence $(n^{-1}\bar{S}_n^2)_{n>0}$, we need to impose additional conditions on some series of residual random variables. As a consequence, this method yields the invariance principle under the \mathbb{L}^q -criterion

$$X_0 \in \mathbb{L}^p, \sum_{n=0}^{\infty} \mathbb{E}_0(X_n) \text{ converges in } \mathbb{L}^q \text{ and } \sum_{n=0}^{\infty} \|X_{-n} - \mathbb{E}_0(X_{-n})\|_q < \infty, \quad (1.4)$$

where q belongs to $[1, 2]$ and p is the conjugate exponent of q . When X_0 is a bounded random variable, criterion (1.4) with $q = 1$ yields the invariance principle for stationary sequences under the \mathbb{L}^1 -criterion of Gordin (1973).

The paper is organized as follows. Section 2 is devoted to background material and to the statement of results. In Section 3, we study the uniform integrability of the sequence $(n^{-1}\bar{S}_n^2)_{n>0}$. The central limit theorems are proved in Section 4. Next, in Section 5, we apply our invariance principle to a class of functional autoregressivemodels which may fail to be irreducible. Finally Section 6 collects the applications of criterion (1.3) to mixing sequences.

2 Statement of results

For any sequence $(X_i)_{i \in \mathbb{Z}}$ of real-valued random variables, we consider the sequences $S_n = X_1 + \dots + X_n$,

$$S_n^* = \max\{0, S_1, \dots, S_n\} \quad \text{and} \quad \bar{S}_n = \max\{|S_1|, |S_2|, \dots, |S_n|\}.$$

In this paper we give nonergodic versions of central limit theorems and invariance principles, as done in Volný (1993). With the same notations as

in the introduction, an element A of \mathcal{A} is said to be invariant if $T(A) = A$. We denote by \mathcal{I} the σ -algebra of all invariant sets. The probability \mathbb{P} is ergodic if each element of \mathcal{I} has measure 0 or 1.

2.1 The adapted case

Our first result is an extension of Doob's inequality for martingales. This maximal inequality is stated in the nonstationary case.

Proposition 1 *Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of square-integrable and centered random variables, adapted to a nondecreasing filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. Let λ be any nonnegative real number and $\Gamma_k = (S_k^* > \lambda)$.*

(a) *We have*

$$\mathbb{E}((S_n^* - \lambda)_+^2) \leq 4 \sum_{k=1}^n \mathbb{E}(X_k^2 \mathbb{1}_{\Gamma_k}) + 8 \sum_{k=1}^{n-1} \|X_k \mathbb{1}_{\Gamma_k} \mathbb{E}(S_n - S_k | \mathcal{F}_k)\|_1.$$

(b) *If furthermore the two-dimensional array $(X_k \mathbb{E}(S_n - S_{k-1} | \mathcal{F}_k))_{1 \leq k \leq n}$ is uniformly integrable then the sequence $(n^{-1} \overline{S}_n^2)_{n > 0}$ is uniformly integrable.*

In the stationary and adapted case, Proposition 1(b) yields the uniform integrability of the sequence $(n^{-1} \overline{S}_n^2)_{n > 0}$ under condition (1.3). This fact will be used in section 4 to prove both the finite dimensional convergence of the normalized Donsker partial sum process and the following nonergodic version of the invariance principle.

Theorem 1 *Let $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ be the nondecreasing filtration introduced in definition 1. Let X_0 be a \mathcal{M}_0 -measurable, square-integrable and centered random variable, and $X_i = X_0 \circ T^i$. Assume that condition (1.3) is satisfied. Then:*

(a) *The sequence $(\mathbb{E}(X_0^2 | \mathcal{I}) + 2\mathbb{E}(X_0 S_n | \mathcal{I}))_{n > 0}$ converges in \mathbb{L}^1 to some nonnegative and \mathcal{I} -measurable random variable η .*

(b) *The sequence $\{n^{-1/2} S_n(t) : t \in [0, 1]\}$ converges in distribution in $C([0, 1])$ to the random process $\sqrt{\eta} W$, where W is a standard brownian motion independent of \mathcal{I} .*

Remark 1. If \mathbb{P} is ergodic then $\eta = \sigma^2 = \mathbb{E}(X_0^2) + 2 \sum_{i > 0} \mathbb{E}(X_0 X_i)$ and the usual invariance principle holds.

2.2 Application to weakly dependent sequences

In this section, we apply Theorem 1 to strongly mixing or absolutely regular sequences. In order to develop our results, we need further definitions.

Definitions 2. Let \mathcal{U} and \mathcal{V} be two σ -algebras of \mathcal{A} . The strong mixing coefficient of Rosenblatt (1956) is defined by

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)| : U \in \mathcal{U}, V \in \mathcal{V}\}. \quad (2.1)$$

Let $\mathbb{P}_{\mathcal{U} \otimes \mathcal{V}}$ be the probability measure defined on $(\Omega \times \Omega, \mathcal{U} \otimes \mathcal{V})$ by $\mathbb{P}_{\mathcal{U} \otimes \mathcal{V}}(U \times V) = \mathbb{P}(U \cap V)$. We denote by $\mathbb{P}_{\mathcal{U}}$ and $\mathbb{P}_{\mathcal{V}}$ the restriction of the probability measure \mathbb{P} to \mathcal{U} and \mathcal{V} respectively. The β -mixing coefficient $\beta(\mathcal{U}, \mathcal{V})$ of Rozanov and Volkonskii (1959) is defined by

$$\beta(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}_{\mathcal{U} \otimes \mathcal{V}}(C) - \mathbb{P}_{\mathcal{U}} \otimes \mathbb{P}_{\mathcal{V}}(C)| : C \in \mathcal{U} \otimes \mathcal{V}\}. \quad (2.2)$$

Both Theorem 1 and the covariance inequality of Rio (1993) yield the nonergodic version of the invariance principle of Doukhan *et al.* (1994) for strongly mixing sequences.

Corollary 1 *Let $(X_i)_{i \in \mathbb{Z}}$ be defined as in Theorem 1, and suppose that*

$$\sum_{n=0}^{+\infty} \int_0^{\alpha(\mathcal{M}_0, \sigma(X_n))} Q^2(u) du < \infty, \quad (2.3)$$

where Q denotes the càdlàg inverse of the function $t \rightarrow \mathbb{P}(|X_0| > t)$. Then the series $\sum_{n>0} \|X_0 \mathbb{E}_0(X_n)\|_1$ converges and Theorem 1 applies.

Remark 2. The L^2 criterion (1.2) leads to the suboptimal strong mixing condition

$$\sum_{n=0}^{+\infty} n \int_0^{\alpha(\mathcal{M}_0, \sigma(X_n))} Q^2(u) du < \infty.$$

This can be shown using Rio's covariance inequality. For more about these mixing conditions, cf. Bradley (1997).

Now, from the covariance inequality of Delyon (1990) we get the following invariance principle for absolutely regular sequences.

Corollary 2 *Let ξ_0 be an \mathcal{M}_0 -measurable variable with values in a measurable space \mathcal{E} , and $\xi_i = \xi_0 \circ T^i$. There exists a sequence of random variables*

$(b_n)_{n>0}$ from $(\Omega, \mathcal{A}, \mathbb{P})$ to $[0, 1]$ with $\mathbb{E}(b_n) = \beta(\mathcal{M}_0, \sigma(\xi_n))$ such that the following statement holds true: set $B = \sum_{n>0} b_n$ and let g be a measurable function from \mathcal{E} to \mathbb{R} . Assume that $X_i = g(\xi_i)$ is a square integrable and centered random variable. If X_0 belongs to $\mathbb{L}^2(B\mathbb{P})$ then the series $\sum_{k>0} \|X_0 \mathbb{E}_0(X_k)\|_1$ converges and Theorem 1 applies.

2.3 Application to Markov chains

In this section, we give an application of Theorem 1 to stationary Markov chains. Let \mathcal{E} be a general state space and K be a transition probability kernel on \mathcal{E} . Let

$$K^n(x, A) = \int_{\mathcal{E}} K(x, dx_1) \int_{\mathcal{E}} K(x_1, dx_2) \cdots \int_A K(x_{n-1}, dx_n).$$

We write Kg and $K^n g$ respectively for the functions $\int g(y)K(x, dy)$ and $\int g(y)K^n(x, dy)$.

Corollary 3 *Let ξ_0 be a random variable with values in a measurable space \mathcal{E} , and $\xi_i = \xi_0 \circ T^i$. Suppose that $(\xi_i)_{i \in \mathbb{Z}}$ is a strictly stationary Markov chain, denote by K its transition kernel and by μ the law of ξ_0 . Let g be a measurable function from \mathcal{E} to \mathbb{R} . Assume that $X_i = g(\xi_i)$ is a square integrable and centered random variable. If the series $\sum_{n=0}^{\infty} gK^n g$ converges in $\mathbb{L}^1(\mu)$, then*

- (a) *the random process $\{n^{-1/2}S_n(t) : t \in [0, 1]\}$ converges in distribution in $C([0, 1])$ to $\sqrt{\eta}W$, where W and η are defined as in Theorem 1.*
- (b) *If furthermore the underlying probability \mathbb{P} is ergodic, then (a) holds with $\eta = \sigma_g^2 = \mu(g^2) + 2 \sum_{n>0} \mu(gK^n g)$ a.s.*

Remark 3. Corollary 3 can be extended to nonstationary positive Harris chains (cf. Meyn and Tweedie (1993), Proposition 17.1.6), with the same expression for σ_g^2 . If furthermore the chain is aperiodic then the usual central limit theorem holds as soon as the series of covariances converges, as shown by Chen (1997). However, in order to prove that the variance of the limiting distribution is equal to σ_g^2 , he has to assume that the series $\sum_{n=0}^{\infty} gK^n g$ converges in $\mathbb{L}^1(\mu)$. Note that the form of σ_g^2 coincides with the one given in Nummelin (1984), Corollary 7.3(ii) (cf. De Acosta (1997), Proposition 2.2).

Remark 4. Many central limit theorems (Maigret (1978), Gordin and Lifšic (1978)) are based upon the identity $g = f - Kf$ with f in $\mathbb{L}^2(\mu)$, known

as the Poisson equation. In fact the \mathbb{L}^2 -criterion (1.2) and the coboundary decomposition (1.1) with both M_0 and Z in \mathbb{L}^2 are equivalent to the existence of a solution f in $\mathbb{L}^2(\mu)$ to the Poisson equation.

Application : Autoregressive Lipschitz model. For δ in $[0, 1[$ and C in $]0, 1]$, let $\mathcal{L}(C, \delta)$ be the class of 1-Lipschitz functions f which satisfy

$$f(0) = 0 \quad \text{and} \quad |f'(t)| \leq 1 - C(1 + |t|)^{-\delta} \quad \text{almost everywhere.}$$

Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of i.i.d. real-valued random variables. For $S \geq 1$ let $ARL(C, \delta, S)$ be the class of Markov chains on \mathbb{R} defined by

$$\xi_n = f(\xi_{n-1}) + \varepsilon_n \quad \text{where} \quad f \in \mathcal{L}(C, \delta) \quad \text{and} \quad \mathbb{E}(|\varepsilon_0|^S) < \infty. \quad (2.4)$$

Proposition 2 *Assume that $(\xi_i)_{i \in \mathbb{Z}}$ belongs to $ARL(C, \delta, S)$. There exists a unique invariant probability μ , and furthermore*

$$\int |x|^{S-\delta} \mu(dx) < +\infty.$$

Let $(\xi_i)_{i \in \mathbb{Z}}$ be a stationary Markov chain belonging to $ARL(C, \delta, S)$ with transition kernel K and invariant probability μ . Consider the configuration space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}^{\mathbb{Z}}, P^\xi)$ where P^ξ is the law of $(\xi_i)_{i \in \mathbb{Z}}$, and the shift operator τ from $\mathbb{R}^{\mathbb{Z}}$ to $\mathbb{R}^{\mathbb{Z}}$ defined by $[\tau(\omega)]_i = \omega_{i+1}$. Since μ is the unique probability invariant by K , P^ξ is invariant by τ and ergodic. Denote by $\pi_i = \pi_0 \circ \tau^i$ the projection from $\mathbb{R}^{\mathbb{Z}}$ to \mathbb{R} defined by $\pi_i(\omega) = \omega_i$. Since $(\pi_i)_{i \in \mathbb{Z}}$ has the same distribution P^ξ as $(\xi_i)_{i \in \mathbb{Z}}$, Corollary 3(b) applied to the Markov chain $(\pi_i)_{i \in \mathbb{Z}}$ provides a sufficient condition on g for the sequence $(g(\xi_i))_{i \in \mathbb{Z}}$ to satisfy the invariance principle. The following proposition gives a condition on the moment of the errors under which Corollary 3 applies to Lipschitz functions.

Proposition 3 *Assume that $(\xi_i)_{i \in \mathbb{Z}}$ is a stationary Markov chain belonging to $ARL(C, \delta, S)$ for some $S \geq 2 + 2\delta$. Denote by K its transition kernel and by μ its invariant probability. Let g be any Lipschitz function such that $\mu(g) = 0$. Then $\sum_{n>0} |gK^n g|$ converges in $\mathbb{L}^1(\mu)$ and the sequence $(g(\xi_i))_{i \in \mathbb{Z}}$ satisfies the invariance principle. Moreover, the variance term σ_g^2 is the same as in Corollary 3(b).*

Remark 5. Arguing as in section 5.2, it can be shown that the \mathbb{L}^2 criterion (1.2) requires the stronger moment condition $S \geq 2 + 3\delta$.

An element of $ARL(C, \delta, S)$ may fail to be irreducible in the general case. However, if the common distribution of the ε_i has an absolutely continuous component which is bounded away from 0 in a neighborhood of the origin, then the chain is irreducible and fits in the example of Tuominen and Tweedie (1994), Section 5.2. In this case, the rate of ergodicity can be derived from Theorem 2.1 in Tuominen and Tweedie (1994) (cf. Ango-Nzé (1994) for exact rates of ergodicity).

2.4 The general case

In this section, we extend the results of Section 2.1 to non-adapted sequences. In order to obtain central limit theorems, we impose some asymptotic conditions on the random variables $X_{-n} - \mathbb{E}_0(X_{-n})$.

Definition 3. Let $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ be the nondecreasing filtration introduced in definition 1. We set $\mathcal{M}_{-\infty} = \bigcap_{i \in \mathbb{Z}} \mathcal{M}_i$ and $\mathcal{M}_{\infty} = \sigma(\bigcup_{i \in \mathbb{Z}} \mathcal{M}_i)$. We denote by $\mathbb{E}_{-\infty}(Y)$ (resp. $\mathbb{E}_{\infty}(Y)$) the conditional expectation of Y with respect to the σ -algebra $\mathcal{M}_{-\infty}$ (resp. \mathcal{M}_{∞}).

Let us start with the central limit theorem.

Theorem 2 *Let $(\mathcal{M}_i)_{i \in \mathbb{Z}}$ be the nondecreasing filtration introduced in definition 1. Let X_0 be a square-integrable and centered random variable, and $X_i = X_0 \circ T^i$. Let*

$$\mathcal{K} = \{k \in \overline{\mathbb{Z}} \text{ such that } \sum_{n=0}^{\infty} \mathbb{E}_k(X_0) \mathbb{E}_k(X_n) \text{ converges in } \mathbb{L}^1\}.$$

Suppose that \mathcal{K} is a nonempty set. If

$$\inf_{k \in \mathcal{K}} \limsup_{n \rightarrow +\infty} \mathbb{E}((X_0 - \mathbb{E}_k(X_0))^2) + 2 \sum_{i=1}^n \mathbb{E}(X_0(X_{-i} - \mathbb{E}_k(X_{-i}))) = 0, \quad (2.5)$$

then, for any $l > 0$ and any (t_1, \dots, t_l) in $[0, 1]^l, n^{-1/2}(S_n(t_1), \dots, S_n(t_l))$ converges in distribution to $\sqrt{\eta}(\varepsilon_1, \dots, \varepsilon_l)$, where η is some nonnegative, integrable and \mathcal{I} -measurable random variable and $(\varepsilon_1, \dots, \varepsilon_l)$ is a Gaussian random vector independent of \mathcal{I} with covariance function $\text{Cov}(\varepsilon_i, \varepsilon_j) = t_i \wedge t_j$.

Remark 6. If $\mathbb{E}_{-\infty}(X_0) = 0$ then $-\infty$ belongs to \mathcal{K} . Conversely, arguing as in Dedecker (1998), Proposition 3, it can be shown that $\mathbb{E}_{-\infty}(X_0) = 0$ as soon as $\mathcal{K} \neq \emptyset$.

In order to obtain the uniform integrability of the sequence $(n^{-1}\overline{S}_n^2)_{n>0}$, we need absolute values in the summands in (2.5).

Proposition 4 *Let $(X_i)_{i \in \mathbb{Z}}$ be defined as in Theorem 2, and suppose that*

$$\mathcal{L} = \{l \in \overline{\mathbb{Z}} \text{ such that } (\mathbb{E}_l(X_0)\mathbb{E}_l(S_n))_{n>0} \text{ is uniformly integrable}\}$$

is a nonempty set. If

$$\inf_{l \in \mathcal{L}} \left(\mathbb{E}((X_0 - \mathbb{E}_l(X_0))^2) + 2 \sum_{n>0} \|(X_0 - \mathbb{E}_l(X_0))(X_{-n} - \mathbb{E}_l(X_{-n}))\|_1 \right) = 0, \quad (2.6)$$

then the sequence $(n^{-1}\overline{S}_n^2)_{n>0}$ is uniformly integrable.

Proposition 4 and Theorem 2 together yield the following invariance principle.

Theorem 3 *Let $(X_i)_{i \in \mathbb{Z}}$ and \mathcal{K} be defined as in Theorem 2, and suppose that \mathcal{K} is a nonempty set. If*

$$\inf_{k \in \mathcal{K}} \left(\mathbb{E}((X_0 - \mathbb{E}_k(X_0))^2) + 2 \sum_{n>0} \|(X_0 - \mathbb{E}_k(X_0))(X_{-n} - \mathbb{E}_k(X_{-n}))\|_1 \right) = 0, \quad (2.7)$$

then $\{n^{-1/2}S_n(t) : t \in [0, 1]\}$ converges in distribution in $C([0, 1])$ to $\sqrt{\eta}W$, where η is some nonnegative, integrable and \mathcal{I} -measurable random variable and W is a standard brownian motion independent of \mathcal{I} .

Now, Hölder's inequality applied to Theorems 2 and 3 gives the following \mathbb{L}^q -criteria.

Corollary 4 *Let $(X_i)_{i \in \mathbb{Z}}$ be defined as in Theorem 2. Suppose furthermore that X_0 belongs to \mathbb{L}^p for some p in $[2, +\infty]$. Let $q = p/(p-1)$.*

(a) *Suppose that*

$$\sum_{n=0}^{\infty} \mathbb{E}_0(X_n) \text{ and } \sum_{n=0}^{\infty} (X_{-n} - \mathbb{E}_0(X_{-n})) \text{ converge in } \mathbb{L}^q.$$

Then (2.5) holds true and Theorem 2 applies.

(b) *Suppose that*

$$\sum_{n=0}^{\infty} \mathbb{E}_0(X_n) \text{ converges in } \mathbb{L}^q \text{ and } \sum_{n=0}^{\infty} \|X_{-n} - \mathbb{E}_0(X_{-n})\|_q < \infty.$$

Then (2.7) holds true and Theorem 3 applies.

Remark 7. To prove Corollary 4, note that assumption (a) as well as (b) implies that X_0 is \mathcal{M}_∞ -measurable.

3 Maximal inequalities, uniform integrability

In this section, we prove Propositions 1 and 4.

Proof of Proposition 1(a). We proceed as in Garsia (1965):

$$(S_n^* - \lambda)_+^2 = \sum_{k=1}^n ((S_k^* - \lambda)_+^2 - (S_{k-1}^* - \lambda)_+^2). \quad (3.1)$$

Since the sequence $(S_k^*)_{k \geq 0}$ is nondecreasing, the summands in (3.1) are non-negative. Now

$$((S_k^* - \lambda)_+ - (S_{k-1}^* - \lambda)_+)((S_k^* - \lambda)_+ + (S_{k-1}^* - \lambda)_+) > 0$$

if and only if $S_k > \lambda$ and $S_k > S_{k-1}^*$. In that case $S_k = S_k^*$, whence

$$(S_k^* - \lambda)_+^2 - (S_{k-1}^* - \lambda)_+^2 \leq 2(S_k - \lambda)((S_k^* - \lambda)_+ - (S_{k-1}^* - \lambda)_+). \quad (3.2)$$

Consequently

$$\begin{aligned} (S_n^* - \lambda)_+^2 &\leq 2 \sum_{k=1}^n (S_k - \lambda)(S_k^* - \lambda)_+ - 2 \sum_{k=1}^n ((S_k - \lambda)(S_{k-1}^* - \lambda)_+) \\ &\leq 2(S_n - \lambda)_+(S_n^* - \lambda)_+ - 2 \sum_{k=1}^n (S_{k-1}^* - \lambda)_+ X_k. \end{aligned} \quad (3.3)$$

Noting that

$$2(S_n - \lambda)_+(S_n^* - \lambda)_+ \leq \frac{1}{2}(S_n^* - \lambda)_+^2 + 2(S_n - \lambda)_+^2,$$

we infer that

$$(S_n^* - \lambda)_+^2 \leq 4(S_n - \lambda)_+^2 - 4 \sum_{k=1}^n (S_{k-1}^* - \lambda)_+ X_k. \quad (3.4)$$

In order to bound $(S_n - \lambda)_+^2$, we adapt the decomposition (3.1) and next we apply Taylor's formula:

$$\begin{aligned} (S_n - \lambda)_+^2 &= \sum_{k=1}^n ((S_k - \lambda)_+^2 - (S_{k-1} - \lambda)_+^2) \\ &= 2 \sum_{k=1}^n (S_{k-1} - \lambda)_+ X_k + 2 \sum_{k=1}^n X_k^2 \int_0^1 (1-t) \mathbb{1}_{S_{k-1} + tX_k > \lambda} dt. \end{aligned} \quad (3.5)$$

Since $\mathbb{1}_{S_{k-1} + tX_k > \lambda} \leq \mathbb{1}_{S_k^* > \lambda}$, it follows that

$$(S_n - \lambda)_+^2 \leq 2 \sum_{k=1}^n (S_{k-1} - \lambda)_+ X_k + \sum_{k=1}^n X_k^2 \mathbb{1}_{S_k^* > \lambda}. \quad (3.6)$$

Hence, by (3.4) and (3.6)

$$(S_n^* - \lambda)_+^2 \leq 4 \sum_{k=1}^n (2(S_{k-1} - \lambda)_+ - (S_{k-1}^* - \lambda)_+) X_k + 4 \sum_{k=1}^n X_k^2 \mathbb{1}_{S_k^* > \lambda}. \quad (3.7)$$

Let $D_0 = 0$ and $D_k = 2(S_k - \lambda)_+ - (S_k^* - \lambda)_+$ for $k > 0$. Clearly

$$D_{k-1} X_k = \sum_{i=1}^{k-1} (D_i - D_{i-1}) X_k. \quad (3.8)$$

Hence

$$(S_n^* - \lambda)_+^2 \leq 4 \sum_{i=1}^{n-1} (D_i - D_{i-1}) (S_n - S_i) + 4 \sum_{k=1}^n X_k^2 \mathbb{1}_{S_k^* > \lambda}. \quad (3.9)$$

Since the random variables $D_i - D_{i-1}$ are \mathcal{F}_i -measurable, we have:

$$\begin{aligned} \mathbb{E}((D_i - D_{i-1})(S_n - S_i)) &= \mathbb{E}((D_i - D_{i-1}) \mathbb{E}(S_n - S_i \mid \mathcal{F}_i)) \\ &\leq \mathbb{E}|(D_i - D_{i-1}) \mathbb{E}(S_n - S_i \mid \mathcal{F}_i)|. \end{aligned} \quad (3.10)$$

It remains to bound $|D_i - D_{i-1}|$. If $(S_i^* - \lambda)_+ = (S_{i-1}^* - \lambda)_+$, then

$$|D_i - D_{i-1}| = 2|(S_i - \lambda)_+ - (S_{i-1} - \lambda)_+| \leq 2|X_i| \mathbb{1}_{S_i^* > \lambda},$$

because $D_i - D_{i-1} = 0$ whenever $S_i \leq \lambda$ and $S_{i-1} \leq \lambda$. Otherwise $S_i = S_i^* > \lambda$ and $S_{i-1} \leq S_{i-1}^* < S_i$, which implies that

$$D_i - D_{i-1} = (S_i - \lambda)_+ + (S_{i-1}^* - \lambda)_+ - 2(S_{i-1} - \lambda)_+.$$

Hence $D_i - D_{i-1}$ belongs to $[0, 2((S_i - \lambda)_+ - (S_{i-1} - \lambda)_+)]$. In any case

$$|D_i - D_{i-1}| \leq 2|X_i| \mathbb{1}_{S_i^* > \lambda}, \quad (3.11)$$

which together with (3.9) and (3.10) implies Proposition 1(a).

Proof of Proposition 1(b). Let $A_k(\lambda) = \{\bar{S}_k > \lambda\}$. From Proposition 1(a) applied to the sequences $(X_i)_{i \in \mathbb{Z}}$ and $(-X_i)_{i \in \mathbb{Z}}$ we get that

$$\mathbb{E}((\bar{S}_n - \lambda)_+^2) \leq 8 \sum_{k=1}^n \left(\mathbb{E}(X_k^2 \mathbb{1}_{A_k(\lambda)}) + 2 \|\mathbb{1}_{A_k(\lambda)} X_k \mathbb{E}(S_n - S_k \mid \mathcal{F}_k)\|_1 \right). \quad (3.12)$$

Now, under the assumptions of Proposition 1(b), both the sequence $(X_k^2)_{k > 0}$ and the array $(X_k \mathbb{E}(S_n - S_k \mid \mathcal{F}_k))_{1 \leq k \leq n}$ are uniformly integrable. It follows

that the \mathbb{L}^1 -norms of the above random variables are each bounded by some positive constant M . Hence, from (3.12) with $\lambda = 0$ we get that

$$\mathbb{E}(\overline{S}_n^2) \leq 24Mn.$$

It follows that

$$\mathbb{P}(A_k(x\sqrt{n})) \leq (nx^2)^{-1}\mathbb{E}(\overline{S}_n^2) \leq 24Mx^{-2}. \quad (3.13)$$

Hence, from (3.13) and the uniform integrability of both the sequence $(X_k^2)_{k>0}$ and the array $(X_k\mathbb{E}(S_n - S_k | \mathcal{F}_k))_{1 \leq k \leq n}$ we get that

$$n^{-1}\mathbb{E}((\overline{S}_n - x\sqrt{n})_+^2) \leq \delta(x)$$

for some nonincreasing function δ satisfying $\lim_{x \rightarrow +\infty} \delta(x) = 0$. This completes the proof of Proposition 1(b).

Proof of Proposition 4. Since the sequence $(-X_i)_{i \in \mathbb{Z}}$ still satisfies criterion (2.6), it is enough to prove that $(n^{-1}S_n^{*2})_{n>0}$ is an uniformly integrable sequence.

Let ε be any positive real number and l be some element of \mathcal{L} such that

$$\mathbb{E}((X_0 - \mathbb{E}_l(X_0))^2) + 2 \sum_{n<0} \|(X_0 - \mathbb{E}_l(X_0))(X_n - \mathbb{E}_l(X_n))\|_1 \leq \varepsilon. \quad (3.14)$$

Notations 1. Let $\mathcal{F}_i = \mathcal{M}_{i+l}$ and $Z_i = \mathbb{E}_{i+l}(X_i)$. Write

$$\begin{aligned} T_n &= Z_1 + \cdots + Z_n, \quad T_n^* = \max\{0, T_1, \dots, T_n\} \quad \text{and} \\ Y_n &= X_n - Z_n, \quad W_n = S_n - T_n, \quad W_n^* = \max\{0, W_1, \dots, W_n\}. \end{aligned}$$

Then $(Z_i)_{i \in \mathbb{Z}}$ is a stationary sequence adapted to the filtration $(\mathcal{F}_i)_i$. Clearly, for each event A ,

$$\mathbb{E}(S_n^{*2} \mathbf{1}_A) \leq 2\mathbb{E}(T_n^{*2} \mathbf{1}_A) + 2\mathbb{E}(W_n^{*2}). \quad (3.15)$$

Now

$$Z_0\mathbb{E}(T_n | \mathcal{F}_0) = \mathbb{E}_l(X_0)\mathbb{E}_l(T_n) = \mathbb{E}_l(X_0)\mathbb{E}_l(S_n).$$

Since l belongs to \mathcal{L} , it follows that the sequence $(Z_0\mathbb{E}(T_n | \mathcal{F}_0))_{n>0}$ is uniformly integrable. This fact and the stationarity of $(Z_i)_{i \in \mathbb{Z}}$ together ensure the uniform integrability of the array $(Z_k\mathbb{E}(T_n - T_{k-1} | \mathcal{F}_k))_{1 \leq k \leq n}$. Now Proposition 1(b) implies the uniform integrability of the sequence $(n^{-1}T_n^{*2})_{n>0}$.

Hence there exists some positive δ such that, for any event A with $\mathbb{P}(A) \leq \delta$ and any positive integer n ,

$$\mathbb{E}(T_n^{*2} \mathbf{1}_A) \leq n\varepsilon. \quad (3.16)$$

It remains to bound $\mathbb{E}(W_n^{*2})$. From (3.4) applied with $\lambda = 0$ we get that

$$\mathbb{E}(W_n^{*2}) \leq 4\mathbb{E}(W_n^2) - 4 \sum_{k=1}^n \text{Cov}(W_{k-1}^*, Y_k). \quad (3.17)$$

By definition of the random variables Y_k ,

$$\sum_{n \in \mathbb{Z}} |\mathbb{E}(Y_0 Y_n)| = \text{Var}(X_0 - \mathbb{E}_l(X_0)) + 2 \sum_{n < 0} |\text{Cov}(X_0 - \mathbb{E}_l(X_0), X_n - \mathbb{E}_{l+n}(X_n))|.$$

Now, for any negative n ,

$$\mathbb{E}\left((X_0 - \mathbb{E}_l(X_0))(\mathbb{E}_{l+n}(X_n) - \mathbb{E}_l(X_n))\right) = 0,$$

whence

$$\sum_{n \in \mathbb{Z}} |\mathbb{E}(Y_0 Y_n)| = \text{Var}(X_0 - \mathbb{E}_l(X_0)) + 2 \sum_{n < 0} |\text{Cov}(X_0 - \mathbb{E}_l(X_0), X_n - \mathbb{E}_l(X_n))|.$$

By (3.14) it follows that

$$n^{-1} \mathbb{E}(W_n^2) \leq \text{Var}(X_0 - \mathbb{E}_l(X_0)) + 2 \sum_{n < 0} \| (X_0 - \mathbb{E}_l(X_0))(X_n - \mathbb{E}_l(X_n)) \|_1 \leq \varepsilon. \quad (3.18)$$

Now let us recall that $\text{Cov}(B, Y_k) = 0$ for any square-integrable and \mathcal{M}_{k+l} -measurable random variable B . Hence it will be convenient to replace the random variables W_{k-1}^* by \mathcal{M}_{k+l} -measurable random variables in (3.17).

Notations 2. For $k \in [1, n]$ and $i \in [1, k[$, let $Y_{i,k} = \mathbb{E}_{k+l}(X_i) - \mathbb{E}_{i+l}(X_i)$. We set $W_{i,k} = Y_{1,k} + \dots + Y_{i,k}$ and $A_{k-1}^* = \max\{W_{1,k}, \dots, W_{k-1,k}\}$.

Since A_{k-1}^* is \mathcal{M}_{k+l} -measurable, we have:

$$\text{Cov}(W_{k-1}^*, Y_k) = \text{Cov}(W_{k-1}^* - A_{k-1}^*, Y_k). \quad (3.19)$$

Now recall that $(a_1, \dots, a_{k-1}) \longrightarrow \max(0, a_1, a_1 + a_2, \dots, a_1 + \dots + a_{k-1})$ is a 1-Lipschitz mapping with respect to the ℓ^1 -norm. Hence

$$|W_{k-1}^* - A_{k-1}^*| \leq \sum_{i=1}^{k-1} |Y_i - Y_{i,k}| = \sum_{i=1}^{k-1} |X_i - \mathbb{E}_{k+l}(X_i)|,$$

which, together with (3.19) implies that

$$\begin{aligned} |\text{Cov}(W_{k-1}^*, Y_k)| &\leq \sum_{i=1}^{k-1} \|Y_k(X_i - \mathbb{E}_{k+l}(X_i))\|_1 \\ &\leq \sum_{i=1}^{k-1} \|(X_0 - \mathbb{E}_l(X_0))(X_{i-k} - \mathbb{E}_l(X_{i-k}))\|_1. \end{aligned} \quad (3.20)$$

Collecting (3.17) (3.18) and (3.20), we obtain that

$$\frac{1}{8n} \mathbb{E}(W_n^{*2}) \leq \text{Var}(X_0 - \mathbb{E}_l(X_0)) + 2 \sum_{n < 0} \|(X_0 - \mathbb{E}_l(X_0))(X_n - \mathbb{E}_l(X_n))\|_1 \leq \varepsilon.$$

Together with (3.15) and (3.16), it implies that $\mathbb{E}(S_n^{*2} \mathbb{1}_A) \leq 18n\varepsilon$ for any event A with $\mathbb{P}(A) \leq \delta$. This completes the proof of Proposition 4.

4 Central limit theorems

4.1 The adapted case

In this section, we prove Theorem 1.

Proof of Theorem 1(a). From assumption (1.3), the sequence of random variables $(\mathbb{E}(X_0^2 | \mathcal{M}_{-\infty}) + 2\mathbb{E}(X_0 S_n | \mathcal{M}_{-\infty}))_{n > 0}$ converges in \mathbb{L}^1 . Theorem 1(a) is then a consequence of part (b) of Claim 1 below:

Claim 1 *We have:*

- (a) Both $\mathbb{E}(X_0 X_k | \mathcal{I})$ and $\mathbb{E}(\mathbb{E}(X_0 X_k | \mathcal{M}_{-\infty}) | \mathcal{I})$ are $\mathcal{M}_{-\infty}$ -measurable.
- (b) $\mathbb{E}(X_0 X_k | \mathcal{I}) = \mathbb{E}(\mathbb{E}(X_0 X_k | \mathcal{M}_{-\infty}) | \mathcal{I})$.

Claim 1(b) is derived from Claim 1(a) via the following elementary fact.

Claim 2 *Let Y be a random variable in $\mathbb{L}^1(\mathbb{P})$ and \mathcal{U}, \mathcal{V} two σ -algebras of $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that $\mathbb{E}(Y | \mathcal{U})$ and $\mathbb{E}(\mathbb{E}(Y | \mathcal{V}) | \mathcal{U})$ are \mathcal{V} -measurable. Then $\mathbb{E}(Y | \mathcal{U}) = \mathbb{E}(\mathbb{E}(Y | \mathcal{V}) | \mathcal{U})$.*

It remains to prove Claim 1(a). The fact that $\mathbb{E}(\mathbb{E}(X_0 X_k | \mathcal{M}_{-\infty}) | \mathcal{I})$ is $\mathcal{M}_{-\infty}$ -measurable follows from the \mathbb{L}^1 -ergodic theorem. Indeed the random variables $\mathbb{E}(X_i X_{k+i} | \mathcal{M}_{-\infty})$ are $\mathcal{M}_{-\infty}$ -measurable and

$$\mathbb{E}(\mathbb{E}(X_0 X_k | \mathcal{M}_{-\infty}) | \mathcal{I}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}(X_i X_{k+i} | \mathcal{M}_{-\infty}) \text{ in } \mathbb{L}^1.$$

Next, from the stationarity of the sequence $(X_i)_{i \in \mathbb{Z}}$, we have

$$\left\| \mathbb{E}(X_0 X_k | \mathcal{I}) - \frac{1}{n} \sum_{i=1}^n X_i X_{i+k} \right\|_1 = \left\| \mathbb{E}(X_0 X_k | \mathcal{I}) - \frac{1}{n} \sum_{i=1-N-k}^{-N-k} X_i X_{i+k} \right\|_1.$$

Both this equality and the \mathbb{L}^1 -ergodic theorem imply that $\mathbb{E}(X_0 X_k | \mathcal{I})$ is the limit in \mathbb{L}^1 of a sequence of \mathcal{M}_{-N} -measurable random variables. Since this is true for any integer N , we infer that $\mathbb{E}(X_0 X_k | \mathcal{I})$ is $\mathcal{M}_{-\infty}$ -measurable. This concludes the proof of Claim 1(a).

Proof of Theorem 1(b). The first step of the proof is a central limit theorem for the normalized sums.

Notations 3. Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of $\mathcal{N}(0, 1)$ -distributed and independent random variables, independent of the sequence $(X_i)_{i \in \mathbb{Z}}$. For any δ in $]0, 1]$, let $q = q(\delta) = \lfloor n\delta \rfloor$ and $p = p(\delta) = \lfloor n/q \rfloor$ for n large enough. For any integer i in $[1, p]$ we set

$$U_i = n^{-1/2}(X_{iq-q+1} + \cdots + X_{iq}), \quad V_i = U_1 + U_2 + \cdots + U_i, \\ \Delta_i = (\eta/n)^{1/2}(\varepsilon_{iq-q+1} + \cdots + \varepsilon_{iq}) \quad \text{and} \quad \Gamma_i = \Delta_i + \Delta_{i+1} + \cdots + \Delta_p.$$

Notations 4. Let g be any function from \mathbb{R} to \mathbb{R} . For k and l in $[1, p+1]$, we set $g_{k,l} = g(V_k + \Gamma_l)$, with the conventions $g_{k,p+1} = g(V_k)$ and $g_{0,l} = g(\Gamma_l)$. Afterwards, we will apply this notation to the successive derivatives of the function h .

Let $B_1^3(\mathbb{R})$ denote the class of three-times continuously differentiable functions h from \mathbb{R} to \mathbb{R} such that $\max(\|h'\|_\infty, \|h''\|_\infty, \|h'''\|_\infty) \leq 1$.

The convergence in distribution of $n^{-1/2}S_n$ is an immediate consequence of the proposition below.

Proposition 5 *Under the assumptions of Theorem 1,*

$$\lim_{n \rightarrow +\infty} \mathbb{E}(h(n^{-1/2}S_n)) = \mathbb{E}(h(\eta^{1/2}\varepsilon))$$

for any h in $B_1^3(\mathbb{R})$, where η is defined as in Theorem 1 and ε is a standard normal random variable independent of \mathcal{I} .

Proof. First, we make the elementary decomposition:

$$\begin{aligned} \mathbb{E}(h(n^{-1/2}S_n) - h(\eta^{1/2}\varepsilon)) &= \mathbb{E}(h(n^{-1/2}S_n) - h(V_p)) \\ &+ \mathbb{E}(h(V_p) - h(\Gamma_1)) + \mathbb{E}(h(\Gamma_1) - h(\eta^{1/2}\varepsilon)). \end{aligned} \quad (4.1)$$

Suppose that $pq \neq n$. Noting that h is 1-Lipschitz, we have

$$|\mathbb{E}(h(n^{-1/2}S_n) - h(V_p))| \leq \|n^{-1/2}S_n - V_p\|_1 \leq \sqrt{\delta} \|(n-pq)^{-1/2}S_{n-pq}\|_2.$$

Since the sequence $(k^{-1}S_k^2)_{k>0}$ is bounded in \mathbb{L}^1 , we infer that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} |\mathbb{E}(h(n^{-1/2}S_n) - h(V_p))| = 0. \quad (4.2)$$

In the same way

$$|\mathbb{E}(h(\Gamma_1) - h(\eta^{1/2}\varepsilon))| \leq n^{-1/2} \|\eta^{1/2}\|_2 \|\varepsilon_{pq+1} + \dots + \varepsilon_n\|_2 \leq \sqrt{\delta} \|\eta\|_1,$$

and consequently

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} |\mathbb{E}(h(\Gamma_1) - h(\eta^{1/2}\varepsilon))| = 0. \quad (4.3)$$

In view of (4.2) and (4.3), it remains to control the second term in (4.1). Here we will use Lindeberg's decomposition:

$$\mathbb{E}(h(V_p) - h(\Gamma_1)) = \sum_{k=1}^p \mathbb{E}(h_{k,k+1} - h_{k-1,k+1}) + \sum_{k=1}^p \mathbb{E}(h_{k-1,k+1} - h_{k-1,k}). \quad (4.4)$$

Now, applying the Taylor integral formula we get that:

$$\begin{cases} h_{k,k+1} - h_{k-1,k+1} &= U_k h'_{k-1,k+1} + \frac{1}{2} U_k^2 h''_{k-1,k+1} + R_k \\ h_{k-1,k+1} - h_{k-1,k} &= -\Delta_k h'_{k-1,k+1} - \frac{1}{2} \Delta_k^2 h''_{k-1,k+1} + r_k \end{cases}$$

where

$$|R_k| \leq U_k^2 (1 \wedge |U_k|) \text{ and } |r_k| \leq \Delta_k^2 (1 \wedge |\Delta_k|). \quad (4.5)$$

Since $\mathbb{E}(\Delta_k h'_{k-1,k+1}) = 0$, it follows that

$$\mathbb{E}(h(V_p) - h(\Gamma_1)) = D_1 + D_2 + D_3, \quad (4.6)$$

where

$$\begin{aligned} D_1 &= \sum_{k=1}^p \mathbb{E}(U_k h'_{k-1,k+1}), \\ D_2 &= \frac{1}{2} \sum_{k=1}^p \mathbb{E}((U_k^2 - \Delta_k^2) h''_{k-1,k+1}), \\ D_3 &= \sum_{k=1}^p \mathbb{E}(R_k + r_k). \end{aligned}$$

Control of D_3 .

By (4.5) and the stationarity of the sequence, we get that

$$\sum_{k=1}^p \|R_k\|_1 \leq p \mathbb{E}(U_1^2(1 \wedge |U_1|)).$$

Bearing in mind the definition of U_1 , we obtain

$$\sum_{k=1}^p \|R_k\|_1 \leq \mathbb{E} \left[\frac{S_q^2}{q} \left(1 \wedge \frac{|S_q|}{\sqrt{pq}} \right) \right] \leq \sup_{q>0} \mathbb{E} \left[\frac{S_q^2}{q} \left(1 \wedge \frac{|S_q|}{\sqrt{pq}} \right) \right].$$

From the uniform integrability of the sequence $(q^{-1}S_q^2)_{q>0}$, the right hand term of the above inequalities tends to zero as δ tends to 0 (i.e. p tends to infinity). Obviously the same holds for $\sum_{k=1}^p \|r_k\|_1$, which entails that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} |D_3| = 0.$$

Control of D_1 .

$$\begin{aligned} \mathbb{E}(U_k h'_{k-1,k+1}) &= \mathbb{E}(U_k h'(\Gamma_{k+1})) \\ &+ \sum_{j=1}^{(k-1)q} \mathbb{E}(U_k (h'(n^{-1/2}S_j + \Gamma_{k+1}) - h'(n^{-1/2}S_{j-1} + \Gamma_{k+1}))). \end{aligned}$$

By definition, we have

$$\mathbb{E}(U_k h'(\Gamma_{k+1})) = \frac{1}{\sqrt{n}} \mathbb{E} \left(h' \left((\eta/n)^{1/2} \sum_{kq+1}^{pq} \varepsilon_i \right) \sum_{(k-1)q+1}^{kq} X_i \right).$$

Note that (1.3) implies that $n^{-1}S_n$ converges to 0 in \mathbb{L}^2 . Hence $\mathbb{E}(X_0|\mathcal{I}) = 0$ by the \mathbb{L}^2 -ergodic theorem. Taking the conditional expectation with respect to \mathcal{I} in the above equation, it follows that $\mathbb{E}(U_k h'(\Gamma_{k+1})) = 0$.

Now, in order to bound the summands in the above decomposition, we proceed as follows: let Υ_j be the random variable obtained by integrating $h'(n^{-1/2}S_j + \Gamma_{k+1}) - h'(n^{-1/2}S_{j-1} + \Gamma_{k+1})$ with respect to the sequence $(\varepsilon_i)_{i>0}$. Since η is $\mathcal{M}_{-\infty}$ -measurable (see Claim 1(a)), we infer that the random

variable Υ_j is \mathcal{M}_j -measurable. Moreover h' is 1-Lipschitz (cf. Notations 4), which implies that

$$|h'(n^{-1/2}S_j + \Gamma_{k+1}) - h'(n^{-1/2}S_{j-1} + \Gamma_{k+1})| \leq n^{-1/2}|X_j|,$$

and therefore $|\Upsilon_j| \leq n^{-1/2}|X_j|$. Hence

$$\begin{aligned} |\mathbb{E}(U_k h'_{k-1,k+1})| &\leq \sum_{j=1}^{(k-1)q} |\mathbb{E}(\mathbb{E}(U_k | \mathcal{M}_j) \Upsilon_j)| \\ &\leq n^{-1/2} \sum_{j=1}^{(k-1)q} \mathbb{E}|\mathbb{E}(U_k | \mathcal{M}_j) X_j|. \end{aligned}$$

Bearing in mind the definition of U_k and using the stationarity of the sequence, we obtain the upper bound:

$$|\mathbb{E}(U_k h'_{k-1,k+1})| \leq n^{-1} \sum_{m=1}^{(k-1)q} \mathbb{E}|X_0 \mathbb{E}_0(S_{q+m-1} - S_{m-1})|.$$

Now, by assumption (1.3)

$$\lim_{m \rightarrow +\infty} \sup_{q > 0} \mathbb{E}|X_0 \mathbb{E}_0(S_{q+m-1} - S_{m-1})| = 0,$$

and consequently

$$\lim_{n \rightarrow +\infty} n^{-1} \sum_{m=1}^{(k-1)q} \mathbb{E}|X_0 \mathbb{E}_0(S_{q+m-1} - S_{m-1})| = 0.$$

Finally, for each integer k in $[1, p]$,

$$\lim_{n \rightarrow +\infty} \mathbb{E}(U_k h'_{k-1,k+1}) = 0,$$

which entails that D_1 converges to 0 as n tends to $+\infty$.

Control of D_2 .

First, note that the random vector $(\varepsilon_{kq-q+1}, \dots, \varepsilon_{kq})$ is independent of the σ -field generated by η , $(\varepsilon_i)_{i > kq}$ and the initial sequence. Now integrating with respect to $(\varepsilon_{kq-q+1}, \dots, \varepsilon_{kq})$ we get that

$$\mathbb{E}(\Delta_k^2 h''_{k-1,k+1}) = (q/n) \mathbb{E}(\eta h''_{k-1,k+1}). \quad (4.7)$$

Here we need some additional notation.

Notations 5. For any positive integer N , we introduce

$$\begin{aligned}\Lambda_{k,N} &= [(k-1)q+1, kq]^2 \cap \{(i,j) \in \mathbb{Z}^2 : |i-j| \leq N\}, \\ \bar{\Lambda}_{k,N} &= [(k-1)q+1, kq]^2 \cap \{(i,j) \in \mathbb{Z}^2 : j-i > N\}\end{aligned}$$

and $\eta_N = \mathbb{E}(X_0^2|\mathcal{I}) + 2(\mathbb{E}(X_0X_1|\mathcal{I}) + \dots + \mathbb{E}(X_0X_N|\mathcal{I}))$.

With these notations, by (4.7) we have:

$$\begin{aligned}\mathbb{E}((U_k^2 - \Delta_k^2)h''_{k-1,k+1}) &= \frac{1}{n}\mathbb{E}\left(\left(\sum_{(i,j) \in \Lambda_{k,N}} X_i X_j - q\eta_N\right)h''_{k-1,k+1}\right) \\ &+ \frac{2}{n}\mathbb{E}\left(\left(\sum_{(i,j) \in \bar{\Lambda}_{k,N}} X_i X_j\right)h''_{k-1,k+1}\right) \\ &+ \frac{q}{n}\mathbb{E}((\eta_N - \eta)h''_{k-1,k+1}).\end{aligned}\tag{4.8}$$

η_N converges in \mathbb{L}^1 to η , and therefore

$$\lim_{N \rightarrow +\infty} \limsup_{n \rightarrow +\infty} (q/n)\mathbb{E}((\eta_N - \eta)h''_{k-1,k+1}) = 0.\tag{4.9}$$

We control now the second term of decomposition (4.8). According to Claim 1(a), the random variable η is $\mathcal{M}_{-\infty}$ -measurable. Hence, integrating $h''_{k-1,k+1}$ with respect to the sequence $(\varepsilon_i)_{i>0}$, we obtain a \mathcal{M}_{kq-q} -measurable random variable with values in $[-1, 1]$. It follows that

$$\begin{aligned}|\mathbb{E}(h''_{k-1,k+1} \sum_{(i,j) \in \bar{\Lambda}_{k,N}} X_i X_j)| &\leq \sum_{i=(k-1)q+1}^{kq} \mathbb{E} \left| X_i \sum_{j=N+i+1}^{kq} \mathbb{E}(X_j|\mathcal{M}_i) \right| \\ &\leq \sum_{i=(k-1)q+1}^{kq} \mathbb{E} \left| X_0 \sum_{j=N+1}^{kq-i} \mathbb{E}(X_j|\mathcal{M}_0) \right|.\end{aligned}$$

Now, by assumption (1.3)

$$\lim_{N \rightarrow +\infty} \sup_{q,i} \mathbb{E} \left| X_0 \sum_{j=N+1}^{kq-i} \mathbb{E}(X_j|\mathcal{M}_0) \right| = 0,$$

and consequently

$$\lim_{N \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{1}{n} \left| \sum_{(i,j) \in \bar{\Lambda}_{k,N}} \mathbb{E}(h''_{k-1,k+1} X_i X_j) \right| = 0.\tag{4.10}$$

To control the first term of decomposition (4.8), we write

$$\left| \mathbb{E} \left(\left(\sum_{(i,j) \in \Lambda_{k,N}} X_i X_j - q\eta_N \right) h''_{k-1,k+1} \right) \right| \leq \mathbb{E} \left| \sum_{(i,j) \in \Lambda_{k,N}} X_i X_j - q\eta_N \right|. \quad (4.11)$$

Here, note that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \left| \sum_{(i,j) \in \Lambda_{k,N}} X_i X_j - \sum_{i=(k-1)q+1}^{kq} \sum_{j=i-N}^{i+N} X_i X_j \right| = 0. \quad (4.12)$$

The L^1 -ergodic theorem applied to the last sum gives

$$\lim_{n \rightarrow +\infty} \frac{q}{n} \mathbb{E} \left| \frac{1}{q} \sum_{i=(k-1)q+1}^{kq} \sum_{j=i-N}^{i+N} X_i X_j - \eta_N \right| = 0. \quad (4.13)$$

From (4.11), (4.12) and (4.13), we obtain

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left| \mathbb{E} \left(\left(\sum_{(i,j) \in \Lambda_{k,N}} X_i X_j - q\eta_N \right) h''_{k-1,k+1} \right) \right| = 0. \quad (4.14)$$

Collecting (4.9), (4.10) and (4.14) we get that

$$\lim_{n \rightarrow +\infty} \mathbb{E}((U_k^2 - \Delta_k^2) h''_{k-1,k+1}) = 0,$$

which entails that D_2 converges to 0 as n tends to $+\infty$.

End of the proof of Proposition 5.

Collecting the above controls, we get that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} |\mathbb{E}(h(V_p) - h(\Gamma_1))| = 0. \quad (4.15)$$

Now Proposition 5 follows from both (4.1), (4.2), (4.3) and (4.15).

Proof of Theorem 1. From the uniform integrability of $(n^{-1} \overline{S}_n^2)_{n>0}$, we know that the sequence of processes $\{n^{-1/2} S_n(t) : t \in [0, 1]\}$ is tight in $C([0, 1])$. It remains to prove the weak convergence of the finite dimensional marginals.

Notation 6. For m and n in \mathbb{N} with $m < n$, let $S_{m,n} = (n-m)^{-1/2}(S_n - S_m)$.

In fact, it suffices to prove that if the differences $n_{i+1} - n_i$ converge to $+\infty$ then the array of random vectors $(S_{0,n_1}, S_{n_1,n_2}, \dots, S_{n_{p-1},n_p})$ converges

in distribution to $\sqrt{\eta}(y_1, \dots, y_p)$, where y_1, \dots, y_p are independent standard normals. For any p -tuple (a_1, \dots, a_p) and any h in $B_1^3(\mathbb{R})$, write

$$h(a_1 S_{0, n_1} + \dots + a_p S_{n_{p-1}, n_p}) - h(\sqrt{\eta}(a_1 y_1 + \dots + a_p y_p)) = \sum_{k=1}^p [g_k(a_k S_{n_{k-1}, n_k}) - g_k(\sqrt{\eta} a_k y_k)],$$

where

$$g_k(x) = h\left(a_1 S_{0, n_1} + \dots + a_{k-1} S_{n_{k-2}, n_{k-1}} + x + \sqrt{\eta}(a_{k+1} y_{k+1} + \dots + a_p y_p)\right).$$

Note that the random functions g_k belong to $B_1^3(\mathbb{R})$ for any ω in Ω . To prove the finite dimensional convergence, it is then sufficient to prove that

$$\lim_{(n_k - n_{k-1}) \rightarrow +\infty} \mathbb{E}(g_k(a_k S_{n_{k-1}, n_k}) - g_k(\sqrt{\eta} a_k y_k)) = 0,$$

which can be done as in the proof of Proposition 5. This completes the proof of Theorem 1.

4.2 The general case

In this section, we prove Theorem 2. Let ψ be a map from \mathbb{N} into \mathcal{K} such that

$$\lim_{k \rightarrow +\infty} \left\{ \mathbb{E}((X_0 - \mathbb{E}_{\psi(k)}(X_0))^2) + 2 \limsup_{n \rightarrow +\infty} \sum_{i=1}^n \mathbb{E}(X_0(X_{-i} - \mathbb{E}_{\psi(k)}(X_{-i}))) \right\} = 0.$$

Notations 7. Let $X_0^{(k)} = \mathbb{E}_{\psi(k)}(X_0)$ and $Y_0^{(k)} = X_0 - X_0^{(k)}$. We set

$$S_n^{(k)} = X_0^{(k)} \circ T + \dots + X_0^{(k)} \circ T^n$$

and we denote by $\{S_n^{(k)}(t) : t \in [0, 1]\}$ the partial sum process associated to the sums $S_n^{(k)}$.

By Theorem 1 applied to the sequence $(X_i^{(k)})_i = (X_0^{(k)} \circ T^i)_i$, the finite dimensional marginals of the process $\{n^{-1/2} S_n^{(k)}(t) : t \in [0, 1]\}$ converge in distribution to the corresponding marginals of the process $\sqrt{\eta^{(k)}} W$, where W is a standard Brownian motion on $[0, 1]$ independent of \mathcal{I} and $\eta^{(k)}$ is the nonnegative, integrable and \mathcal{I} -measurable random variable defined by

$$\eta^{(k)} = \mathbb{E}((X_0^{(k)})^2 | \mathcal{I}) + 2 \sum_{i>0} \mathbb{E}(X_0^{(k)} X_i^{(k)} | \mathcal{I}). \quad (4.16)$$

Hence Theorem 2 follows from Proposition 6 below via the triangle inequality.

Proposition 6 *Under the assumptions of Theorem 2,*

(a) *we have:*

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^{-1/2} \|S_n - S_n^{(k)}\|_2 = 0.$$

(b) *The sequence $(\sqrt{\eta^{(k)}})_k$ converges in \mathbb{L}^2 to some nonnegative and \mathcal{I} -measurable random variable $\sqrt{\eta}$.*

Proof. We start by proving (a). Let $Y_i^{(k)} = Y_0^{(k)} \circ T^i$. Since $Y_0^{(k)}$ is orthogonal to $\mathbb{L}^2(\mathcal{M}_{\psi^{(k)}})$, we have for any positive i ,

$$\mathbb{E}(Y_0^{(k)} Y_{-i}^{(k)}) = \mathbb{E}(Y_0^{(k)}(X_{-i} - \mathbb{E}_{\psi^{(k)}}(X_{-i}))) = \mathbb{E}(X_0(X_{-i} - \mathbb{E}_{\psi^{(k)}}(X_{-i}))).$$

Hence

$$\frac{1}{n} \|S_n - S_n^{(k)}\|_2^2 = \frac{1}{n} \sum_{N=0}^{n-1} \left(\mathbb{E}\left((Y_0^{(k)})^2\right) + 2 \sum_{i=1}^{N-1} \mathbb{E}\left(X_0(X_{-i} - \mathbb{E}_{\psi^{(k)}}(X_{-i}))\right) \right).$$

Now Proposition 6(a) follows from (2.5) and the above inequality via the Cesaro mean convergence theorem.

In order to prove (b), we will use the following elementary Lemma.

Lemma 1 *Let $(B, \|\cdot\|)$ be a Banach space. Assume that the sequences $(u_{n,k})$, (u_n) and (v_k) of elements of B satisfy*

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|u_{n,k} - u_n\| = 0 \text{ and } \lim_{n \rightarrow +\infty} u_{n,k} = v_k.$$

Then the sequence (v_k) converges in B .

Let $B = \mathbb{L}^2(\mathcal{I})$. We now apply Lemma 1 with

$$v_k = \sqrt{\eta^{(k)}}, \quad u_n = n^{-1/2} \mathbb{E}^{1/2}(S_n^2 | \mathcal{I}) \quad \text{and} \quad u_{n,k} = n^{-1/2} \mathbb{E}^{1/2}((S_n^{(k)})^2 | \mathcal{I}).$$

From the triangle inequality applied conditionally to \mathcal{I} , we get that

$$n^{-1/2} \|S_n - S_n^{(k)}\|_2 \geq \|u_{n,k} - u_n\|_2.$$

Hence, by Proposition 6(a),

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|u_{n,k} - u_n\|_2 = 0.$$

Now Theorem 1(a) and the Cesaro mean convergence theorem together imply that $u_{n,k}^2$ converges to v_k^2 in $\mathbb{L}^1(\mathcal{I})$. Since the random variables $u_{n,k}$ and u_n are nonnegative, it follows that $u_{n,k}$ converges to v_k in $\mathbb{L}^2(\mathcal{I})$, which completes the proof of Proposition 6(b).

5 Markov chains

5.1 Proof of Proposition 2

Existence of μ . Since f is continuous, the chain ξ_i is weak Feller (cf. Meyn and Tweedie (1993) Chapter 6). Therefore, to prove the existence of an invariant probability μ it suffices to show (cf. Meyn and Tweedie, Theorem 12.3.4) that $KV \leq V - 1 + b\mathbb{1}_F$ for some positive function V , some compact set F and some positive constant b .

Let $V(x) = |x|$. By definition, $KV(x) = \mathbb{E}(|\xi_{n+1}| \mid \xi_n = x)$. Hence

$$\begin{aligned} KV(x) &\leq |f(x)| + \mathbb{E}|\varepsilon_0| \\ &\leq |x| + C(1 - \delta)^{-1}(1 - (1 + |x|)^{1-\delta}) + \mathbb{E}|\varepsilon_0|. \end{aligned}$$

Let R be a positive real such that

$$C(1 - \delta)^{-1}[1 - (1 + |x|)^{1-\delta}] + \mathbb{E}|\varepsilon_0| \leq -1 \text{ for any } |x| \geq R.$$

Then $KV \leq V - 1 + b\mathbb{1}_{[-R,R]}$ and the existence of μ follows.

Uniqueness of μ . We denote by $(\xi_n^x)_{n \geq 0}$ the chain starting from $\xi_0 = x$. To prove the uniqueness of the invariant probability μ , it suffices to show (see Duflo (1996) Proposition 1.IV.22) that for any (x, y) in \mathbb{R}^2 :

$$\lim_{n \rightarrow +\infty} \mathbb{E}|\xi_n^x - \xi_n^y| = 0. \quad (5.1)$$

Since $|\xi_n^x - \xi_n^y| = |f(\xi_{n-1}^x) - f(\xi_{n-1}^y)|$, we have

$$|\xi_n^x - \xi_n^y| \leq \left(1 - \frac{C}{(1 + \max(|\xi_{n-1}^x|, |\xi_{n-1}^y|))^\delta}\right) |\xi_{n-1}^x - \xi_{n-1}^y|. \quad (5.2)$$

Set $\alpha(t) = 1 - C(1+t)^{-\delta}$ and $\Sigma_k = |\varepsilon_1| + \dots + |\varepsilon_k|$. Noting that, for (x, y) in \mathbb{R}^2 , $\max(|\xi_{n-1}^x|, |\xi_{n-1}^y|) \leq |x| + |y| + \Sigma_{n-1}$, and iterating (5.2) n times, we get

$$|\xi_n^x - \xi_n^y| \leq \alpha^n(|x| + |y| + \Sigma_{n-1})|x - y|.$$

So, it remains to control $I_n := \mathbb{E}(\alpha^n(|x| + |y| + \Sigma_{n-1}))$. With this aim in view, we write

$$\begin{aligned} I_n &= n \int_0^1 \mathbb{P}\left(\alpha(|x| + |y| + \Sigma_{n-1}) > \lambda\right) \lambda^{n-1} d\lambda \\ &= n \int_0^1 \mathbb{P}\left(1 + |x| + |y| + \Sigma_{n-1} > [C/u]^{1/\delta}\right) (1-u)^{n-1} du. \end{aligned} \quad (5.3)$$

Clearly,

$$I_n \leq \left[1 - \frac{C}{(2[1 + |x| + |y|])^\delta} \right]^n + n \int_0^1 \mathbb{P}\left(2\Sigma_{n-1} > [C/u]^{1/\delta}\right) (1-u)^{n-1} du.$$

The first term on the right hand side tends to zero as n tends to infinity. To control the second term, which we denote by $I_n^{(1)}$, we apply Markov's inequality:

$$\begin{aligned} I_n^{(1)} &\leq \frac{2n(n-1)\mathbb{E}|\xi_0|}{C^{1/\delta}} \int_0^1 u^{1/\delta} (1-u)^{n-1} du \\ &\leq \frac{2n\mathbb{E}|\xi_0|}{C^{1/\delta}} \int_0^{n-1} \left[\frac{v}{n-1} \right]^{1/\delta} \left[1 - \frac{v}{n-1} \right]^{n-1} dv \\ &\leq \frac{2n\mathbb{E}|\xi_0|}{[C(n-1)]^{1/\delta}} \int_0^{+\infty} v^{1/\delta} e^{-v} dv. \end{aligned} \quad (5.4)$$

Since $\delta < 1$, $I_n^{(1)}$ tends to zero as n tends to infinity. Consequently (5.1) holds, and the invariant probability μ is unique.

Moment of μ . Let us consider the function $V(x) = |x|^S$. Since

$$KV(x) = \mathbb{E}(|\xi_{n+1}|^S | \xi_n = x) = \mathbb{E}(|f(x) + \varepsilon_{n+1}|^S),$$

we have

$$\begin{aligned} \frac{[KV(x)]^{1/S}}{|x|} &\leq \frac{(|f(x)| + \|\varepsilon_0\|_S)}{|x|} \\ &\leq 1 + \frac{1}{|x|^\delta} \left(\frac{C}{(1-\delta)} \left[\frac{1}{|x|^{1-\delta}} - \left(1 + \frac{1}{|x|}\right)^{1-\delta} \right] + \frac{\|\varepsilon_0\|_S}{|x|^{1-\delta}} \right). \end{aligned}$$

From this inequality, we infer that there exist two positive constants R and c such that $|x|^{-1}[KV(x)]^{1/S} \leq 1 - c|x|^{-\delta}$ for any $|x| \geq R$. It follows that

$$KV(x) \leq V(x) - c|x|^{S-\delta} + b\mathbb{1}_{[-R,R]}(x).$$

Iterating this inequality n times gives

$$\frac{c}{n} \sum_{k=1}^n \int |y|^{S-\delta} K^k(x, dy) \leq \frac{1}{n} KV(x) + \frac{b}{n} \sum_{k=1}^n K^k([-R, R])(x).$$

Letting $n \rightarrow +\infty$, we get that

$$\int |x|^{S-\delta} \mu(dx) \leq \frac{b}{c} \mu([-R, R]) < +\infty.$$

5.2 Proof of Proposition 3

Let g be any L -Lipschitz function. We have:

$$\begin{aligned} |K^n g(x) - \mu(g)| &\leq \left| \int [K^n g(x) - K^n g(y)] \mu(dy) \right| \\ &\leq \int \mathbb{E} |g(\xi_n^x) - g(\xi_n^y)| \mu(dy) \\ &\leq L \int \mathbb{E} |\xi_n^x - \xi_n^y| \mu(dy). \end{aligned}$$

Using the same notations as in the proof of the uniqueness of μ , we have:

$$|K^n g(x) - \mu(g)| \leq L \int \mathbb{E}(\alpha^n(|x| + |y| + \Sigma_{n-1})) |x - y| \mu(dy). \quad (5.5)$$

Here again, we need to control the term $I_n := \mathbb{E}(\alpha^n(|x| + |y| + \Sigma_{n-1}))$. Set $\Gamma_{n-1} = \Sigma_{n-1} - (n-1)\mathbb{E}|\varepsilon_0|$. Starting from (5.3), we write:

$$I_n = n \int_0^1 \mathbb{P}\left(\Gamma_{n-1} > [C/u]^{1/\delta} - [1 + |x| + |y| + (n-1)\mathbb{E}|\varepsilon_0|]\right) (1-u)^{n-1} du.$$

Set $A_n(x, y) = C[2(1 + |x| + |y| + (n-1)\mathbb{E}|\varepsilon_0|)]^{-\delta}$. We have

$$I_n \leq (1 - A_n(x, y))^n + n \int_0^{A_n(x, y)} \mathbb{P}(2\Gamma_{n-1} > [C/u]^{1/\delta}) (1-u)^{n-1} du$$

To control the second term on the right hand side, which we denote by $I_n^{(2)}$, we use a Fuk-Nagaev type inequality (cf. Petrov (1995), Lemma 2.3). For any $r \geq 1$,

$$\begin{aligned} \mathbb{P}(2\Gamma_{n-1} > [C/u]^{1/\delta}) &\leq (n-1) \mathbb{P}\left(|\varepsilon_0| > \frac{1}{2r} \left[\frac{C}{u}\right]^{1/\delta}\right) \\ &\quad + 2e^r \left[1 + \frac{C^{2/\delta}}{4ru^{2/\delta}(n-1)\mathbb{E}(\varepsilon_0^2)}\right]^{-r}. \end{aligned}$$

Integrating this inequality, we obtain

$$I_n^{(2)} \leq n(n-1) \int_0^1 \mathbb{P}\left(|\varepsilon_0| > \frac{1}{2r} \left[\frac{C}{u}\right]^{1/\delta}\right) (1-u)^{n-1} du + 2 \left[\frac{er\mathbb{E}(\varepsilon_0^2)}{(n-1)\mathbb{E}^2|\varepsilon_0|}\right]^r.$$

Proceeding as in (5.4), we write:

$$\begin{aligned} I_n^{(2)} &\leq \frac{(2r)^S \mathbb{E}(|\varepsilon_0|^S) n(n-1)}{C^{S/\delta}} \int_0^1 u^{S/\delta} (1-u)^{n-1} du + 2 \left[\frac{er\mathbb{E}(\varepsilon_0^2)}{(n-1)\mathbb{E}^2|\varepsilon_0|}\right]^r \\ &\leq \frac{(2r)^S \mathbb{E}(|\varepsilon_0|^S) n}{C^{S/\delta} (n-1)^{S/\delta}} \int_0^{+\infty} v^{S/\delta} e^{-v} dv + 2 \left[\frac{er\mathbb{E}(\varepsilon_0^2)}{(n-1)\mathbb{E}^2|\varepsilon_0|}\right]^r. \end{aligned}$$

Taking $r = S/\delta - 1$ provides $I_n^{(2)} = O(n^{1-S/\delta})$. Consequently, there exists a constant M such that:

$$I_n \leq [1 - A_n(x, y)]^n + Mn^{1-S/\delta}.$$

Hence from (5.5) we obtain

$$\begin{aligned} |K^n g(x) - \mu(g)| &\leq LMn^{1-S/\delta} \int |x - y| \mu(dy) \\ &+ L \int [1 - A_n(x, y)]^n |x - y| \mu(dy). \end{aligned} \quad (5.6)$$

Set $B_n(x) = C[4(1 + |x| + (n-1)\mathbb{E}|\varepsilon_0|)]^{-\delta}$ and denote by J_n the second term on the right hand side. We have

$$\begin{aligned} J_n &\leq L(1 - B_n(x))^n \int |x - y| \mu(dy) \\ &+ nL \int_0^{B_n(x)} \left[\int |x - y| \mathbb{I}_{(4|y| > [C/u]^{1/\delta})} \mu(dy) \right] (1 - u)^{n-1} du. \end{aligned}$$

Since $x \rightarrow |x|^{S-\delta}$ belongs to $\mathbb{L}^1(\mu)$, we have the finite upper bound

$$\int |x - y| \mathbb{I}_{(4|y| > [C/u]^{1/\delta})} \mu(dy) \leq \left[\frac{4^\delta u}{C} \right]^{\frac{(S-1)}{\delta}-1} \int |x - y| |y|^{S-\delta-1} \mu(dy).$$

Proceeding as in (5.4) we find:

$$\begin{aligned} J_n &\leq L(1 - B_n(x))^n \int |x - y| \mu(dy) \\ &+ \left[\frac{4^\delta}{C(n-1)} \right]^{\frac{(S-1)}{\delta}-1} \frac{nL}{n-1} \left[\int_0^{+\infty} v^{\frac{(S-1)}{\delta}-1} e^{-v} dv \right] \int |x - y| |y|^{S-\delta-1} \mu(dy). \end{aligned}$$

From the above inequality and (5.6), we infer that there exists a constant R such that

$$\begin{aligned} |K^n g(x) - \mu(g)| &\leq LMn^{1-S/\delta} \int |x - y| \mu(dy) \\ &+ L(1 - B_n(x))^n \int |x - y| \mu(dy) \\ &+ Rn^{1-(S-1)/\delta} \int |x - y| |y|^{S-\delta-1} \mu(dy). \end{aligned} \quad (5.7)$$

Since g is L -Lipschitz $|g(x)| \leq L(|x| + a)$ for some positive constant a . Now $\mu(g) = 0$, which gives

$$\begin{aligned} \mu(|gK^n g|) &\leq L^2 M n^{1-S/\delta} \iint (|x| + a)|x - y| \mu(dx) \mu(dy) \\ &+ L^2 \iint (1 - B_n(x))^n (|x| + a)|x - y| \mu(dx) \mu(dy) \quad (5.8) \\ &+ LR n^{1-(S-1)/\delta} \iint (|x| + a)|x - y| |y|^{S-\delta-1} \mu(dx) \mu(dy). \end{aligned}$$

Since $S \geq 2 + \delta$, from Proposition 2 we obtain that x^2 is μ -integrable. This implies that the first integral on the right hand side is finite. Moreover $|y|^{S-\delta}$ is μ -integrable, which implies the convergence of the third integral on the right hand side. Hence we deduce from (5.8) that a sufficient condition for the convergence of the series $\sum_{n>0} \mu(|gK^n g|)$ is

$$\sum_{n=1}^{+\infty} \iint (1 - B_n(x))^n (|x| + a)|x - y| \mu(dx) \mu(dy) < +\infty.$$

By Fubini's theorem and the fact that $(1 - B_n(x))^n \leq \exp(-nB_n(x))$ it suffices to prove that

$$\iint \left(\sum_{n=1}^{+\infty} e^{-nB_n(x)} \right) (|x| + a)|x - y| \mu(dx) \mu(dy) < +\infty. \quad (5.9)$$

If $|x| \leq 1$

$$\sum_{n=1}^{+\infty} e^{-nB_n(x)} \leq \sum_{n=1}^{+\infty} \exp\left(-Cn(8 + 4n\mathbb{E}|\varepsilon_0|)^{-\delta}\right) < +\infty. \quad (5.10)$$

If $|x| > 1$

$$\begin{aligned} \sum_{n=1}^{+\infty} e^{-nB_n(x)} &\leq \int_0^{+\infty} \exp\left(-Cy(4[1 + |x| + y\mathbb{E}|\varepsilon_0|])^{-\delta}\right) dy \\ &\leq \int_0^{+\infty} \exp\left(-Cy|x|^{-\delta}(8 + 4y|x|^{-\delta}\mathbb{E}|\varepsilon_0|)^{-\delta}\right) dy \\ &\leq |x|^\delta \int_0^{+\infty} \exp\left(-Cz(8 + 4z\mathbb{E}|\varepsilon_0|)^{-\delta}\right) dz. \quad (5.11) \end{aligned}$$

From (5.9) (5.10) and (5.11) we conclude that a sufficient condition for the convergence of $\sum_{n>0} \mu(|gK^n g|)$ is

$$\iint |x|^\delta (|x| + a)|x - y| \mu(dx) \mu(dy) < +\infty,$$

which is realized as soon as $x \rightarrow |x|^{2+\delta}$ belongs to $\mathbb{L}^1(\mu)$. Now, by Proposition 2, the function $|x|^{2+\delta}$ is μ -integrable if $S \geq 2 + 2\delta$, which concludes the proof of Proposition 3.

6 Weakly dependent sequences

In this section, we prove Corollaries 1 and 2. First, write

$$\mathbb{E}|X_0\mathbb{E}_0(X_k)| = \text{Cov}(|X_0|(\mathbb{1}_{\mathbb{E}_0(X_k)>0} - \mathbb{1}_{\mathbb{E}_0(X_k)\leq 0}), X_k). \quad (6.1)$$

Now, by the covariance inequality of Rio (1993),

$$\text{Cov}(|X_0|(\mathbb{1}_{\mathbb{E}_0(X_k)>0} - \mathbb{1}_{\mathbb{E}_0(X_k)\leq 0}), X_k) \leq 4 \int_0^{\alpha(\mathcal{M}_0, \sigma(X_k))} Q_{X_0}^2(u) du. \quad (6.2)$$

Collecting (6.1) and (6.2) we obtain Corollary 1. Before proving Corollary 2, let us recall the covariance inequality of Delyon (1990).

Proposition 7 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and \mathcal{U}, \mathcal{V} two σ -algebras of \mathcal{A} . There exist two random variables $d_{\mathcal{U}}$ and $d_{\mathcal{V}}$ from $(\Omega, \mathcal{A}, \mathbb{P})$ to $[0, 1]$ respectively \mathcal{U} and \mathcal{V} -measurable, with $\mathbb{E}(d_{\mathcal{U}}) = \mathbb{E}(d_{\mathcal{V}}) = \beta(\mathcal{U}, \mathcal{V})$ and such that the following holds: For any conjugate exponents p and q , and any random vector (X, Y) in $\mathbb{L}^p(\mathcal{U}) \times \mathbb{L}^q(\mathcal{V})$,*

$$|\text{Cov}(X, Y)| \leq 2\mathbb{E}^{1/p}(d_{\mathcal{U}}|X|^p)\mathbb{E}^{1/q}(d_{\mathcal{V}}|Y|^q).$$

Proceeding as in Viennet (1997), we apply Proposition 7 to the sequence $X_i = g(\xi_i)$, where the variable ξ_0 is \mathcal{M}_0 -measurable. There exist two random variables d_{k, \mathcal{M}_0} and $d_{\mathcal{M}_0, k}$ respectively \mathcal{M}_0 and $\sigma(\xi_k)$ -measurable, with $\mathbb{E}(d_{k, \mathcal{M}_0}) = \mathbb{E}(d_{\mathcal{M}_0, k}) = \beta(\mathcal{M}_0, \sigma(\xi_k))$ and such that

$$\begin{aligned} \text{Cov}(|X_0|(\mathbb{1}_{\mathbb{E}_0(X_k)>0} - \mathbb{1}_{\mathbb{E}_0(X_k)\leq 0}), X_k) &\leq 2\mathbb{E}^{1/2}(d_{k, \mathcal{M}_0}|X_0|^2)\mathbb{E}^{1/2}(d_{\mathcal{M}_0, k}|X_k|^2) \\ &\leq \mathbb{E}(d_{k, \mathcal{M}_0}|X_0|^2) + \mathbb{E}(d_{\mathcal{M}_0, k}|X_k|^2). \end{aligned}$$

Since $d_{\mathcal{M}_0, k}$ is $\sigma(\xi_k)$ -measurable, it may be written as $d_{\mathcal{M}_0, k} = D_{\mathcal{M}_0, k}(\xi_k)$. Using (6.1) and the stationarity of the sequence $(\xi_i)_{i \in \mathbb{Z}}$, we obtain

$$\mathbb{E}|X_0\mathbb{E}_0(X_k)| \leq \mathbb{E}([d_{k, \mathcal{M}_0} + D_{\mathcal{M}_0, k}(\xi_0)]|X_0|^2). \quad (6.3)$$

Put $b_k = [d_{k, \mathcal{M}_0} + D_{\mathcal{M}_0, k}(\xi_0)]/2$ and $B = \sum_{n>0} b_n$. The sequence $(b_k)_{k>0}$ satisfies $\mathbb{E}(b_k) = \beta(\mathcal{M}_0, \sigma(\xi_k))$. Moreover, we deduce from (6.3) that if X_0 belongs to $\mathbb{L}^2(B\mathbb{P})$ then the series $\sum_{k>0} \|X_0\mathbb{E}_0(X_k)\|_1$ is convergent. This completes the proof of Corollary 2.

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