Exponential inequalities and functional central limit theorems for random fields

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Abstract

We establish new exponential inequalities for partial sums of random fields. Next, using classical chaining arguments, we give sufficient conditions for partial sum processes indexed by large classes of sets to converge to a set-indexed Brownian motion. For stationary fields of bounded random variables, the condition is expressed in terms of a series of conditional expectations. For non-uniform ϕ-mixing random fields, we require both finite fourth moments and an algebraic decay of the mixing coefficients.

Résumé

Nous établissons des inégalités exponentielles pour des sommes partielles issues de champs aléatoires. En utilisant des arguments de chaînage classiques, nous donnons ensuite des conditions suffisantes pour que des processus de sommes partielles indexés par de grandes classes d'ensembles convergent vers un mouvement Brownien. Pour les champs stationnaires de variables aléatoires bornées, la condition fait intervenir une série d'espérances conditionnelles. Dans le cas des champs non uniformément ϕ-mélangeants, nous supposons l'existence de moments d'ordre quatre ainsi qu'une decroissance algébrique des coefficients de mélange.

1 Introduction

Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary field of real-valued random variables with mean zero and finite variance. If $\mathcal{A}$ is a collection of Borel subsets of $[0,1]^d$, define the smoothed partial sum process $\{S_n(A) : A \in \mathcal{A}\}$ by

$$S_n(A) = \sum_{i \in \mathbb{Z}^d} \lambda(nA \cap R_i) X_i \,, \tag{1.1}$$

where $R_i =]i_1 - 1, i_1] \times \cdots \times]i_d - 1, i_d]$ is the unit cube with upper corner at i and λ is the Lebesgue measure on $\mathbb{R}^d$. In a recent paper (cf. Dedecker (1998)) we prove that the sequence $n^{-d/2} S_n(A)$ converges in distribution to a mixture of Gaussian laws provided that the following $\mathbb{L}^1$-projective criterion is satisfied

$$\sum_{k \in \mathbb{Z}^d} \|X_k \mathbb{E}(X_0 | \mathcal{F}_k)\|_1 < \infty \quad \text{where} \quad \mathcal{F}_k = \sigma(X_i, |i| \geq |k|) \,. \tag{1.2}$$

This condition is weaker than martingale-type assumptions and provides optimal results for mixing random fields.

The next step is to study the asymptotic behavior of the sequence of processes $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$. To be precise we focus on the following property: the sequence $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$ is said to satisfy a functional central limit theorem if it converges in distribution to a mixture of Brownian motions in the space $C(\mathcal{A})$ of continuous real functions on $\mathcal{A}$ equipped with the metric of uniform convergence.

To measure the size of $\mathcal{A}$ one usually considers the metric entropy with respect to the Lebesgue measure. Dudley (1973) proves the existence of a standard Brownian motion with sample paths in $C(\mathcal{A})$ as soon as $\mathcal{A}$ has finite entropy integral (i.e. Condition (2.1) of Section 2 holds). Using the more restrictive notion of entropy with inclusion, Bass (1985) and simultaneously Alexander and Pyke (1986) establish a functional central limit theorem for partial sums of i.i.d. random fields. Bass's approach is mainly based on Bernstein's inequality for sums of independent random variables, which allows an adaptative truncation of the variables in the chaining procedure.

More generally, the problem of establishing tightness for Banach-valued random sequences is strongly related, via chaining arguments, to the existence of exponential bounds (see e.g. Ledoux and Talagrand (1991)). Therefore our first objective is to build tractable inequalities for partial sums of random fields. In Proposition 1, we establish upper bounds for $\mathbb{L}^p$-norms of

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partial sums by adapting a scheme of our own (cf. Dedecker (1998) Section 5.2). Proposition 1(a) is an extension of Burkholder's inequality: the upper bound consists in a series of conditional expectations which reduces to a single term in the particular case of martingale-differences random fields. Proposition 1(b) is comparable to Rosenthal-type inequality: The upper bound consists in a variance term and in several sums of conditional expectations.

Next, optimizing in p these inequalities (as done in Doukhan, León and Portal (1984)), we obtain exponential bounds for partial sums of bounded random fields. Corollary 3(a) generalizes Azuma's inequality, while Corollary 3(b) is comparable to Bernstein's. In particular, these inequalities apply to non-uniform ϕ-mixing random fields under fairly mild conditions: we obtain Hoeffding and Bernstein-type bounds by assuming only an algebraic decay of the coefficients (see Corollary 4).

We now go back to our original interest. Denote by "$\mathbb{L}^p$ criterion" the projective condition obtained from (1.2) by replacing $\mathbb{L}^1$-norms by $\mathbb{L}^p$-norms. With the help of the above inequalities, we are in position to prove the tightness of the sequence of processes $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$ in the following situations:

- When $\mathcal{A}$ is the collection of lower-left quadrants, we establish in Theorem 1 a functional central limit theorem for random fields satisfying a $\mathbb{L}^p$ criterion for some $p > 1$. As a straightforward consequence, we deduce an α-mixing condition which improves on Chen's (1991) result.

- When X_0 is bounded and $\mathcal{A}$ satisfies Dudley's entropy condition, we prove in Theorem 2 a functional central limit theorem under the $\mathbb{L}^\infty$ criterion. Applied to non-uniform ϕ-mixing random fields, this criterion provides the same condition as for finite-dimensional convergence: we only require that the sum over $\mathbb{Z}^d$ of the coefficients is finite. Moreover, the $\mathbb{L}^\infty$ criterion is satisfied for bounded lattice spin systems in the so-called *weak-mixing* region. In particular it applies to the 2D-Ising model with external field in the whole interior of the uniqueness region.

- When $(X_i)_{i \in \mathbb{Z}^d}$ is a non-uniform ϕ-mixing random field and $\mathcal{A}$ satisfies Dudley's entropy condition (using entropy with inclusion), we prove in Theorem 3 a functional central limit theorem by assuming an algebraic decay of the coefficients. More precisely the mixing rate is related to the moments of the variables: if the ϕ-mixing rate is $O(k^{-b})$ for some b in $]d, 2d[$, we require finite $(2b/(b-d))$-moments. The fastest rate is obtained for random variables

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having only finite fourth moments.

Before presenting our results in more details, let us explain why a ϕ-mixing setting seems to be well adapted to our problem. First, note that the summands in (1.2) are easily controled with the help of ϕ-mixing coefficients (see Section 3.1). In fact, since a mixing coefficient measures the dependence between two σ-algebras $\mathcal{U}$ and $\mathcal{V}$, it allows to bound the conditional expectation of any integrable and $\mathcal{V}$-measurable function with respect to $\mathcal{U}$. As we shall see in Section 5, this property will play an essential role to build a tractable exponential inequality. The functional central limit question for mixing random fields has been already investigated in an early work of Goldie and Greenwood (1986). The mixing coefficients they consider in this paper are uniform ϕ-mixing and β-mixing coefficients (by uniform we mean that the supremum is taken over a collection of $(\mathcal{U}, \mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ may be each generated by an infinite number of variables). The idea is first to apply coupling techniques in order to come down to the independant case and then to adapt Bass's approach. However, as pointed out by Dobrushin (1968, page 205), the notion of uniform mixing may be too restictive: for instance, it is too strong in general for applications to Gibbs random fields (see Remark 4, Section 2.4 for further comments on this question). The coefficients we use in this paper are non-uniform (more precisely the σ-algebra $\mathcal{V}$ is generated by at most two variables).

The paper is organized as follows: Section 2 is devoted to background material and to the functional central limit question. The tools are presented in Section 3: moment inequalities are stated in Proposition 1 and exponential inequalities in Corollaries 3 and 4. The former are proved in Section 4 and the latter in Section 5. In Section 6 we explain how to obtain the finite dimentional convergence from Theorem 2 in Dedecker (1998). Tightness of the partial sum process is proved in Section 7 by combining some of our inequalities with classical chaining arguments.

2 Functional central limit theorems

Let $\mathcal{A}$ be a collection of Borel subsets of $[0, 1]^d$. We focus on the process $\{S_n(A) : A \in \mathcal{A}\}$ defined by (1.1). As a function of A, this process is continuous with respect to the pseudo-metric $d(A, B) = \sqrt{\lambda(A \Delta B)}$.

Denote by $H(\mathcal{A}, \varepsilon)$ the logarithm of the smallest number of open balls of

radius ε with respect to d which form a covering of $\mathcal{A}$. Let $C(\mathcal{A})$ be the space of continuous real functions on $\mathcal{A}$, equipped with the norm $\|.\|_{\mathcal{A}}$ defined by

$$\|f\|_{\mathcal{A}} = \sup_{A \in \mathcal{A}} |f(A)|.$$

A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process with sample paths in $C(\mathcal{A})$ and $\mathrm{Cov}(W(A), W(B)) = \lambda(A \cap B)$. From Dudley (1973), we know that such a process exists as soon as

$$\int_0^1 \sqrt{H(\mathcal{A}, x)} \, dx < \infty. \tag{2.1}$$

We say that the sequence $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$ satisfies a functional central limit theorem if it converges in distribution to a mixture of set-indexed Brownian motions in the space $C(\mathcal{A})$ (which means that the limiting process is of the form ηW, where W is a standard Brownian motion and η is a nonnegative random variable independent of W).

2.1 Preliminary notations

Let us consider the space $\mathbb{R}$ with its borel σ-algebra $\mathcal{B}$. By a real random field we mean a probability space $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mathbb{P})$. Denote by X the identity application from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$, and by X_i the projection from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ defined by $X_i(\omega) = \omega_i$, for any ω in $\mathbb{R}^{\mathbb{Z}^d}$ and i in $\mathbb{Z}^d$. From now on, the application X, or the field of all projections $(X_i)_{i \in \mathbb{Z}^d}$ will designate the whole random field $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mathbb{P})$.

For k in $\mathbb{Z}^d$, define the translation operator T_k from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$ by: $[T_k(\omega)]_i = \omega_{i+k}$. An element A of $\mathcal{B}^{\mathbb{Z}^d}$ is said to be invariant if $T_k(A) = A$ for any k in $\mathbb{Z}^d$. We denote by $\mathcal{I}$ the σ-algebra of all invariant sets. A random field is said to be strictly stationary if $T_k \circ \mathbb{P} = \mathbb{P}$, for any k in $\mathbb{Z}^d$.

On $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, i_2, ..., i_d)$ and $j = (j_1, j_2, ..., j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, ..., d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \leq q < p$. Note that the lexicographic order provides a total ordering of $\mathbb{Z}^d$. Let the sets $\{V_i^k : i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$ be defined as follows: $V_i^1 = \{j \in \mathbb{Z}^d : j <_{lex} i\}$, and for $k \geq 2$:

$$V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d : |i - j| \geq k\} \quad \text{where } |i - j| = \max_{1 \leq k \leq d} |i_k - j_k|.$$

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For any Γ in $\mathbb{Z}^d$, define $\mathcal{F}_\Gamma = \sigma(X_i : i \in \Gamma)$. If $f(X_i)$ belongs to $\mathbb{L}^1(\mathbb{P})$, set

$$\mathbb{E}_k(f(X_i)) = \mathbb{E}(f(X_i)|\mathcal{F}_{V_i^k}).\tag{2.2}$$

The lexicographical ordering appears not veary natural, because it is asymmetric. There are two reasons why we use the σ-field $\mathcal{F}_{V_0^k}$ instead of $\mathcal{F}_k$. Firstly the former is included in the latter, so that for any $p \geq 1$, $\|\mathbb{E}_k(X_0)\|_p$ is smaller than $\|\mathbb{E}(X_0|\mathcal{F}_k)\|_p$ and the $\mathbb{L}^p$ criterion (2.3) below is weaker than the $\mathbb{L}^p$ criterion derived from (1.2). Secondly, when $d = 1$ the σ-field $\mathcal{F}_{V_0^k}$ coincides with the past σ-algebras $\mathcal{M}_k = \sigma(X_i, i \leq k)$, which are the natural ones in that case.

Mixing coefficients for random fields. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Given two σ-algebras $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{A}$, define the ϕ-mixing coefficient and the strong mixing coefficient α by

$$\begin{aligned}
\phi(\mathcal{U}, \mathcal{V}) &= \sup\{\|\mathbb{P}(V|\mathcal{U}) - \mathbb{P}(V)\|_\infty, V \in \mathcal{V}\}, \\
\alpha(\mathcal{U}, \mathcal{V}) &= \sup\{|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)|; U \in \mathcal{U}, V \in \mathcal{V}\}.
\end{aligned}$$

Now, let $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mathbb{P})$ be a real random field and denote by $|\Gamma|$ the cardinality of any subset Γ of $\mathbb{Z}^d$. The coefficients we shall use in the sequel are defined by: for any (k, n) in $\mathbb{N}^2$,

$$\begin{aligned}
\phi_k(n) &= \sup\{\phi(\mathcal{F}_{\Gamma_1}, \mathcal{F}_{\Gamma_2}), |\Gamma_2| \leq k, d(\Gamma_1, \Gamma_2) \geq n\}, \\
\alpha_k(n) &= \sup\{\alpha(\mathcal{F}_{\Gamma_1}, \mathcal{F}_{\Gamma_2}), |\Gamma_2| \leq k, d(\Gamma_1, \Gamma_2) \geq n\},
\end{aligned}$$

where the distance d is defined by $d(\Gamma_1, \Gamma_2) = \min\{|j - i|, i \in \Gamma_1, j \in \Gamma_2\}$. See Notations 3, Section 3.1 for more general mixing coefficients and some of their properties.

2.2 The case of lower-left quadrants

For any p in $[1, \infty]$, consider the following $\mathbb{L}^p$-projective criterion, slightly less restrictive than the $\mathbb{L}^p$ criterion derived from (1.2):

$$\sum_{k \in V_0^1} \|X_k \mathbb{E}_{|k|}(X_0)\|_p < \infty.\tag{2.3}$$

When $d = 1$, the summands are equal to $\|X_0 \mathbb{E}(X_k|\mathcal{M}_0)\|_p$, where $\mathcal{M}_0$ is the past σ-algebra $\mathcal{M}_0 = \sigma(X_i, i \leq 0)$. In that case, Dedecker and Rio (2000)

obtained a functional central limit theorem for the Donsker line under the $\mathbb{L}^1$ criterion. In this section, we consider the general case $d \geq 1$.

If (2.3) holds with $p = 1$, the finite-dimensional convergence of $n^{-d/2}S_n(A)$ is a consequence of a central limit theorem established in Dedecker (1998). We shall see that Criterion (2.3) with $p > 1$ implies the tightness of the sequence $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$ in $C(\mathcal{A})$ when $\mathcal{A}$ is the family of lower-left quadrants.

For any t in $[0,1]^d$, define the lower-left quadrant $[0,t]$ with upper corner at t by: $[0,t] = [0,t_1] \times \cdots \times [0,t_d]$. Denote by $\mathcal{Q}_d$ the collection of lower-left quadrants in $[0,1]^d$, and write $f(t)$ for $f([0,t])$. Obviously $\mathcal{Q}_d$ satisfies condition (2.1).

Theorem 1 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary field of centered random variables. Assume that there exists $p > 1$ such that $\|X_0^2\|_p$ is finite and the $\mathbb{L}^p$ criterion (2.3) is satisfied. Then*

(a) *For the σ-algebra $\mathcal{I}$ of invariant sets defined in Section 2.1, we have*

$$\sum_{k \in \mathbb{Z}^d} \|\mathbb{E}(X_0 X_k | \mathcal{I})\|_p < \infty.$$

We denote by η the nonnegative and $\mathcal{I}$-measurable random variable $\eta = \sum_{k \in \mathbb{Z}^d} \mathbb{E}(X_0 X_k | \mathcal{I})$.

(b) *The sequence $\{n^{-d/2}S_n(t) : t \in [0,1]^d\}$ converges in distribution in $C(\mathcal{Q}_d)$ to $\sqrt{\eta}W$, where W is a standard Brownian motion indexed by $\mathcal{Q}_d$ and independent of $\mathcal{I}$.*

Remark 1. When $d = 1$, Dedecker and Rio (2000) prove that Theorem 1 holds with $p = 1$. Note also that Theorem 2 requires $2 + \epsilon$ moments, whereas Basu and Dorea (1979) show that (b) holds for square-integrable martingale-difference random fields. Consequently, we conjecture that Theorem 2 remains valid for $p = 1$.

For α-mixing random fields, we control the summands in the $\mathbb{L}^p$ criterion (2.3) by combining Rio's inequality (cf. Rio (1994), Theorem 1.1) with a duality argument. We obtain the bound

$$\|X_k \mathbb{E}_{|k|}(X_0)\|_p \leq 4 \Big(\int_0^{\alpha_1(|k|)} Q_{X_0}^{2p}(u) du \Big)^{\frac{1}{p}},$$

where Q_{X_i} is the inverse càdlàg of the tail function $t \to \mathbb{P}(|X_i| > t)$. This leads to the following corollary for α-mixing random fields:

Corollary 1 *Theorem 1 holds if we replace the $\mathbb{L}^p$ criterion (2.3) by*

$$\sum_{k \in \mathbb{Z}^d} \left(\int_0^{\alpha_1(|k|)} Q_{X_0}^{2p}(u) du \right)^{\frac{1}{p}} < \infty \, . \tag{2.4}$$

Let δ be any positive real such that $\mathbb{E}(|X_0|^{2+\delta}) < \infty$. Condition (2.4) is satisfied if there exists a positive real number ϵ such that

$$\sum_{k=1}^{\infty} k^{d-1} \alpha_1^{\frac{\delta}{2+\delta} - \epsilon}(k) < \infty \, .$$

Remark 2. Define the coefficients

$$\alpha_{2,2}(n) = \sup\{\alpha(\sigma(X_i, X_j), \sigma(X_k, X_l)) : d(\{i,j\}, \{k,l\}) \geq n\} \, .$$

If $\alpha_{2,2}(n)$ tends to zero as n tends to infinity, then the σ-algebras $\sigma(X_0, X_k)$ are independent of $\mathcal{I}$ and consequently $\eta = \sigma^2 = \sum_{k \in \mathbb{Z}^d} \mathbb{E}(X_0 X_k)$. This follows from Corollary 2 and Remark 4 in Dedecker (1998).

2.3 The bounded case

In this section, we shall see that the $\mathbb{L}^\infty$ criterion (2.3) implies the tightness of the sequence $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$ in $C(\mathcal{A})$ under Dudley's entropy condition. For any Borel set A in $[0,1]^d$, let ∂A be the boundary of A. We say that A is regular if $\lambda(\partial A) = 0$.

Theorem 2 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary field of bounded and centered random variables. Let $\mathcal{A}$ be a collection of regular Borel sets of $[0,1]^d$ satisfying the entropy condition (2.1). Assume that the $\mathbb{L}^\infty$ criterion (2.3) holds. Then the sequence $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$ converges in distribution in $C(\mathcal{A})$ to $\sqrt{\eta} W$, where W is a standard Brownian motion indexed by $\mathcal{A}$ and independent of $\mathcal{I}$ and η is the nonnegative $\mathcal{I}$-measurable random variable defined in Theorem 1(a).*

Remark 3. As shown in Perera (1997), a regularity assumption on the boundary of A is necessary to ensure the asymptotic normality of $n^{-1/2} S_n(A)$.

Applying an inequality due to Serfling (1968) (cf. inequality (3.2) Section 3.1), we obtain the following corollary for bounded ϕ-mixing random fields:

Corollary 2 *Theorem 2 holds if we replace the $\mathbb{L}^\infty$ criterion (2.3) by*

$$\sum_{k=1}^{\infty} k^{d-1} \phi_1(k) < \infty \, . \tag{2.5}$$

Application to bounded spin systems: Let $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mathbb{P})$ be a strictly stationary random field. Assume that the random variable X_0 is bounded and that $\mathbb{P}$ is a Gibbs measure associated to a finite-range potential (see for instance Martinelli and Olivieri (1994) for a definition of Gibbs measures). For any finite subset Γ of $\mathbb{Z}^d$ define the Gibbs specifications $\pi_{\Gamma,X}$ by

$$\pi_{\Gamma,X} = \mathbb{P}(.|\sigma(X_i : i \in \Gamma^c)). \tag{2.6}$$

Suppose now that the family π satisfies the *weak mixing* condition introduced by Dobrushin and Shlosman (1985) (see also Martinelli and Olivieri (1994), inequality (2.5)). In that case $\mathbb{P}$ is the unique solution of equation (2.6) and the σ-algebra $\mathcal{I}$ is $\mathbb{P}$-trivial. Moreover, there exist two positive constants C_1 and C_2 such that

$$\|\mathbb{E}_k(X_0) - \mathbb{E}(X_0)\|_\infty \le C_1 \exp(-C_2 k). \tag{2.7}$$

Set $Y = (X_i - \mathbb{E}(X_i))_{i \in \mathbb{Z}^d}$. From inequality (2.7) we infer that the $\mathbb{L}^\infty$ criterion is satisfied. Consequently Theorem 2 applies to the stationary random field Y, with $\eta = \sigma^2 = \sum_{k \in \mathbb{Z}^d} \mathrm{Cov}(X_0, X_k)$.

In many interesting cases, the Gibbs specifications may be deduced from the physical properties of the system. The first problem is then to find a probability measure solution of (2.6), which will be a possible law for the whole system (if there are several solutions, one says that there is phase coexistence). In what follows, we present an example of such a system, namely the nearest neighbor Ising model, and recall some recent results concerning this model.

Example : Ising model with external field. For each element x of $\mathbb{Z}^d$, define the ℓ^1-norm $\|x\|_1 = |x_1| + \ldots + |x_d|$. Given a finite subset Γ of $\mathbb{Z}^d$, consider

$$
\begin{aligned}
\mathbb{B}_\Gamma &= \{\{x,y\} : x,y \in \Gamma \text{ and } \|x-y\|_1 = 1\}, \\
\partial \mathbb{B}_\Gamma &= \{\{x,y\} : x \in \Gamma, y \notin \Gamma \text{ and } \|x-y\|_1 = 1\}.
\end{aligned}
$$

For σ and τ in $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ and h in $\mathbb{R}$, define the Hamiltonian by

$$H_{\Gamma,\tau,h}(\sigma) = -\frac{1}{2} \sum_{\{x,y\} \in \mathbb{B}_\Gamma} \sigma(x)\sigma(y) - \frac{1}{2} \sum_{\substack{\{x,y\} \in \partial \mathbb{B}_\Gamma \\ y \notin \Gamma}} \sigma(x)\tau(y) - \frac{h}{2} \sum_{x \in \Gamma} \sigma(x).$$

The Gibbs probability in Γ with boundary condition τ under external field h and at temperature $T = \beta^{-1}$ is defined on Ω as

$$\mu_{\Gamma,\tau,T,h}(\sigma) = \begin{cases} \dfrac{\exp(-\beta H_{\Gamma,\tau,h}(\sigma))}{Z_{\Gamma,\tau,T,h}} & \text{if } \sigma(x) = \tau(x) \text{ for each } x \text{ in } \Gamma^c, \\ 0 & \text{otherwise,} \end{cases}$$

where the partition function $Z_{\Gamma,\tau,T,h}$ is the appropriate normalization. It is well known that for high enough temperature, the influence of the boundary conditions becomes negligible as the size of Γ increases. More precisely, there exists a critical temperature T_c and a uniqueness region $\mathcal{U}$

$$\mathcal{U} = \{(h,T) \in \mathbb{R} \times [0,\infty) \ : \ h \neq 0 \ \text{ or } \ T > T_c\}$$

such that: for any (h,T) in $\mathcal{U}$ and any τ in Ω, the sequence $\mu_{[-n,n]^d,\tau,T,h}$ converges weakly to a strictly stationary and ergodic probability $\mu_{T,h}$ as n tends to infinity. Moreover, if X is a random field with probability $\mu_{T,h}$, the probabilities $\mu_{\Gamma,X,T,h}$ are the Gibbs specifications of $\mu_{T,h}$.

The family $\mu_{\Gamma,X,T,h}$ is weak mixing in the following regions of $\mathcal{U}$:

(a) for any temperature $T > T_c$.

(b) for low temperature and arbitrarily small (not vanishing) field h provided that h/T is large enough.

(c) for any (h,T) in $\mathcal{U}$ if $d = 2$.

Part (a) is due to Higuchi (1993), Theorem 2(i). Part (b) has been proved by Martinelli and Olivieri (1994), Theorems 3.1 and 5.1. Complete analiticity for two-dimensional Ising model (which implies weak mixing) has been established by Schonmann and Shlosman (1995). We refer to the latter for a clear and detailed description of the Ising model.

2.4 The unbounded case

Assume now that $\mathcal{A}$ is totally bounded with inclusion: for each positive ε there exists a finite collection such that for any A in $\mathcal{A}$, there exists A^+ and A^- in $\mathcal{A}(\varepsilon)$ with $A^- \subseteq A \subseteq A^+$ and $d(A^-, A^+) \leq \varepsilon$. Denote by $\mathbb{H}(\mathcal{A}, \varepsilon)$ the logarithm of the cardinality of the smallest such subcollection $\mathcal{A}(\varepsilon)$. The function $\mathbb{H}(\mathcal{A}, .)$ is the entropy with inclusion (or bracketing entropy) of the class $\mathcal{A}$. Assume that $\mathcal{A}$ has a convergent bracketing entropy integral:

$$\int_0^1 \sqrt{\mathbb{H}(\mathcal{A}, x)} \, dx < \infty. \tag{2.8}$$

Theorem 3 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary field of random variables with mean zero and finite variance. Let $\mathcal{A}$ be a collection of regular Borel sets of $[0,1]^d$ satisfying the bracketing entropy condition (2.8). Consider the following assumptions*

(i) $\mathbb{E}(|X_0|^4) < \infty$ *and* $\sum_{k>0} k^{2d-1} \phi_2(k) < \infty$.

(ii) *For some b in $]\,d, 2d\,[$, $\mathbb{E}(|X_0|^{2b/(b-d)}) < \infty$ and $\phi_2(k) = O(k^{-b})$.*

Suppose that one of the two assumption (i) or (ii) is satisfied and define $\sigma^2 = \sum_{k \in \mathbb{Z}^d} \mathbb{E}(X_0 X_k)$. Then the sequence $\{n^{-d/2} S_n(A) : A \in \mathcal{A}\}$ converges in distribution in $C(\mathcal{A})$ to σW, where W is a standard Brownian motion indexed by $\mathcal{A}$.

Remark 4. The functional central limit question for mixing random fields has been already investigated by Goldie and Greenwood (1986) who give conditions in terms of uniform ϕ_∞ and β_∞ coefficients (the latter being less restictive than the former). See also Goldie and Morrow (1986) for a detailed discussion of this question and further references. The main idea is to apply coupling techniques related to β_∞ coefficients in order to approximate sums of dependent random variables by sums of independant variables. However, as first pointed out by Dobrushin (1968, p. 205), uniform mixing is too strong in general for applications to Gibbs fields when $d > 1$. For β-mixing fields, this point has been definitively enlightened by Bradley (1989), who proves in Theorem 1(ii) of his paper that if $\beta_\infty(n)$ tends to zero as n goes to infinity then the random field is m-dependent, even if $d = 1$. He also proves in Theorem 1(i) that this fact remains true for $d > 1$ when considering weaker coefficients, which are natural generalization of classical β-mixing coefficients for random sequences to higher dimension. This means that for $d > 1$, the use of "natural" uniform ϕ or β-mixing coefficients is forbidden. See again Doukhan (1994), Sections 1.3 and 2.2 for more informartions on this subject.

Remark 5. Conditions (2.6) and (i) are in some sense the boundary of condition (ii). The rate $\phi_2(k) = O(k^{-2d})$, close to Condition (i), seems to have a particular signification for Gibbs measures. Indeed, for the covariances decay (which is controlled by the decay of ϕ-mixing coefficients), Laroche (1995) proves that there is no transitory rate between an algebraic decay as k^{-2d} and exponential decay. More precisely, outside the weak mixing region (in particular in the phase transition region), the coefficient $\phi_2(k)$ cannot decrease faster than k^{-2d}.

3 Upper bounds for partial sums

In this section we establish new moment inequalities for partial sums of random fields, which are comparable to classical Burkholder's and Rosenthal's. These inequalities are the main tools to prove tightness of the partial sum process $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$, as we shall see in Section 7. More precisely, Theorem 1 (resp. Theorem 2 and Theorem 3) of the preceding section is a consequence of Proposition 1(a) below (resp. Corollary 3(a) and Corollary 4(b)). Before stating these results, we need more notations.

Notations 1. Define the sets $\{W_{i,j}^k : i \in \mathbb{Z}^d , j \in V_i^1 , k \geq |i-j|\}$ as follows

$$W_{i,j}^k = \{l \in V_i^1 \ : \ d(l, \{i, j\}) \geq k\} \, .$$

Write $W_{i,j}$ for the set $W_{i,j}^{|i-j|}$ and define $W_{i,j}^c = V_i^{|i-j|} \backslash W_{i,j}$. For any measurable function g from $\mathbb{R}^2$ to $\mathbb{R}$ such that $g(X_i, X_j)$ belongs to $\mathbb{L}^1$, set

$$\mathbb{E}_k(g(X_i, X_j)) = \mathbb{E}(g(X_i, X_j)|\mathcal{F}_{W_{i,j}^k}) \, . \tag{3.1}$$

If $j = i$ set $W_{i,j}^k = V_i^k$, so that Notation (3.1) is a natural extension of Notation (2.2).

Notations 2. For any i in $\mathbb{Z}^d$ and any $\alpha \geq 1$, let

$$
\begin{aligned}
b_{i,\alpha}(X) &= \|X_i^2\|_\alpha + \sum_{k \in V_i^1} \|X_k \mathbb{E}_{|k-i|}(X_i)\|_\alpha \\
c_i(X) &= \frac{1}{2}\mathbb{E}(X_i^2) + \sum_{j \in V_i^1} |\mathbb{E}(X_i X_j)| \\
d_{i,\alpha}^{(1)}(X) &= \sum_{j \in V_i^1} \sum_{k \in W_{i,j}^c} \|X_k X_j \mathbb{E}_{|i-j|}(X_i)\|_\alpha \\
d_{i,\alpha}^{(2)}(X) &= \sum_{j \in V_i^1} \sum_{k \in W_{i,j}} \|X_k \mathbb{E}_{d(k,\{i,j\})}(X_j X_i - \mathbb{E}(X_j X_i))\|_\alpha \\
d_{i,\alpha}^{(3)}(X) &= \frac{1}{2} \sum_{j \in V_i^1} \|X_j \mathbb{E}_{|i-j|}(X_i^2 - \mathbb{E}(X_i^2))\|_\alpha \\
\text{and } d_{i,\alpha}(X) &= d_{i,\alpha}^{(1)}(X) + d_{i,\alpha}^{(2)}(X) + d_{i,\alpha}^{(3)}(X) + \|X_i^3\|_\alpha \, .
\end{aligned}
$$

We are now in position to state our main result.

Proposition 1 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a field of centered and square-integrable random variables. Let Γ be a finite subset of $\mathbb{Z}^d$ and set $S_\Gamma(X) = \sum_{i \in \Gamma} X_i$. The following inequalities hold:*

(a) *For any $p \geq 2$,*

$$\|S_\Gamma(X)\|_p \leq \left(2p \sum_{i \in \Gamma} b_{i,p/2}(X)\right)^{\frac{1}{2}}.$$

(b) *For any $p \geq 3$,*

$$\|S_\Gamma(X)\|_p \leq \left(2p \sum_{i \in \Gamma} c_i(X)\right)^{\frac{1}{2}} + \left(3p^2 \sum_{i \in \Gamma} d_{i,p/3}(X)\right)^{\frac{1}{3}}.$$

Remark 6. Assume that the martingale-type condition $\mathbb{E}_1(X_i) = 0$ holds for any i in $\mathbb{Z}^d$. Then $b_{i,p/2} = \|X_i^2\|_{p/2}$ and Proposition 1(a) reduces to

$$\|S_\Gamma(X)\|_p \leq \left(2p \sum_{i \in \Gamma} \|X_i^2\|_{\frac{p}{2}}\right)^{\frac{1}{2}}.$$

Hence Proposition 1(a) is an extension of Burkholder's inequality for martingales (see for instance Hall and Heyde (1980), Theorem 2.10). Note that the constant $\sqrt{p}$ in the above inequality is optimal (see for instance Theorem 4.3 in Pinelis (1994)), and hence it is also optimal for Proposition 1(a). This fact is essential to derive "good" exponential bounds from these inequalities by applying first Markov's inequality of order p and then choosing the optimal p (cf. Corrolary 3(a) and its proof in Section 5).

Inequality of Proposition 1(b) is comparable to Rosenthal's inequality: the first term behaves like a variance term, and the second one involves moments of order p. However, in the martingale case, our inequality has a different structure than the classical Rosenthal's (see again Hall and Heyde Theorem 2.12). In our case the first term is more precise, since we obtain a variance term instead of the conditional expectation of the X_i^2's with respect to the past σ-algebras. Conversely, the second term cannot reduce to the sum of the $\mathbb{L}^p$-norm of the variables. Once again, the constant $\sqrt{p}$ in the first term seems to be the good one (see Pinelis (1994), Theorem 4.1). The second term being distinct from classical Rosenthal-type bounds, it is not clear whether the constant $p^{2/3}$ is optimal or not. However, considering the weaker inequality (5.1) of Section 5, one may think that it has the right behavior.

Optimizing these inequalities in p provides exponential inequalities for partial sums of bounded random fields.

Corollary 3 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a field of bounded and centered random variables.*

(a) *Set $b = \sum_{i \in \Gamma} b_{i,\infty}(X)$. For any positive real x,*

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq \exp\left(\frac{1}{e} - \frac{x^2}{4eb}\right).$$

(b) *Let M and V be two positive numbers such that*

$$M^3 \geq 3 \sum_{i \in \Gamma} d_{i,\infty}(X), \quad and \quad V \geq 2 \sum_{i \in \Gamma} c_i(X).$$

For any positive real x,

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq \exp\left(3 - \frac{x^2}{4e^2V + 2exM^3V^{-1}}\right).$$

Remark 7. Corollary 3(a) is an extension of Azuma's inequality (1967) for martingales. The next step would be to obtain a Bernstein-type bound under a projective criterion involving $b_{i,\infty}(X)$. Unfortunately, such an inequality may fail to hold even in the martingale case (see for instance Pinelis (1994) where optimal bounds for martingales are given). Nevertheless, inequality of Corollary 3(b) is easily comparable to Bernstein's. To be precise, setting $v = \sum_{i \in \Gamma} \|X_i\|_2^2$ and $m = \max\{\|X_i\|_\infty, i \in \Gamma\}$, the denominator in the exponent of Bernstein's inequality is given by $v + xm$ (up to some positive constants), whereas in our case it has the form $V + xM^3V^{-1}$. This loss leads to impose finite fourth moments in order to prove tightness of the partial sum process under ϕ-mixing assumptions (cf. Section 2, Theorem 3).

3.1   Exponential inequalities for ϕ-mixing random fields

Notations 3. Let us introduce more general coefficients than in Section 2.1. For any (k, l) in $(\mathbb{N} \cup \{\infty\})^2$, the double indexed coefficients $\phi_{k,l}$ are defined by:

$$\phi_{k,l}(n) = \sup\{\phi(\mathcal{F}_{\Gamma_1}, \mathcal{F}_{\Gamma_2}), |\Gamma_1| \leq k, |\Gamma_2| \leq l, d(\Gamma_1, \Gamma_2) \geq n\}.$$

Note that these new coefficients are related to the single indexed coefficients of Section 2.1 via the equality $\phi_k = \phi_{\infty,k}$. With the help of these coefficients, we control conditional expectations as well as covariances: From Serfling (1968), we have the upper bounds

$$\|\mathbb{E}_k(f(X_i)) - \mathbb{E}(f(X_i))\|_\infty \leq 2\|f(X_i)\|_\infty \phi_{\infty,1}(k) \qquad (3.2)$$
$$\|\mathbb{E}_k(g(X_i, X_j)) - \mathbb{E}(g(X_i, X_j))\|_\infty \leq 2\|g(X_i, X_j)\|_\infty \phi_{\infty,2}(k). \qquad (3.3)$$

From the covariance inequality of Peligrad (1983), we have

$$|\text{Cov}(X_i, X_j)| \leq 2\phi_{1,1}(|i-j|)\|X_i\|_2\|X_j\|_2. \tag{3.4}$$

For more about these definitions and the mixing properties of random fields, we refer to Doukhan (1994), Sections 1.3 and 2.2.

Combining Corollary 3 with inequalities (3.2), (3.3) and (3.4), we obtain the following corollary for stationary and ϕ-mixing random fields:

Corollary 4 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a strictly stationary field of bounded and centered random variables. Take $m \geq \|X_0\|_\infty$ and $v \geq \|X_0\|_2^2$. For any $(a_i)_{i\in\mathbb{Z}^d}$ in $[-1,1]^{\mathbb{Z}^d}$, write aX for the random field $(a_iX_i)_{i\in\mathbb{Z}^d}$. Set*

$$
\begin{aligned}
B(\phi) &= 1 + \sum_{j\in\mathbb{Z}^d\setminus 0} \phi_{\infty,1}(|j|), \quad C(\phi) = \sum_{j\in\mathbb{Z}^d} \phi_{1,1}(|j|) \quad and \\
D(\phi) &= \sum_{j\in\mathbb{Z}^d\setminus 0} \left((2|j|-1)^d + \frac{1}{2}\right)\phi_{\infty,1}(|j|) + 2\sum_{j\in\mathbb{Z}^d}(2|j|+1)^d\phi_{\infty,2}(|j|).
\end{aligned}
$$

(a) *The following upper bounds hold*

$$b_{i,\infty}(aX) \leq B(\phi)|a_i|m^2, \quad c_i(aX) \leq C(\phi)|a_i|v, \quad d_{i,\infty}(aX) \leq D(\phi)|a_i|m^3.$$

(b) *Set $A(\Gamma) = \sum_{i\in\Gamma}|a_i|$. For any positive real x we have the bounds:*

$$
\begin{aligned}
(i) \quad & \mathbb{P}(|S_\Gamma(aX)| > x) \leq \exp\left(\frac{1}{e} - \frac{x^2}{4B(\phi)A(\Gamma)em^2}\right). \\
(ii) \quad & \mathbb{P}(|S_\Gamma(aX)| > x) \leq \exp\left(3 - \frac{x^2}{8D(\phi)A(\Gamma)e^2v + 3exm^3v^{-1}}\right).
\end{aligned}
$$

Remark 8. Note that inequality (ii) is non-trivial as soon as the series $\sum_{k>0} k^{2d-1}\phi_{\infty,2}(k)$ converges (which implies that $D(\phi)$ is finite). This assumption is much weaker than the one usually required to obtain a Bernstein-type inequality under mixing conditions. For instance Lezaud (1998) and Samson (1998) establish such a bound, respectively for ρ-mixing Markov chains and uniformly ϕ-mixing sequences: in both cases they require an exponential decay of the coefficients.

4 Moment inequality

In this section we prove Proposition 1. We proceed by induction on the cardinality of the set Γ, which we denote by $|\Gamma|$. In each case (a) and (b),

we verify that the result holds when $\Gamma = \{i\}$. Suppose now that the result is true for any random field X and any subset of $\mathbb{Z}^d$ with cardinality $n - 1$, and let Γ be such that $|\Gamma| = n$. To describe Γ we define the one to one map f from $[1, n] \cap \mathbb{N}^*$ to Γ by: f is the unique function such that for $1 \leq m < n \leq |\Gamma|$, we have $f(m) <_{lex} f(n)$. We set $S_{f(k)} = \sum_{i=1}^{k} X_{f(i)}$.

The proof is adapted from Rio (2000). For any real t in [0,1], let

$$u(t) = |S_{f(n-1)} + t X_{f(n)}|^p \quad \text{and} \quad v(t) = \mathbb{E}(u(t)). \qquad (4.1)$$

4.1   Proof of Proposition 1(a)

For the sake of brevity, write b_i for $b_{i, p/2}(X)$. Without loss of generality we may assume that b_i is finite for each i in Γ. If $\Gamma = \{i\}$ then $\mathbb{E}|X_i|^p \leq (2pb_i)^{p/2}$ and (a) holds.

Define $\psi_p(x) = |x|^p(\mathbb{1}_{x>0} - \mathbb{1}_{x\leq 0})$. Using Taylor's expansion, we write

$$u(t) = |S_{f(n-1)}|^p + pt X_{f(n)} \psi_{p-1}(S_{f(n-1)}) + p(p-1) \int_0^1 (1-s) t^2 X_{f(n)}^2 (u(st))^{\frac{p-2}{p}} ds,$$

and consequently

$$u(t) \leq |S_{f(n-1)}|^p + pt X_{f(n)} \psi_{p-1}(S_{f(n-1)}) + p^2 \int_0^t X_{f(n)}^2 (u(s))^{\frac{p-2}{p}} ds. \qquad (4.2)$$

To handle the second term on right hand, we proceed as in Dedecker (1998) Section 5.2. Let m be any one to one map from $[1, n-1] \cap \mathbb{N}^*$ to $\Gamma \backslash \{f(n)\}$ such that $|m(k) - f(n)| \leq |m(k-1) - f(n)|$. Set $S_{m(k)} = \sum_{i=1}^{k} X_{m(i)}$ with the convention $S_{m(0)} = 0$. The above choice of m ensures that $S_{m(k)}$ and $S_{m(k-1)}$ are $\mathcal{F}_{V_{f(n)}^{|m(k)-f(n)|}}$-measurable. Now

$$X_{f(n)} \psi_{p-1}(S_{f(n-1)}) = \sum_{k=1}^{n-1} X_{f(n)}(\psi_{p-1}(S_{m(k)}) - \psi_{p-1}(S_{m(k-1)}))$$

$$= (p-1) \sum_{k=1}^{n-1} X_{f(n)} X_{m(k)} \int_0^1 |S_{m(k-1)} + s X_{m(k)}|^{p-2} ds.$$

Taking the conditional expectation of $X_{f(n)}$ with respect to $\mathcal{F}_{V_{f(n)}^{|m(k)-f(n)|}}$, and applying Hölder's inequality, we obtain

$$\mathbb{E}(X_{f(n)} \psi_{p-1}(S_{f(n-1)}))$$

$$\leq p \sum_{k=1}^{n-1} \|X_{m(k)} \mathbb{E}_{|m(k)-f(n)|}(X_{f(n)})\|_{\frac{p}{2}} \int_0^1 \|S_{m(k-1)} + s X_{m(k)}\|_p^{p-2} ds.$$

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Now, our induction hypothesis yields

$$\|S_{m(k-1)} + sX_{m(k)}\|_p^{p-2} \le \left(2p \sum_{i=1}^{k} b_{m(i)}\right)^{\frac{p}{2}-1} \le \left(2p \sum_{i=1}^{n-1} b_{f(i)}\right)^{\frac{p}{2}-1}.$$

Hence

$$\mathbb{E}(X_{f(n)}\psi_{p-1}(S_{f(n-1)})) \le p \sum_{k=1}^{n-1} \|X_{m(k)}\mathbb{E}_{|m(k)-f(n)|}(X_{f(n)})\|_{\frac{p}{2}} \left(2p \sum_{i=1}^{n-1} b_{f(i)}\right)^{\frac{p}{2}-1}$$

$$\le p \left(b_{f(n)} - \|X_{f(n)}^2\|_{\frac{p}{2}}\right)\left(2p \sum_{i=1}^{n-1} b_{f(i)}\right)^{\frac{p}{2}-1}.$$

Since $p \ge 2$, we infer that

$$pt\mathbb{E}(X_{f(n)}\psi_{p-1}(S_{f(n-1)}))$$

$$\le (2p)^{\frac{p}{2}} \left(\frac{b_{f(n)} - \|X_{f(n)}^2\|_{\frac{p}{2}}}{b_{f(n)}}\right) \left[\left(\sum_{i=1}^{n-1} b_{f(i)} + tb_{f(n)}\right)^{\frac{p}{2}} - \left(\sum_{i=1}^{n-1} b_{f(i)}\right)^{\frac{p}{2}}\right],$$

which yields via (4.1) and (4.2),

$$v(t) \le -(2p)^{\frac{p}{2}} \frac{\|X_{f(n)}^2\|_{\frac{p}{2}}}{b_{f(n)}} \left[\left(\sum_{i=1}^{n-1} b_{f(i)} + tb_{f(n)}\right)^{\frac{p}{2}} - \left(\sum_{i=1}^{n-1} b_{f(i)}\right)^{\frac{p}{2}}\right]$$

$$+ (2p)^{\frac{p}{2}} \left(\sum_{i=1}^{n-1} b_{f(i)} + tb_{f(n)}\right)^{\frac{p}{2}} + p^2 \|X_{f(n)}^2\|_{\frac{p}{2}} \int_0^t (v(s))^{\frac{p-2}{p}} ds. \quad (4.3)$$

Note that the function

$$w(t) = (2p)^{\frac{p}{2}} \left(\sum_{i=1}^{n-1} b_{f(i)} + tb_{f(n)}\right)^{\frac{p}{2}}$$

solves the equation associated to inequality (4.3). The following lemma ensures that $v(t) \le w(t)$ for any t in [0,1], which completes the proof of Proposition 1(a).

Lemma 1 *For any t in $[0,1]$ and any $\beta > 1$, we have $v(t) \le \beta w(t)$.*

Proof. If $X_i = 0$ almost surely for each i in $\Gamma \backslash \{f(n)\}$, then $v(t) \le w(t)$ and the result follows. Else, note that $\beta w(0) > v(0)$. Set

$$t_0 = \sup\{t \in [0,1] : v(s) \le \beta w(s) \text{ for any } s \text{ in } [0,t]\}.$$

We have

$$\begin{aligned}
v(t_0) - \beta w(t_0) &\leq p^2 \|X^2_{f(n)}\|_{\frac{p}{2}} \int_0^{t_0} (v(s))^{\frac{p-2}{p}} - \beta(w(s))^{\frac{p-2}{p}} ds \\
&\leq p^2 \|X^2_{f(n)}\|_{\frac{p}{2}} \int_0^{t_0} (v(s))^{\frac{p-2}{p}} - (\beta w(s))^{\frac{p-2}{p}} ds. \quad (4.4)
\end{aligned}$$

Since v and w are continuous and since $\beta w(0) > v(0)$, we infer that t_0 is positive, and (4.4) implies that $v(t_0) < \beta w(t_0)$. But if $t_0 < 1$ then necessary $v(t_0) = \beta w(t_0)$. Finally $t_0 = 1$ and Lemma 1 is proved.

4.2 Proof of Proposition 1(b)

Instead of Proposition 1(b), we shall prove the following more general result:

Proposition 2 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a field of centered and square-integrable random variables, and N be a fixed positive integer. For any i in $\mathbb{Z}^d$, let*

$$\begin{aligned}
\gamma_i(X) &= \frac{1}{2}\mathbb{E}(X_i^2) + \sum_{j \in V_i^1 \setminus V_i^N} |\mathbb{E}(X_i X_j)| + \sum_{j \in V_i^N} \|X_j \mathbb{E}_{|j-i|}(X_i)\|_{\frac{p}{2}} \\
\delta_i^{(1)}(X) &= \sum_{j \in V_i^1 \setminus V_i^N} \sum_{k \in W_{i,j}^c} \|X_k X_j \mathbb{E}_{|i-j|}(X_i)\|_{\frac{p}{3}} \\
\delta_i^{(2)}(X) &= \sum_{j \in V_i^1 \setminus V_i^N} \sum_{k \in W_{i,j}} \|X_k \mathbb{E}_{d(k,\{i,j\})}(X_j X_i - \mathbb{E}(X_j X_i))\|_{\frac{p}{3}} \\
\delta_i^{(3)}(X) &= \frac{1}{2} \sum_{j \in V_i^1} \|X_j \mathbb{E}_{|i-j|}(X_i^2 - \mathbb{E}(X_i^2))\|_{\frac{p}{3}} \\
and \quad \delta_i(X) &= \delta_i^{(1)} + \delta_i^{(2)} + \delta_i^{(3)} + \|X_i^3\|_{\frac{p}{3}}.
\end{aligned}$$

For any $p \geq 3$, the following inequality holds:

$$\|S_\Gamma(X)\|_p \leq \left(2p \sum_{i \in \Gamma} \gamma_i(X)\right)^{\frac{1}{2}} + \left(3p^2 \sum_{i \in \Gamma} \delta_i(X)\right)^{\frac{1}{3}}. \quad (4.5)$$

Remark 9. Note that Proposition 1(b) follows by letting $N \to +\infty$. Proposition 2 will be used later on to prove Theorem 3 under assumption (ii).

For the sake of brevity, write γ_i for $\gamma_i(X)$ and δ_i for $\delta_i(X)$. Without loss of generality we may assume that γ_i and δ_i are finite for each i in Γ.

If $\Gamma = \{i\}$ then $\mathbb{E}|X_i|^p \leq \delta_i^{p/3}$ and (4.5) holds.

Recall that $\psi_p(x) = |x|^p(\mathbb{1}_{x>0} - \mathbb{1}_{x\leq 0})$. From Taylor's expansion, we have

$$u(t) = |S_{f(n-1)}|^p + ptX_{f(n)}\psi_{p-1}(S_{f(n-1)}) + p(p-1)t^2\frac{X_{f(n)}^2}{2}|S_{f(n-1)}|^{p-2}$$

$$+ p(p-1)(p-2)\int_0^1 \frac{(1-s)^2}{2}t^3 X_{f(n)}^3 \psi_{p-3}(S_{f(n-1)} + stX_{f(n)})ds. \quad (4.6)$$

Starting from this equality, we control each of the terms by applying the induction hypothesis.

The second order terms

First, we make the elementary decomposition $X_{f(n)}^2|S_{f(n-1)}|^{p-2} = I_1 + I_2$, where I_1 and I_2 are defined by

$$I_1 = [X_{f(n)}^2 - \mathbb{E}(X_{f(n)}^2)]|S_{f(n-1)}|^{p-2} \quad \text{and} \quad I_2 = \mathbb{E}(X_{f(n)}^2)|S_{f(n-1)}|^{p-2}.$$

To handle I_1, we use the one to one map m as done in Section 5.1.

$$
\begin{aligned}
I_1 &= \sum_{k=1}^{n-1}[X_{f(n)}^2 - \mathbb{E}(X_{f(n)}^2)](|S_{m(k)}|^{p-2} - |S_{m(k-1)}|^{p-2}) \\
&= (p-2)\sum_{k=1}^{n-1}[X_{f(n)}^2 - \mathbb{E}(X_{f(n)}^2)]X_{m(k)}\int_0^1 \psi_{p-3}(S_{m(k-1)} + sX_{m(k)})\, ds.
\end{aligned}
$$

Taking the conditional expectation of $[X_{f(n)}^2 - \mathbb{E}(X_{f(n)}^2)]$ with respect to $\mathcal{F}_{V_{f(n)}^{|m(k)-f(n)|}}$ and applying Hölder's inequality, we infer that $|\mathbb{E}(I_1)|$ is bounded by

$$p\sum_{k=1}^{n-1}\|X_{m(k)}\mathbb{E}_{|m(k)-f(n)|}([X_{f(n)}^2 - \mathbb{E}(X_{f(n)}^2)])\|_{\frac{p}{3}}\int_0^1\|S_{m(k-1)} + sX_{m(k)}\|_p^{p-3}ds.$$
$$(4.7)$$

Now, our induction hypothesis yields

$$
\begin{aligned}
\|S_{m(k-1)} + sX_{m(k)}\|_p^{p-3} &\leq \left[\left(2p\sum_{i=1}^k \gamma_{m(i)}\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^k \delta_{m(i)}\right)^{\frac{1}{3}}\right]^{p-3} \\
&\leq \left[\left(2p\sum_{i\in\Gamma}\gamma_i\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p-3}. \quad (4.8)
\end{aligned}
$$

Bearing in mind the definition of $\delta_i^{(3)}$, we infer from (4.7) and (4.8) that

$$\frac{1}{2}p^2t^2|\mathbb{E}(I_1)| \leq p^3t\delta_{f(n)}^{(3)}\left[\left(2p\sum_{i\in\Gamma}\gamma_i\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p-3}. \quad (4.9)$$

Let y be the function defined by

$$y(t) = \left[\left(2p \sum_{i \in \Gamma} \gamma_i \right)^{\frac{1}{2}} + \left(3p^2 \left(\sum_{i=1}^{n-1} \delta_{f(i)} + t\delta_{f(n)} \right) \right)^{\frac{1}{3}} \right]^p . \qquad (4.10)$$

Obvious computations show that

$$p^3 \delta_{f(n)} \left[\left(2p \sum_{i \in \Gamma} \gamma_i \right)^{\frac{1}{2}} + \left(3p^2 \left(\sum_{i=1}^{n-1} \delta_{f(i)} + t\delta_{f(n)} \right) \right)^{\frac{1}{3}} \right]^{p-3} \le y'(t) , \qquad (4.11)$$

which together with (4.9) yields

$$\frac{1}{2} p^2 t^2 |\mathbb{E}(I_1)| \le \frac{\delta_{f(n)}^{(3)}}{\delta_{f(n)}} \left(y(t) - y(0) \right) . \qquad (4.12)$$

The first order terms

Notations 4. For any positive integer N, set

$$E_n^N = \Gamma \cap V_{f(n)}^N \quad \text{and} \quad S_{f(n)}^N = \sum_{i \in E_n^N} X_i .$$

We first make the decomposition $X_{f(n)} \psi_{p-1}(S_{f(n-1)}) = I_3 + I_4$, where

$$I_3 = X_{f(n)}(\psi_{p-1}(S_{f(n-1)}) - \psi_{p-1}(S_{f(n)}^N)) \quad \text{and} \quad I_4 = X_{f(n)} \psi_{p-1}(S_{f(n)}^N) .$$

Using again the map m, we have

$$I_3 = \sum_{k=|E_n^N|+1}^{n-1} X_{f(n)}(\psi_{p-1}(S_{m(k)}) - \psi_{p-1}(S_{m(k-1)})) .$$

Applying Taylor's expansion, we write $I_3 = J_1 + J_2$, where

$$J_1 = \sum_{k=|E_n^N|+1}^{n-1} (p-1)X_{f(n)}X_{m(k)}|S_{m(k-1)}|^{p-2} \quad \text{and}$$

$$J_2 = \sum_{k=|E_n^N|+1}^{n-1} (p-1)(p-2)X_{f(n)}X_{m(k)}^2 \int_0^1 (1-s)\psi_{p-3}(S_{m(k-1)} + sX_{m(k)})ds .$$

Notation 5. Define the set

$$G_{n,k} = \{ i \in m([1, k-1] \cap \mathbb{N}^*) \ : \ d(i, \{f(n), m(k)\}) \ge |f(n) - m(k)| \}$$

Let h_k be a one to one map from $[1, k-1] \cap \mathbb{N}^*$ to $m([1, k-1] \cap \mathbb{N}^*)$ such that $d(h_k(i), \{f(n), m(k)\}) \leq d(h_k(i-1), \{f(n), m(k)\})$. For the sake of brevity, we write h for h_k. Set $S_{h(i)} = \sum_{j=1}^{i} X_{h(j)}$ and $S_{h(0)} = 0$.

Now, write $J_1 = K_1 + K_2 + K_3$, where

$$
K_1 = (p-1) \sum_{k=|E_n^N|+1}^{n-1} X_{f(n)} X_{m(k)} (|S_{m(k-1)}|^{p-2} - |S_{h(|G_{n,k}|)}|^{p-2})
$$

$$
K_2 = (p-1) \sum_{k=|E_n^N|+1}^{n-1} [X_{f(n)} X_{m(k)} - \mathbb{E}(X_{f(n)} X_{m(k)})]|S_{h(|G_{n,k}|)}|^{p-2}
$$

$$
K_3 = (p-1) \sum_{k=|E_n^N|+1}^{n-1} \mathbb{E}(X_{f(n)} X_{m(k)})|S_{h(|G_{n,k}|)}|^{p-2} .
$$

Control of K_2

We have

$$
\frac{K_2}{(p-1)} = \sum_{k=|E_n^N|+1}^{n-1} \sum_{i=1}^{|G_{n,k}|} [X_{f(n)} X_{m(k)} - \mathbb{E}(X_{f(n)} X_{m(k)})](|S_{h(i)}|^{p-2} - |S_{h(i-1)}|^{p-2}) ,
$$

and

$$
|S_{h(i)}|^{p-2} - |S_{h(i-1)}|^{p-2} = (p-2) X_{h(i)} \int_0^1 \psi_{p-3}(S_{h(i-1)} + s X_{h(i)}) ds . \quad (4.13)
$$

Write $d(n, k, i)$ for the distance $d(h(i), \{f(n), m(k)\})$ and $W(n, k, i)$ for the set $W_{f(n), m(k)}^{d(n,k,i)}$ (see Notations 1, Section 3, for the definition of this last set). The choice of h ensures that $S_{h(i)}$ and $S_{h(i-1)}$ are $\mathcal{F}_{W(n,k,i)}$-measurable. Taking the conditional expectation of $[X_{f(n)} X_{m(k)} - \mathbb{E}(X_{f(n)} X_{m(k)})]$ with respect to $\mathcal{F}_{W(n,k,i)}$ and applying Hölder's inequality, we infer that

$$
\left| \mathbb{E}\left([X_{f(n)} X_{m(k)} - \mathbb{E}(X_{f(n)} X_{m(k)})] X_{h(i)} \int_0^1 \psi_{p-3}(S_{h(i-1)} + s X_{h(i)}) ds \right) \right|
$$

is bounded by

$$
\| X_{h(i)} \mathbb{E}_{d(n,k,i)}([X_{f(n)} X_{m(k)} - \mathbb{E}(X_{f(n)} X_{m(k)})]) \|_{\frac{p}{3}} \int_0^1 \| S_{h(i-1)} + s X_{h(i)} \|_p^{p-3} ds .
$$

Arguing as for I_1, we use first the induction hypothesis and second the definition of $\delta_i^{(2)}$ to conclude that

$$
pt |\mathbb{E}(K_2)| \leq \frac{\delta_{f(n)}^{(2)}}{\delta_{f(n)}} \Big(y(t) - y(0) \Big) , \quad (4.14)
$$

where y is the function defined by (4.10). This completes the control of K_2.

<center><u>*Control of K_1 and J_2*</u></center>

Again, we write

$$\frac{K_1}{(p-1)} = \sum_{k=|E_n^N|+1}^{n-1} \sum_{i=|G_{n,k}|+1}^{k-1} X_{f(n)} X_{m(k)} \big(|S_{h(i)}|^{p-2} - |S_{h(i-1)}|^{p-2}\big),$$

and we use the expansion (4.13). Since the one to one map h describes the set $m([1, k-1] \cap \mathbb{N}^*)$, the variables $S_{h(i)}$ and $S_{h(i-1)}$ are $\mathcal{F}_{V_{f(n)}^{|m(k)-f(n)|}}$-measurable. Taking the conditional expectation of $X_{f(n)}$ with respect to $\mathcal{F}_{V_{f(n)}^{|m(k)-f(n)|}}$, and applying Hölder's inequality, we infer that $p^{-2}|\mathbb{E}(K_1)|$ is bounded by

$$\sum_{k=|E_n^N|+1}^{n-1} \sum_{i=|G_{n,k}|+1}^{k-1} \|X_m(k) X_{h(i)} \mathbb{E}_{|m(k)-f(n)|}(X_{f(n)})\|_{\frac{p}{3}} \int_0^1 \|S_{h(i-1)} + sX_{h(i)}\|_p^{p-3} ds.$$

$$(4.15)$$

In the same way, $p^{-2}|\mathbb{E}(J_2)|$ is bounded by

$$\sum_{k=|E_n^N|+1}^{n-1} \|X_{m(k)}^2 \mathbb{E}_{|m(k)-f(n)|}(X_{f(n)})\|_{\frac{p}{3}} \int_0^1 \|S_{m(k-1)} + sX_{m(k)}\|_p^{p-3} ds. \quad (4.16)$$

Collecting (4.15) and (4.16) and arguing as for I_1, we first use the induction hypothesis and second the definition of $\delta_i^{(1)}$ to conclude that

$$pt|\mathbb{E}(K_1 + J_2)| \le \frac{\delta_{f(n)}^{(1)}}{\delta_{f(n)}} \Big(y(t) - y(0)\Big), \quad (4.17)$$

where y is the function defined by (4.10). This completes the control of K_1 and J_2.

The remainder terms

Collecting (4.12), (4.14) and (4.17), we have shown that

$$\frac{1}{2}p^2 t^2 |\mathbb{E}(I_1)| + pt|\mathbb{E}(K_1 + K_2 + J_2)| \le \frac{\delta_{f(n)} - \|X_{f(n)}^3\|_{\frac{p}{3}}}{\delta_{f(n)}} \Big(y(t) - y(0)\Big). \quad (4.18)$$

In this section we focus on the remainder terms I_2, I_4 and K_3. We start by I_4. Using again the map m we write

$$
\begin{aligned}
I_4 &= \sum_{k=1}^{|E_n^N|} X_{f(n)} \big(\psi_{p-1}(S_{m(k)}) - \psi_{p-1}(S_{m(k-1)})\big) \\
&= (p-1) \sum_{k=1}^{|E_n^N|} X_{f(n)} X_{m(k)} \int_0^1 |S_{m(k-1)} + sX_{m(k)}|^{p-2} ds.
\end{aligned}
$$

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Taking the conditional expectation of $X_{f(n)}$ with respect to $\mathcal{F}_{V_{f(n)}^{|m(k)-f(n)|}}$, and applying Hölder's inequality, we obtain

$$|\mathbb{E}(I_4)| \leq p \sum_{k=1}^{|E_n^N|} \|X_{m(k)}\mathbb{E}_{|m(k)-f(n)|}(X_{f(n)})\|_{\frac{p}{2}} \int_0^1 \|S_{m(k-1)} + sX_{m(k)}\|_p^{p-2} ds \,.$$

(4.19)

Next, for $|\mathbb{E}(I_2)|$ and $|\mathbb{E}(K_3)|$ we have the upper bounds

$$|\mathbb{E}(I_2)| \;\leq\; \mathbb{E}(X_{f(n)}^2)\|S_{f(n-1)}\|_p^{p-2}$$

(4.20)

$$|\mathbb{E}(K_3)| \;\leq\; p \sum_{k=|E_n^N|+1}^{n-1} |\mathbb{E}(X_{f(n)}X_{m(k)})|\|S_{h(|G_{n,k}|)}\|_p^{p-2} \,.$$

(4.21)

From the induction hypothesis, the terms $\|S_{m(k-1)}+sX_{m(k)}\|_p^{p-2}$, $\|S_{f(n-1)}\|_p^{p-2}$ and $\|S_{h(|G_{n,k}|)}\|_p^{p-2}$ are each bounded by

$$\left[\left(2p\sum_{i=1}^{n-1}\gamma_{f(i)}\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p-2} \,.$$

Bearing in mind the definition of γ_i, we infer from (4.19), (4.20) and (4.21) that

$$\frac{1}{2}p^2t^2|\mathbb{E}(I_2)| + pt|\mathbb{E}(I_4 + K_3)|$$

$$\leq p^2\gamma_{f(n)}\left[\left(2p\sum_{i=1}^{n-1}\gamma_{f(i)}\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p-2} \,. \quad (4.22)$$

Define the function z by

$$z(t) = \left[\left(2p\Big(\sum_{i=1}^{n-1}\gamma_{f(i)} + t\gamma_{f(n)}\Big)\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p} \,.$$

Obvious computations show that

$$p^2\gamma_{f(n)}\left[\left(2p\sum_{i=1}^{n-1}\gamma_{f(i)}\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p-2} \leq z'(t) \,. \quad (4.23)$$

Note that if y is the function defined by (4.10), we have $z(1) = y(0)$. Consequently we conclude from (4.22) and (4.23) that

$$\frac{1}{2}p^2t^2|\mathbb{E}(I_2)| + pt|\mathbb{E}(I_4 + K_3)| \leq y(0) - \left[\left(2p\sum_{i=1}^{n-1}\gamma_{f(i)}\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^{p}$$

(4.24)

which completes the control of the remainder terms.

End of the proof

The induction hypothesis provides the upper bound:

$$\mathbb{E}(|S_{f(n-1)}|^p) \leq \left[\left(2p\sum_{i=1}^{n-1}\gamma_{f(i)}\right)^{\frac{1}{2}} + \left(3p^2\sum_{i=1}^{n-1}\delta_{f(i)}\right)^{\frac{1}{3}}\right]^p. \qquad (4.25)$$

Now if v is the function defined by (4.1), we infer from equation (4.6) and the upper bounds (4.18), (4.24) and (4.25) that

$$v(t) \leq y(t) - \frac{\|X_{f(n)}^3\|_{\frac{p}{3}}}{\delta_{f(n)}}\left(y(t) - y(0)\right) + p^3\|X_{f(n)}^3\|_{\frac{p}{3}}\int_0^t (v(s))^{\frac{p-3}{p}}\,ds. \qquad (4.26)$$

According to inequality (4.11), the function y satisfies

$$y(t) \geq y(t) - \frac{\|X_{f(n)}^3\|_{\frac{p}{3}}}{\delta_{f(n)}}\left(y(t) - y(0)\right) + p^3\|X_{f(n)}^3\|_{\frac{p}{3}}\int_0^t (y(s))^{\frac{p-3}{p}}\,ds. \qquad (4.27)$$

Arguing as in Lemma 1, we conclude from (4.26) and (4.27) that $v(t) \leq y(t)$ for any t in $[0,1]$. This completes the proof of Proposition 1(b).

5 Exponential inequalities

Proof of Corollary 3(a). Without loss of generality, we may assume that b is finite. Applying Markov's inequality, we have, for any positive x and any $p \geq 2$,

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq \min\left(1, \frac{\mathbb{E}|S_\Gamma(X)|^p}{x^p}\right) \leq \min\left(1, \left(\frac{2pb}{x^2}\right)^{\frac{p}{2}}\right).$$

Obvious computations show that the function $p \to (2pbx^{-2})^{p/2}$ has an unique minimum in $p_0 = (2eb)^{-1}x^2$ and is increasing on the interval $[p_0, +\infty]$. By comparing p_0 and 2, we infer that

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq h\left(\frac{x^2}{4eb}\right),$$

where h is the function from $\mathbb{R}_+$ to $\mathbb{R}_+$ defined by

$$h(y) = \begin{cases} 1 & \text{if } y \leq e^{-1} \\ (ey)^{-1} & \text{if } e^{-1} < y \leq 1 \\ e^{-y} & \text{if } y > 1 \end{cases}$$

Finally, Corollary 3(a) follows by noting that $h(y) \leq \exp(-y + e^{-1})$ for any positive y.

Proof of Corollary 3(b). Take M and V as in Corollary 3(b). Note that $p^{2/3}M = (pM^3V^{-1})^{1/3}(\sqrt{pV})^{2/3}$ and consequently $p^{2/3}M \leq pM^3V^{-1} + \sqrt{pV}$. Now, applying Proposition 1(b), we obtain for any real $p \geq 3$,

$$\|S_\Gamma(X)\|_p \leq \sqrt{pV} + p^{2/3}M \leq 2\sqrt{pV} + pM^3V^{-1}, \tag{5.1}$$

and Hölder's inequality yields

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq \min\Big(1, \frac{\mathbb{E}|S_\Gamma(X)|^p}{x^p}\Big) \leq \min\Big(1, \Big(\frac{2\sqrt{pV} + pM^3V^{-1}}{x}\Big)^p\Big). \tag{5.2}$$

Note that if $x = \sqrt{4e^2pV} + epM^3V^{-1}$ then $p \geq x^2(4e^2V + 2exM^3V^{-1})^{-1}$. From this fact and inequality (5.2) we infer that, for any positive x such that $x^2(4e^2V + 2exM^3V^{-1})^{-1} \geq 3$,

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq \exp\Big(\frac{-x^2}{4e^2V + 2exM^3V^{-1}}\Big).$$

In any cases, we conclude that

$$\mathbb{P}(|S_\Gamma(X)| > x) \leq \exp\Big(\frac{-x^2}{4e^2V + 2exM^3V^{-1}} + 3\Big).$$

Proof of Corollary 4. First note that Corollary 4(b) follows straightforwardly from Corollary 4(a) and Corollary 3 (for inequality (ii), take M and V such that $M^3 = 3D(\phi)A(\Gamma)m^3$ and $V = 2D(\phi)A(\Gamma)v$).

In order to prove Corollary 4(a), we bound $b_{i,\infty}(aX)$, $c_i(aX)$ and $d_{i,\infty}(aX)$ with the help of the ϕ-mixing inequalities (3.2), (3.3) and (3.4). From (3.4) and the fact that $(a_i)_{i \in \mathbb{Z}^d}$ belongs to $[-1, 1]^{\mathbb{Z}^d}$, we obtain

$$c_i(aX) \leq \frac{|a_i|}{2} \sum_{j \in \mathbb{Z}^d} |\mathbb{E}(X_iX_j)| \leq |a_i|v \sum_{j \in \mathbb{Z}^d} \phi_{1,1}(|j|). \tag{5.3}$$

This gives the expression of the constant $C(\phi)$. In the same way, we obtain from (3.2) the upper bound

$$b_{i,\infty}(aX) \leq |a_i|\Big(m^2 + \sum_{j \in V_i^1} \|X_j\mathbb{E}_{|i-j|}(X_i)\|_\infty\Big) \leq |a_i|m^2\Big(1 + \sum_{j \in \mathbb{Z}^d \setminus 0} \phi_{\infty,1}(|j|)\Big) \tag{5.4}$$

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which gives the expression of $B(\phi)$. Next, using again (3.2), we get

$$d_{i,\infty}^{(3)}(aX) \leq \frac{|a_i|}{2} \sum_{j \in V_i^1} \|X_j \mathbb{E}_{|i-j|}(X_i^2 - \mathbb{E}(X_i^2))\|_\infty \leq \frac{|a_i|m^3}{2} \sum_{j \in \mathbb{Z}^d \setminus 0} \phi_{\infty,1}(|j|)\,.$$

(5.5)

From (3.2) and the fact that $|W_{i,j}^c| \leq (2|i-j|-1)^d$, we obtain

$$\begin{aligned}
d_{i,\infty}^{(1)}(aX) &\leq |a_i| \sum_{j \in V_i^1} \sum_{k \in W_{i,j}^c} \|X_k X_j \mathbb{E}_{|i-j|}(X_i)\|_\infty \\
&\leq |a_i|m^3 \sum_{j \in \mathbb{Z}^d \setminus 0} (2|j|-1)^d \phi_{\infty,1}(|j|)\,.
\end{aligned}$$

(5.6)

It remains to bound up $d_{i,\infty}^{(2)}(aX)$. Note that, for a fixed positive integer l and (i,j) in $(\mathbb{Z}^d)^2$, there exist at most $2[(2l+1)^d - (2l-1)^d]$ elements of $\mathbb{Z}^d$ such that $d(k, \{i,j\}) = l$. From (3.3) and the definition of $W_{i,j}$, we write

$$\begin{aligned}
d_{i,\infty}^{(2)}(aX) &\leq |a_i| \sum_{j \in V_i^1} \sum_{k \in W_{i,j}} \|X_k \mathbb{E}_{d(k,\{i,j\})}(X_j X_i - \mathbb{E}(X_j X_i))\|_\alpha \\
&\leq 4|a_i|m^3 \sum_{j \in V_i^1} \sum_{l>0} \mathbb{1}_{|i-j| \leq l}\,[(2l+1)^d - (2l-1)^d]\phi_{\infty,2}(l)\,.
\end{aligned}$$

Now, the cardinality of the set $\{j \in V_i^1 : |i-j| \leq l\}$ is less than $(2l+1)^d/2$, and we obtain

$$\begin{aligned}
d_{i,\infty}^{(2)}(aX) &\leq 2|a_i|m^3 \sum_{l>0} (2l+1)^d [(2l+1)^d - (2l-1)^d]\phi_{\infty,2}(l) \\
&\leq 2|a_i|m^3 \sum_{j \in \mathbb{Z}^d \setminus 0} (2|j|+1)^d \phi_{\infty,2}(|j|)\,.
\end{aligned}$$

(5.7)

Recall that $d_{i,\infty}(aX) = d_{i,\infty}^{(1)}(aX) + d_{i,\infty}^{(2)}(aX) + d_{i,\infty}^{(3)}(aX) + \|(a_i X_i)^3\|_\infty$. Since $|a_i| \leq 1$ it follows that $\|(a_i X_i)^3\|_\infty \leq 2|a_i|m^3\phi_{\infty,2}(0)$. This inequality together with (5.5), (5.6) and (5.7) gives the expression of the constant $D(\phi)$ and the proof of Corollary 4 is complete.

6 Finite dimensional convergence

For any subset Γ of $\mathbb{Z}^d$ we consider

$$\partial\Gamma = \{i \in \Gamma : \exists j \notin \Gamma \text{ such that } |i-j| = 1\}\,.$$

For any Borel set A of $[0,1]^d$, we denote by $\Gamma_n(A)$ the finite subset of $\mathbb{Z}^d$ defined by $\Gamma_n(A) = nA \cap \mathbb{Z}^d$.

Lemma 2 *Let A be a regular Borel set of $[0,1]^d$ with $\lambda(A) > 0$. We have*

(a)
$$(i) \quad \lim_{n \to +\infty} \frac{|\Gamma_n(A)|}{n^d} = \lambda(A) \quad and \quad (ii) \quad \lim_{n \to +\infty} \frac{|\partial\Gamma_n(A)|}{|\Gamma_n(A)|} = 0 \,.$$

(b) *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary random field with mean zero and finite variance. Assume that $\sum_{k \in \mathbb{Z}^d} |\mathbb{E}(X_0 X_k)| < \infty$. Then*

$$\lim_{n \to +\infty} n^{-d/2} \|S_n(A) - S_{\Gamma_n(A)}(X)\|_2 = 0 \,. \tag{6.1}$$

The finite dimensional convergence follows straightforwardly from Lemma 2 and Theorem 2 in Dedecker (1998).

Proof of Lemma 2. We start by proving (a). We introduce the subsets of $\mathbb{Z}^d$

$$A_1 = \{i : R_i \subset nA\}, \quad A_2 = \{i : R_i \cap nA \neq \emptyset\}, \quad A_3 = A_2 \cap \{i : R_i \cap (nA)^c \neq \emptyset\}$$

and for any positive real ε, we set

$$(\partial A)^\varepsilon = \{x \in \mathbb{R}^d : \sup_{1 \leq k \leq d} |x_k - y_k| \leq \varepsilon \text{ for some } y \in \partial A\} \,.$$

Clearly $|A_1| \leq |\Gamma_n(A)| \leq |A_2|$ and consequently

$$|A_2| - |A_3| \leq |\Gamma_n(A)| \leq |A_1| + |A_3| .$$

First, note that $|A_2| \geq n^d \lambda(A)$ and $|A_1| \leq n^d \lambda(A)$. Since A_3 is included in the set $\{i : R_i \subset (\partial nA)^1\}$ we infer that $|A_3| \leq n^d \lambda((\partial A)^{1/n})$ and therefore

$$n^d \lambda(A) - n^d \lambda((\partial A)^{1/n}) \leq |\Gamma_n(A)| \leq n^d \lambda(A) + n^d \lambda((\partial A)^{1/n}) \,. \tag{6.2}$$

By assumption the set A is regular, and hence $\lambda((\partial A)^{1/n})$ tends to zero as n tends to infinity. This fact together with (6.2) imply Lemma 2(a)(i). To prove (a)(ii), note that $\partial\Gamma_n(A)$ is included in the set $\{i : R_i \subset (\partial nA)^2\}$. Hence $|\partial\Gamma_n(A)| \leq n^d \lambda((\partial A)^{2/n})$, and we conclude as in the proof of (i).

It remains to prove (b). Set $a_i = \lambda(nA \cap R_i) - \mathbb{1}_{i \in \Gamma_n(A)}$. Since a_i equals zero if i belongs to A_1, we have

$$S_n(A) - S_{\Gamma_n(A)}(X) = \sum_{i \in A_3} a_i X_i \,.$$

Using both the fact that $|a_i| \leq 1$ and the stationarity of the random field, we obtain

$$
\begin{aligned}
\|S_n(A) - S_{\Gamma_n(A)}(X)\|_2^2 &\leq \sum_{(i,j) \in A_3 \times A_3} |\mathbb{E}(X_i X_j)| \\
&\leq |A_3| \sum_{k \in \mathbb{Z}^d} |\mathbb{E}(X_0 X_k)|
\end{aligned}
$$

From the proof of (a) we know that $n^{-d}|A_3|$ tends to zero as n tends to infinity, and finally (6.1) holds.

7 Tightness

To complete the proof of Theorems 1, 2 and 3, we shall prove as usual that the sequence $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$ is tight in $C(\mathcal{A})$.

7.1 End of the proof of Theorem 1

According to the assumptions of Theorem 1, there exists $p > 1$ such that $b_{0,p}(X)$ is finite. For such a p, define the measure μ on $[0,1]^d$ by

$$
\mu = 4p b_{0,p}(X)\lambda \, .
$$

In this section we shall prove that $(n^{-d/2}S_n, \mu)$ belongs to the class $\mathcal{C}(p, 2p)$, where $\mathcal{C}(\beta, \gamma)$ has been defined by Bickel and Wichura (1971) for any $\beta > 1$ and $\gamma > 0$. The tightness of the sequence $\{n^{-d/2}S_n(t) : t \in [0,1]^d\}$ will then follow by applying Theorem 3 of the above paper. For any s and t in $[0,1]^d$ such that $s_k \leq t_k$ for all k, define the subset $B =]s_1, t_1] \times \cdots \times]s_d, t_d]$. Let $a_i = \lambda(nB \cap R_i)$ and write aX for the random field $(a_i X_i)_{i \in \mathbb{Z}^d}$.

Now from Proposition 1(a), we have

$$
\begin{aligned}
\mathbb{E}(|n^{-d/2}S_n(B)|^{2p}) &\leq \left(4p \sum_{i \in \mathbb{Z}^d} n^{-d} b_{i,p}(aX)\right)^p \\
&\leq \left(4p b_{0,p}(X) \sum_{i \in \mathbb{Z}^d} n^{-d}|a_i|\right)^p \\
&\leq (\mu(B))^p \, .
\end{aligned}
$$

From inequality (3) in Bickel and Wichura, this implies that $(n^{-d/2}S_n, \mu)$ belongs to the class $\mathcal{C}(p, 2p)$ and the proof of Theorem 1 is complete.

7.2 End of the proof of Theorem 2

For any A and B in $\mathcal{A}$, define $a_i = \lambda(nA \cap R_i) - \lambda(nB \cap R_i)$ and write aX for the random field $(a_i X_i)_{i \in \mathbb{Z}^d}$. Set $M = \|X_0^2\|_\infty + \sum_{k \in V_0^1} \|X_k \mathbb{E}_{|k|}(X_0)\|_\infty$. We have

$$\sum_{i \in \mathbb{Z}^d} b_{i,\infty}(aX) \leq M \sum_{i \in \mathbb{Z}^d} |a_i| \leq n^d M \lambda(A\Delta B)$$

Applying Corollary 3(a) to the random fields aX, we obtain

$$\mathbb{P}(|S_n(A) - S_n(B)| > n^{d/2}x) \leq \exp\left(\frac{-x^2}{4eM\lambda(A\Delta B)} + \frac{1}{e}\right).$$

This means that for each n the process $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$ is subgaussian (cf Ledoux and Talagrand (1991) page 322). Now suppose that (2.1) holds. Applying Theorem 11.6 in Ledoux and Talagrand (1991), we infer that the sequence $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$ satisfies the following property: for each positive ε there exists a positive real δ, depending only on ε and of the value of the entropy integral, such that

$$\mathbb{E}\left(\sup_{d(A,B)<\delta} |S_n(A) - S_n(B)|\right) < n^{d/2}\varepsilon.$$

This proves that the sequence $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$ is tight in $C(\mathcal{A})$, and the proof of Theorem 2 is complete.

7.3 End of the proof of Theorem 3

In Lemma 3 below, we establish an upper bound on the maximum of $S_\Gamma(aX)$ when a describes a finite collection of elements of $[-1,1]^{\mathbb{Z}^d}$. Next, we adapt the chaining method of Bass (1985) and we use the upper bound of Lemma 3 to control each terms of the decomposition.

Maximal inequalities for partial sums

Lemma 3 *Let X, a, $A(\Gamma)$, v and m be defined as in Corollary 4. Let $\mathcal{G}$ be any finite collection of elements of $[-1,1]^{\mathbb{Z}^d}$, denote by $|\mathcal{G}|$ its cardinality and set $H = \log(|\mathcal{G}|)$. Assume that $H \geq 1$ and take δ such that for any a in $\mathcal{G}$ we have $\sqrt{A(\Gamma)} \leq \delta$.*

(i) *If $\sum_{k>0} k^{2d-1}\phi_{\infty,2}(k) < \infty$, then*

$$\mathbb{E}\left(\max_{a \in \mathcal{G}} |S_\Gamma(aX)|\right) \leq K_1\left(\sqrt{Hv}\delta + Hm^3 v^{-1}\right).$$

(ii) *If $\phi_{\infty,2}(k) = O(k^{-b})$ for some b in $]d, 2d[$, then*

$$\mathbb{E}\left(\max_{a \in \mathcal{G}} |S_\Gamma(aX)|\right) \leq K_2\left(\sqrt{Hv}\delta + H(m^3 v^{-1} \vee m^{\frac{b+d}{b-d}} v^{\frac{d}{d-b}})\right).$$

Proof. Assume that $\sum_{k>0} k^{2d-1}\phi_{\infty,2}(k) < \infty$. From inequality (5.1) and Corollary 4(a), we infer that there exists a constant C_1 such that, for any $p \geq 3$,

$$\|S_\Gamma(aX)\|_p \leq C_1\left(\sqrt{pv}\delta + pm^3 v^{-1}\right). \tag{7.1}$$

Now, if we only assume that $\phi_{\infty,2}(k) = O(k^{-b})$ for some b in $]d, 2d[$, we can obtain from Proposition 2 an inequality similar to (7.1) by providing upper bounds for $\gamma_i(aX)$ and $\delta_i(aX)$. Let N be a positive integer. From inequality (5.3) and the fact that

$$\sum_{j \in V_i^N} \|X_j \mathbb{E}_{|j-i|}(X_i)\|_{\frac{p}{2}} \leq \frac{m^2}{2} \sum_{j \in \mathbb{Z}^d,\, |j| \geq N} \phi_{\infty,1}(|j|),$$

we infer that there exists a constant D_1 such that

$$\left(2p \sum_{i \in \Gamma} \gamma_i(aX)\right)^{\frac{1}{2}} \leq D_1\left(\sqrt{pv}\delta + \sqrt{p}N^{\frac{d-b}{2}}m\delta\right). \tag{7.2}$$

The term $\delta_i^{(3)}(aX)$ is controlled by (5.4). Next arguing as for inequalities (5.5) and (5.6), we have

$$\delta_i^{(1)}(aX) \;\leq\; |a_i|m^3 \sum_{j \in \mathbb{Z}^d,\, |j| < N} (2|j| - 1)^d \phi_{\infty,1}(|j|)$$

$$\delta_i^{(2)}(aX) \;\leq\; 2|a_i|m^3 \sum_{j \in \mathbb{Z}^d} (2(|j| \wedge N) + 1)^d \phi_{\infty,2}(|j|).$$

This implies that there exists a constant D_2 such that

$$\left(3p^2 \sum_{i \in \Gamma} \delta_i(aX)\right)^{\frac{1}{3}} \leq D_2\left(p^{2/3}N^{\frac{2d-b}{3}}m\delta^{2/3}\right) \tag{7.3}$$

Taking $N = [(\delta^2/p)^{1/(b+d)}] + 1$ in (7.2) and (7.3) yields

$$\left(2p \sum_{i \in \Gamma} \gamma_i(aX)\right)^{\frac{1}{2}} \;\leq\; D_1\left(\sqrt{pv}\delta + mp^{\frac{b}{b+d}}\delta^{\frac{2d}{b+d}}\right) \tag{7.4}$$

$$\left(3p^2 \sum_{i \in \Gamma} \delta_i(aX)\right)^{\frac{1}{3}} \;\leq\; D_3\left(mp^{2/3}\delta^{2/3} + mp^{\frac{b}{b+d}}\delta^{\frac{2d}{b+d}}\right) \tag{7.5}$$

for a certain constant D_3. Now

$$
\begin{aligned}
mp^{\frac{b}{b+d}}\delta^{\frac{2d}{b+d}} &= \left(pm^{\frac{b+d}{b-d}}v^{\frac{d}{d-b}}\right)^{\frac{b-d}{b+d}}\left(\sqrt{pv}\delta\right)^{\frac{2d}{b+d}} \\
&\leq pm^{\frac{b+d}{b-d}}v^{\frac{d}{d-b}} + \sqrt{pv}\delta\,, \quad\quad\quad\quad (7.6)
\end{aligned}
$$

and (7.6) remains valid with $b = 2d$. From Proposition 2 and the upper bounds (7.4), (7.5) and (7.6), we infer that there exists a constant C_2 such that, for any $p \geq 3$,

$$
\|S_\Gamma(aX)\|_p \leq C_2\left(\sqrt{pv}\delta + p(m^3 v^{-1} \vee m^{\frac{b+d}{b-d}}v^{\frac{d}{d-b}})\right). \quad\quad (7.7)
$$

Now we are in position to prove Lemma 3. Write

$$
\begin{aligned}
\mathbb{E}\left(\max_{a\in\mathcal{G}}|S_\Gamma(aX)|\right) &\leq \left\|\max_{a\in\mathcal{G}}|S_\Gamma(aX)|\right\|_p \\
&\leq \left(\sum_{a\in\mathcal{G}}\mathbb{E}|S_\Gamma(aX)|^p\right)^{\frac{1}{p}} \\
&\leq |\mathcal{G}|^{\frac{1}{p}}\max_{a\in\mathcal{G}}\|S_\Gamma(aX)\|_p\,. \quad\quad\quad\quad (7.8)
\end{aligned}
$$

Combining (7.8) with (7.1) (resp. (7.7)) and taking $p = 3H$, we obtain Lemma 3(i) (resp. Lemma 3(ii)).

Chaining

In the sequel, we write $\mathbb{H}(x)$ for $\mathbb{H}(\mathcal{A}, x)$ and we assume (without loss of generality) that X_0 has variance 1.

Following Bass (1985), we introduce the notations:

Notations 6. For b in $]d, 2d]$ and $0 \leq a \leq c \leq \infty$, let

$$
X_i(n, a, c) = \begin{cases} X_i & \text{if } 2n^{\frac{d(b-d)}{2(b+d)}}a \leq |X_i| < 2n^{\frac{d(b-d)}{2(b+d)}}c \\ 0 & \text{otherwise} \end{cases}
$$

and for any Borel set A of $[0,1]^d$, define

$$
\begin{aligned}
Z_n(A, a, c) &= n^{-d/2}\sum_{i\in\mathbb{Z}^d}\lambda(nA \cap R_i)\left(X_i(n, a, c) - \mathbb{E}\left(X_i(n, a, c)\right)\right) \\
U_n(A, a, c) &= n^{-d/2}\sum_{i\in\mathbb{Z}^d}\lambda(nA \cap R_i)|X_i(n, a, c)|\,.
\end{aligned}
$$

Now from the basic inequality

$$
\begin{aligned}
n^{-d/2}|S_n(A) - S_n(B)| \leq\ & |Z_n(A, 0, a_0) - Z_n(B, 0, a_0)| \\
& + 2U_n\left([0,1]^d, a_0, \infty\right) + 2\mathbb{E}\left(U_n\left([0,1]^d, a_0, \infty\right)\right)
\end{aligned}
$$

we infer that

$$n^{-d/2}\mathbb{E}\left(\sup_{(A,B)\in\mathcal{A}^2,\, d(A,B)\leq\delta}|S_n(A)-S_n(B)|\right)\leq E_1+E_2, \qquad (7.9)$$

where E_1 and E_2 are defined by

$$
\begin{aligned}
E_1 &= \mathbb{E}\left(\sup_{(A,B)\in\mathcal{A}^2,\, d(A,B)\leq\delta}|Z_n(A,0,a_0)-Z_n(B,0,a_0)|\right)\\
E_2 &= 4\mathbb{E}\left(U_n\left([0,1]^d,a_0,\infty\right)\right).
\end{aligned}
$$

<u>Control of E_1</u>

Let $\delta_i=2^{-i}\delta$. If A and B are any sets in $\mathcal{A}$, there exists sets A_i, A_i^+, B_i, B_i^+ in $\mathcal{A}(\delta_i)$ such that $A_i\subseteq A\subseteq A_i^+$ and $d(A_i,A_i^+)\leq\delta_i$, and similarly for B, B_i, B_i^+. For any sequence $(a_i)_{i\in\mathbf{N}}$ of positive numbers decreasing to 0, we have

$$
\begin{aligned}
Z_n(A,0,a_0) = Z_n(A_0,0,a_0) &+ \sum_{i=0}^{+\infty}\left(Z_n(A_{i+1},0,a_i)-Z_n(A_i,0,a_i)\right)\\
&+ \sum_{i=1}^{+\infty}\left(Z_n(A,a_i,a_{i-1})-Z_n(A_i,a_i,a_{i-1})\right) \quad (7.10)
\end{aligned}
$$

From (7.10) we obtain the bound $E_1\leq F_1+F_2+F_3$, where

$$
\begin{aligned}
F_1 &= \mathbb{E}\left(\max_{\substack{(A_0,B_0)\in(\mathcal{A}(\delta_0))^2\\ d(A_0,B_0)\leq 3\delta_0}}|Z_n(A_0,0,a_0)-Z_n(B_0,0,a_0)|\right)\\
F_2 &= 2\sum_{i=0}^{+\infty}\mathbb{E}\left(\max_{\substack{A_i\in\mathcal{A}(\delta_i),\, A_{i+1}\in\mathcal{A}(\delta_{i+1})\\ d(A_i,A_{i+1})\leq 2\delta_i}}|Z_n(A_{i+1},0,a_i)-Z_n(A_i,0,a_i)|\right)\\
F_3 &= 2\sum_{i=1}^{+\infty}\mathbb{E}\left(\max_{\substack{(A_i,A_i^+)\in(\mathcal{A}(\delta_i))^2\\ d(A_i,A_i^+)\leq\delta_i}}\sup_{A_i\subseteq A\subseteq A_i^+}|Z_n(A,a_i,a_{i-1})-Z_n(A_i,a_i,a_{i-1})|\right).
\end{aligned}
$$

To control F_1, we apply Lemma 3. Set $\Lambda_n=[1,n]^d\cap\mathbb{Z}^d$, and for any i in $\mathbb{Z}^d$,

$$\alpha_i=\lambda(nA_0\cap R_i)-\lambda(nB_0\cap R_i)\quad\text{and}\quad Y_i=X_i(n,0,a_0)-\mathbb{E}(X_i(n,0,a_0)).$$

With those notations, we have

$$n^{d/2}|Z_n(A_0,0,a_0)-Z_n(B_0,0,a_0)|=|S_{\Lambda_n}(\alpha Y)|.$$

Since A_0 and B_0 belong to $\mathcal{A}(\delta_0)$, α describes a set whose log-cardinality is less than $2\mathbb{H}(\delta_0)$. Moreover it is clear that for each α in that set, we have

$$\sum_{i\in\Lambda_n}|\alpha_i|\leq n^d\lambda(A_0\Delta B_0)\leq 9\delta_0^2 n^d.$$

Consequenlty, Lemma 3 with $v = 1$, $m = n^{d(b-d)/2(b+d)}a_0$, $H = 2\mathbb{H}(\delta_0)$, and $\delta = 3\delta_0 n^{d/2}$ gives

$$F_1 \leq K\left(3\sqrt{2\mathbb{H}(\delta_0)}\delta_0 + 2\mathbb{H}(\delta_0)\left(a_0^3 n^{\frac{d(b-2d)}{b+d}} \vee a_0^{\frac{b+d}{b-d}}\right)\right), \tag{7.11}$$

where the constant K is equal either to K_1 when $b = 2d$ and we assume that $\sum_{k>0} k^{2d-1}\phi_{\infty,2}(k) < \infty$ or to K_2 when b belongs to $]d, 2d[$ and we assume that $\phi_{\infty,2}(k) = O(k^{-b})$.

In the same way, we get

$$F_2 \leq \sum_{i=0}^{+\infty} 2K\left(2\sqrt{2\mathbb{H}(\delta_{i+1})}\delta_i + 2\mathbb{H}(\delta_{i+1})\left(a_0^3 n^{\frac{d(b-2d)}{b+d}} \vee a_0^{\frac{b+d}{b-d}}\right)\right). \tag{7.12}$$

To control F_3, note that

$$\sup_{A_i \subseteq A \subseteq A_i^+} |Z_n(A, a_i, a_{i-1}) - Z_n(A_i, a_i, a_{i-1})| \leq U_n(A_i^+ \setminus A_i, a_i, a_{i-1})$$

$$+ \mathbb{E}(U_n(A_i^+ \setminus A_i, a_i, a_{i-1})),$$

and consequently

$$\sup_{A_i \subseteq A \subseteq A_i^+} |Z_n(A, a_i, a_{i-1}) - Z_n(A_i, a_i, a_{i-1})| \leq G_1(i) + G_2(i), \tag{7.13}$$

where

$$G_1(i) = |U_n(A_i^+ \setminus A_i, a_i, a_{i-1}) - \mathbb{E}(U_n(A_i^+ \setminus A_i, a_i, a_{i-1}))|$$
$$G_2(i) = 2\mathbb{E}(U_n(A_i^+ \setminus A_i, a_i, a_{i-1}))$$

Arguing as for F_1 and F_2, we have

$$\mathbb{E}\left(\max_{\substack{(A_i, A_i^+) \in (\mathcal{A}(\delta_i))^2 \\ d(A_i, A_i^+) \leq \delta_i}} G_1(i)\right) \leq K\left(2\sqrt{2\mathbb{H}(\delta_i)}\delta_i + 2\mathbb{H}(\delta_i)\left(a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}}\right)\right). \tag{7.14}$$

On the other hand, since

$$E|X_j(n, a_i, a_{i-1})| \leq \inf\left(\mathbb{E}|X_0|^4 n^{-\frac{3d(b-d)}{b+d}}a_i^{-3}, \mathbb{E}|X_0|^{\frac{2b}{b-d}}n^{-\frac{d}{2}}a_i^{\frac{b+d}{d-b}}\right),$$

we infer that, setting $M = \mathbb{E}|X_0|^4 \vee \mathbb{E}|X_0|^{2b/(b-d)}$,

$$G_2(i) \leq \frac{2\delta_i^2 M}{a_i^3 n^{\frac{d(b-2d)}{b+d}} \vee a_i^{\frac{b+d}{b-d}}}. \tag{7.15}$$

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Collecting (7.13), (7.14) and (7.15), we obtain

$$
F_3 \leq \sum_{i=1}^{+\infty} 4K \left(\sqrt{2\mathbb{H}(\delta_i)}\delta_i + \mathbb{H}(\delta_i) \left(a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}} \right) \right) + \frac{4\delta_i^2 M}{a_i^3 n^{\frac{d(b-2d)}{b+d}} \vee a_i^{\frac{b+d}{b-d}}} \cdot
$$
$$(7.16)$$

From inequalities (7.11), (7.12), (7.16) and the facts that $\mathbb{H}(\delta_0) \leq \mathbb{H}(\delta_1)$ and $\delta_i = 2\delta_{i-1}$, we conclude that there exists a constant C such that

$$
E_1 \leq C \sum_{i=1}^{+\infty} \sqrt{\mathbb{H}(\delta_i)}\delta_{i-1} + \mathbb{H}(\delta_i) \left(a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}} \right) + \frac{\delta_{i-1}^2}{a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}}} \cdot
$$
$$(7.17)$$

We now choose the sequence $(a_i)_{i \in \mathbb{N}}$ by setting

$$
\text{for } i \geq 1, \quad a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}} = \frac{\delta_{i-1}}{\sqrt{\mathbb{H}(\delta_i)}}, \tag{7.18}
$$

so that

$$
\sqrt{\mathbb{H}(\delta_i)}\delta_{i-1} = \mathbb{H}(\delta_i) \left(a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}} \right) = \frac{\delta_{i-1}^2}{a_{i-1}^3 n^{\frac{d(b-2d)}{b+d}} \vee a_{i-1}^{\frac{b+d}{b-d}}}. \tag{7.19}
$$

According to (7.17), (7.19) and the decrease of the function $\mathbb{H}$, we have

$$
E_1 \leq 12C \sum_{i=1}^{+\infty} \sqrt{\mathbb{H}\left(\frac{\delta}{2^i}\right)} \frac{\delta}{2^{i+1}} \leq 12C \int_0^\delta \sqrt{H(x)}dx. \tag{7.20}
$$

Recall that one of the assumptions of Theorem 1 is that the collection $\mathcal{A}$ has a convergent entropy integral (i.e (4.7) holds). Therefore, it follows from (7.20) that

$$
\lim_{\delta \to 0} \limsup_{n \to +\infty} \mathbb{E} \left(\sup_{(A,B) \in \mathcal{A}^2, \, d(A,B) \leq \delta} |Z_n(A, 0, a_0) - Z_n(B, 0, a_0)| \right) = 0 \tag{7.21}
$$

Control of E_2

The proof is adapted from Bass (1985), Proposition 4.1. We first state the following lemma:

Lemma 4 *Assume that* $\mathbb{E}(|X_0|^{2b/(b-d)}) < \infty$ *for some b in $]d, 2d]$. For any positive real a, let $X_0(|k|, a, \infty)$ be defined as in Notations 6. We have*

$$
\sum_{k \in \mathbb{Z}^d \setminus 0} |k|^{-d/2} \mathbb{E}|X_0(|k|, a, \infty)| < \infty.
$$

Now to control E_2 we write, for any positive integer N,

$$\mathbb{E}\left(U_n\left([0,1]^d, a_0, \infty\right)\right) \leq \sum_{k \in \mathbb{Z}^d, |k| \leq N} n^{-d/2}\mathbb{E}|X_0(n, a_0, \infty)|$$
$$+ \sum_{k \in \mathbb{Z}^d, |k| > N} |k|^{-d/2}\mathbb{E}|X_0(|k|, a_0, \infty)|, \quad (7.22)$$

where we use the stationarity of the random field X.

From (7.18) we know that, for n large enough, $a_0 = (\delta_0/\sqrt{\mathbb{H}(\delta_1)})^{(b-d)/(b+d)}$ (in particulary, it does not depend on n). Therefore, according to (7.22), we have

$$\limsup_{n \to +\infty} \mathbb{E}\left(U_n\left([0,1]^d, a_0, \infty\right)\right) \leq \sum_{k \in \mathbb{Z}^d, |k| > N} |k|^{-d/2}\mathbb{E}\left|X_0\left(|k|, \left(\frac{\delta_0}{\sqrt{\mathbb{H}(\delta_1)}}\right)^{\frac{(b-d)}{(b+d)}}, \infty\right)\right|$$

which together with Lemma 4 yields

$$\limsup_{n \to +\infty} \mathbb{E}\left(U_n\left([0,1]^d, a_0, \infty\right)\right) = 0. \quad (7.23)$$

From inequalities (7.9), (7.21) and (7.23) we infer that the sequence of processes $\{n^{-d/2}S_n(A) : A \in \mathcal{A}\}$ is tight in the space $C(\mathcal{A})$, and the proof of Theorem 3 is complete.

Proof of Lemma 4. Since the number of k in $\mathbb{Z}^d$ with $|k| = i$ is less than ci^{d-1} for a constant c, we have

$$\sum_{k \in \mathbb{Z}^d \backslash 0} |k|^{-d/2}\mathbb{E}|X_0(|k|, a, \infty)| \leq c\sum_{i=1}^{+\infty} i^{\frac{d}{2}-1}\mathbb{E}|X_0(i, a, \infty)|,$$

and the definition of $X_0(i, a, \infty)$ leads to

$$\sum_{k \in \mathbb{Z}^d \backslash 0} |k|^{-d/2}\mathbb{E}|X_0(|k|, a, \infty)| \leq c\mathbb{E}\left(|X_0|\sum_{i=1}^{+\infty} i^{\frac{d}{2}-1}\mathbb{1}_{i^{d(b-d)} \leq (|X_0|/2a)^{2(b+d)}}\right)$$
$$\leq c\mathbb{E}(|X_0|^{\frac{2b}{b-d}})(2a)^{\frac{b-d}{b+d}},$$

which concludes the proof of Lemma 4.

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