Necessary and sufficient conditions for the conditional central limit theorem

Jérôme DEDECKER and Florence MERLEVÈDE

Université Paris 6

Abstract

Following Lindeberg's approach, we obtain a new condition for a stationary sequence of square-integrable and real-valued random variables to satisfy the central limit theorem. In the adapted case, this condition is weaker than any projective criterion derived from Gordin's theorem (1969) about approximating martingales. Moreover, our criterion is equivalent to the *conditional central limit theorem*, which implies stable convergence (in the sense of Rényi) to a mixture of normal distributions. We also establish functional and triangular versions of this theorem. From these general results, we derive sufficient conditions which are easier to verify and may be compared to other results in the literature. To be complete, we present an application to kernel density estimators for some classes of discrete time processes.

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1 Introduction

In 1963 Rényi introduced the concept of stable convergence of random variables. This notion is more precise than convergence in distribution and may be useful in several contexts, especially in connection with random normalization (this fact was first pointed out by Smith (1945) in the particular case of sums of Rademacher random variables). Aldous and Eagleson (1978) made clear the equivalence between stability and weak-$\mathbb{L}^1$ convergence of some functions of the variables, and proposed some powerful tools to establish stability of limit theorems. Further, adapting a result of McLeish (1974), they gave sufficient conditions for a sequence of martingale differences to converge stably to a mixture of normal distributions. Their results among many others have been used and developed by Hall and Heyde (1980, Chapter 3) to provide an elegant and rather complete contribution to martingale central limit theory.

Some of these results have been extended to general sequences by providing strong enough conditions to ensure that the partial sums behave asymptotically like a martingale. In this context, McLeish (1975b, 1977) used the concept of mixingale, while Peligrad (1981) followed Gordin's approach. The common feature of these works is the application of Theorem 19.4 in Billinglsey (1968): firstly they prove tightness of the partial sum process and secondly they identify the limit by using a suitable characterization of the Wiener process. They obtain *mixing* convergence of the partial sum process to a Brownian motion, which coincides with stable convergence provided that the conditional variance of the partial sums with respect to the past σ-algebra is asymptotically constant (cf. Remark 5, Section 2 for the relation between stability and mixing).

In this paper we focus on the central limit question for strictly stationary sequences indexed by $\mathbb{Z}$. We propose in Theorem 1 a simple criterion which implies stable convergence of the normalized partial sums to a mixture of normal distributions. More precisely, we show that this criterion is necessary and sufficient to obtain a stronger result than stable convergence. We shall see that this new type of convergence to a mixture of normal distributions, close to the one introduced by Touati (1993, Theorem 3, H-2), is satisfied for a wide class of stationary sequences.

Notations 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. An element A is said to be invariant if $T(A) = A$. We denote by $\mathcal{I}$ the σ-algebra of all invariant sets. The probability $\mathbb{P}$ is ergodic if each element of $\mathcal{I}$ has measure 0 or 1. Finally, let $\mathcal{H}$ be the space of continuous real functions φ such that $x \to |(1+x^2)^{-1}\varphi(x)|$ is bounded.

Theorem 1 *Let $\mathcal{M}_0$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$ and define the nondecreasing filtration $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ by $\mathcal{M}_i = T^{-i}(\mathcal{M}_0)$. Let X_0 be a $\mathcal{M}_0$-measurable, square integrable and centered random variable. Define the sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$, and $S_n = X_1 + \cdots + X_n$. The following statements are equivalent:*

s1 *There exists a nonnegative $\mathcal{M}_0$-measurable random variable η such that, for any φ in $\mathcal{H}$ and any positive integer k,*

$$\lim_{n\to\infty}\Big\|\mathbb{E}\Big(\varphi(n^{-1/2}S_n) - \int \varphi(x\sqrt{\eta})g(x)dx\ \Big|\mathcal{M}_k\Big)\Big\|_1 = 0$$

where g is the distribution of a standard normal.

s2 *(a) the sequence $(n^{-1}S_n^2)_{n>0}$ is uniformly integrable.*

(b) the sequence $\|\mathbb{E}(n^{-1/2}S_n|\mathcal{M}_0)\|_1$ tends to 0 as n tends to infinity.

(c) there exists a nonnegative $\mathcal{M}_0$-measurable random variable η such that $\|\mathbb{E}(n^{-1}S_n^2 - \eta|\mathcal{M}_0)\|_1$ tends to 0 as n tends to infinity.

Moreover the random variable η satisfies $\eta = \eta \circ T$ almost surely.

A stationary sequence $(X \circ T^i)_{i\in\mathbb{Z}}$ of random variables is said to satisfy the conditional central limit theorem (CCLT for short) if it verifies **s1**.

Before presenting the applications of Theorem 1, let us compare Condition **s2** with Theorem 9.5 in Jakubowski (1993) (taking $B_n = \sqrt{n}$ therein), where necessary and sufficient conditions for the usual CLT are given. There are two distinct sets of conditions in this Theorem: on one hand Condition B and on the other hand Conditions (9.5), (9.6) and (9.7). Firstly, it is clear that **s2**(a) is stronger than Conditions (9.5) and (9.6). Secondly, **s2**(a) together with **s2**(c) imply (9.7) with $\sigma^2 = \mathbb{E}(\eta)$. Since Jakubowski's result only deals with pure Gaussian limit, we infer from the two preceding remarks that **s2** implies Condition B *if and only if* the random variable η is constant. This means that the two results are of a different nature and that Theorem 1 may not be derived by using Jakubowski's result (an other reason is that the latter does not necessarily imply stable convergence). In fact, Condition B is a kind of mixing property involving the whole past and the whole future of the sequence, so that random variances are forbidden (this is also the case when considering classical mixing coefficients). Note also that to obtain Condition B from **s2** in the case where η is constant seems as difficult as to prove **s1** directly.

We now give sufficient conditions for the CCLT to hold. Note first that criterion **s2** is satisfied for stationary sequences of martingale differences: indeed, in that case, $\mathbf{s2}(a)$ follows from Doob's maximal inequality, $\mathbf{s2}(b)$ is straightforward and $\mathbf{s2}(c)$ is a consequence of the $\mathbb{L}^1$-ergodic theorem. Now, as first noticed by Gordin (1969), it is often possible to approximate the partial sums of a stationary process by a naturally related martingale with stationary differences. Such an approximation provides a possible approach to obtain sufficient conditions for the CCLT, as shown by Proposition 1:

Proposition 1 *Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. Let H_i be the Hilbert space of $\mathcal{M}_i$-measurable, centered and square integrable functions. For all integer j less than i, denote by $H_i \ominus H_j$, the orthogonal of H_j into H_i. Let Q be the set of all functions from $H_i \ominus H_j$ where $-\infty < j \le i < \infty$. If*

$$\inf_{f\in Q} \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \Big\| \sum_{i=1}^{n} X_0 \circ T^i - f \circ T^i \Big\|_2 = 0\,, \tag{1.1}$$

then **s2** *(hence* **s1***) holds.*

Remark 1. Condition (1.1) is due to Gordin (1969). However, compared to Gordin's theorem, we have to restrict ourselves to adapted sequences (which means that X_i is $\mathcal{M}_i$-measurable). Note that, as in Eagleson (1975), we do not require $\mathbb{P}$ to be ergodic.

It follows from Proposition 1 that any projective criterion derived from (1.1) leads to the CCLT. For instance, Conditions (1) and (2) of Theorem 1 in Heyde (1974) yield (1.1), and so does Condition (3.15) in Dürr and Goldstein (1986). From Theorem 2 in Heyde (1974) we know that (1.1) is satisfied as soon as

$$\sum_{i=0}^{n} P_0(X_i) \text{ converges in } \mathbb{L}^2 \text{ to } g \text{ and } \lim_{n\to\infty} n^{-1/2}\|S_n\|_2 = \|g\|_2\,, \tag{1.2}$$

where P_0 is the projection operator onto $H_0 \ominus H_{-1}$. Volný (1993), Theorems 5 and 6, proposed sufficients conditions based upon the sequence $(P_0(X_i))_{i\ge 0}$

for (1.2) to hold. From these conditions, we easily infer (cf. Section 6.2) that (1.2) is satisfied as soon as there exists a sequence $(L_k)_{k>0}$ of positive numbers such that

$$\sum_{i>0}\Big(\sum_{k=1}^{i} L_k\Big)^{-1} < \infty \quad \text{and} \quad \sum_{k>0} L_k \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2^2 < \infty\,. \tag{1.3}$$

According to McLeish's definition (1975a), (1.3) is a mixingale-type condition. Note that it is close to optimality, since the choice $L_k = 1$ is not strong enough to imply weak convergence of $n^{-1/2}S_n$ (see Proposition 7 Section 6).

We now turn to the functional version of Theorem 1. As usual, to prove tightness of the normalized Donsker line, we need to control the maximum of partial sums. More precisely, under the condition

$$\mathbf{s2}^* \begin{cases} (b) \text{ and } (c) \text{ of } \mathbf{s2} \text{ hold, and } (a) \text{ is replaced by} \\ (a^*)\ :\ \dfrac{1}{n}\Big(\max\limits_{1\le i\le n}|S_i|\Big)^2 \text{ is uniformly integrable,} \end{cases}$$

we obtain the functional version of the CCLT described in Theorem 3, Section 2. Note that if η is constant, the usual functional central limit theorem follows from $\mathbf{s2}^*$ by applying Theorem 19.4 in Billingsley (1968). Note also that if the convergence rate to zero in $\mathbf{s2}(c)$ is of order $n^{-\theta}$ for some positive θ and the variables have finite $2+\delta$-moments, then it follows from Theorem 2.1 in Serfling (1968) that $\mathbf{s2}(a^*)$ is automatically satisfied. Assuming that the same rate holds for $\mathbf{s2}(b)$, Eberlein (1986a) has obtained strong invariance principles with order of approximation $O(t^{1/2-\kappa})$ for some positive κ.

Once again, there exists a large class of stationary sequences satisfying $\mathbf{s2}^*$, as shown by the two propositions below.

Proposition 2 *Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. Condition (1.3) implies* $\mathbf{s2}^*$.

The key for Proposition 2 is to establish a maximal inequality which together with (1.3) implies $\mathbf{s2}(a^*)$. This can be done by following McLeish's approach for nonstationary mixingales (cf. McLeish (1975b), Lemma (6.3)). In the particular case of stationary and adapted sequences, Proposition 2 improves on McLeish's results in three ways: Firstly, Condition (1.3) is realized if either Condition (2.5) in McLeish (1977) holds or $(X_n, \mathcal{M}_n)$ is a mixingale of size -1/2 (cf. McLeish (1975b) Definitions (2.1) and (2.5)). Secondly, the extra condition in Theorem (2.6) of the latter may be removed (in fact it may be replaced by the weaker condition $\mathbf{s2}(c)$, which follows from (1.3) as we have already noticed). Thirdly, we obtain a stronger result in terms of convergence, the functional CCLT implying *mixing*-convergence as soon as η is constant, which we do not require here. To get an idea of the wide range of applications of mixingales, we refer to McLeish (1975a, 1975b, 1977) and Hall and Heyde (1980) Section 2.3. See also Eberlein (1986b) for a survey of results concerning mixingales and other generalizations of martingales.

The second condition (Condition (1.4) below) has a different structure and is not obtained via martingale approximations, although many results in that field may be derived from it.

Proposition 3 *Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$, $(X_i)_{i\in\mathbb{Z}}$ and S_n be as in Theorem 1. Consider the condition:*

$$\sum_{k=1}^{n} X_0\mathbb{E}(X_k|\mathcal{M}_0) \text{ converges in } \mathbb{L}^1 \,. \tag{1.4}$$

If (1.4) is satisfied, then $\mathbf{s2}^*$ holds and the sequence $\mathbb{E}(X_0^2|\mathcal{I}) + 2\mathbb{E}(X_0S_n|\mathcal{I})$ converges in $\mathbb{L}^1$ to η.

Condition (1.4) was introduced by Dedecker and Rio (2000) to obtain the usual functional central limit theorem. In the adapted case, Condition (1.4) is weaker than the $\mathbb{L}^2$-criterion of Gordin (1969)

$$\sum_{k=1}^{n} \mathbb{E}(X_k|\mathcal{M}_0) \text{ converges in } \mathbb{L}^2 \,, \tag{1.5}$$

which has been extended to nonstationary sequences by Peligrad (1981). If $X_i = g(\xi_i)$ is a function of a stationary Markov chain $(\xi_i)_{i\in\mathbb{Z}}$ with transition kernel K and marginal distribution μ, condition (1.4) becomes

$$\sum_{k=0}^{n} gK^k g \text{ converges in } \mathbb{L}^1(\mu) \,, \tag{1.6}$$

and improves on classical results based upon the Poisson equation. Condition (1.6) was simultaneously discovered by Chen (1999) in the particular case of positive Harris Chain. Finally the application of (1.4) to strongly mixing sequences leads to the conditional and nonergodic version of the invariance principle of Doukhan, Massart and Rio (1994), whose optimality is discussed in Bradley (1997).

The paper is organized as follows. In Section 2, we state the conditional central limit theorem (Theorem 2) and its functional version (Theorem 3) in the more general context of triangular arrays. As a consequence, we derive in Corollary 1 the stable convergence of the normalized partial sums to a mixture of normal distributions. The proofs of these results are postponed to Sections 3, 4 and 5 respectively. Section 6 is devoted to the applications of Theorem 1: we prove Propositions 1, 2 and 3 in Sections 6.1, 6.2 and 6.3 respectively. In Section 7, we give the analogous of Propositions 2 and 3 for triangular arrays. In Section 8, we explain how to apply these results to kernel density estimators.

2 Main results

Theorem 1 presented in the introduction is a straightforward consequence of the following theorem for triangular arrays with stationary rows.

Theorem 2 *For each positive integer n, let $\mathcal{M}_{0,n}$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{M}_{0,n} \subseteq T^{-1}(\mathcal{M}_{0,n})$. Define the nondecreasing filtration $(\mathcal{M}_{i,n})_{i\in\mathbb{Z}}$ by $\mathcal{M}_{i,n} = T^{-i}(\mathcal{M}_{0,n})$ and $\mathcal{M}_{i,\inf} = \sigma\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\mathcal{M}_{i,k}\right)$. Let $X_{0,n}$ be a $\mathcal{M}_{0,n}$-measurable and square integrable random variable and define the sequence $(X_{i,n})_{i\in\mathbb{Z}}$ by $X_{i,n} = X_{0,n}\circ T^i$. Finally, for any t in $[0,1]$, write $S_n(t) = X_{1,n} + \cdots + X_{[nt],n}$. Suppose that $n^{-1/2}X_{0,n}$ converges in probability to zero as n tends infinity. The following statements are equivalent:*

S1 *There exists a nonnegative $\mathcal{M}_{0,\inf}$-measurable random variable η such that, for any φ in $\mathcal{H}$, any t in $[0,1]$ and any positive integer k,*

$$\mathbf{S1}(\varphi): \quad \lim_{n\to\infty}\left\|\mathbb{E}\Big(\varphi(n^{-1/2}S_n(t)) - \int\varphi(x\sqrt{t\eta})g(x)dx\;\Big|\mathcal{M}_{k,n}\Big)\right\|_1 = 0$$

where g is the distribution of a standard normal.

S2 *(a)*

$$\lim_{t\to 0}\limsup_{n\to\infty}\mathbb{E}\left(\frac{S_n^2(t)}{nt}\Big(1\wedge\frac{|S_n(t)|}{\sqrt{n}}\Big)\right) = 0\,.$$

(b) For any t in $[0,1]$, the sequence $\|\mathbb{E}(n^{-1/2}S_n(t)|\mathcal{M}_{0,n})\|_1$ tends to 0 as n tends to infinity.

(c) There exists a nonnegative $\mathcal{M}_{0,\inf}$-measurable random variable η such that, for any t in $[0,1]$, the sequence $\|\mathbb{E}(n^{-1}S_n^2(t) - t\eta|\mathcal{M}_{0,n})\|_1$ tends to 0 as n tends to infinity.

Moreover the random variable η satisfies $\eta = \eta\circ T$ almost surely.

Remark 2. An assumption on the asymptotic negligibility of the variables $n^{-1/2}X_{0,n}$ seems to be natural in the context of triangular arrays (see for instance the discussion p. 53 in Hall and Heyde (1980)).

We now turn to the functional version of Theorem 2. Denote by $\mathcal{H}^*$ the space of continuous functions φ from $(C([0,1]), \|.\|_\infty)$ to $\mathbb{R}$ such that $x \to |(1+\|x\|_\infty^2)^{-1}\varphi(x)|$ is bounded.

Theorem 3 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 2. For any t in $[0,1]$, set $U_n(t) = S_n(t) + (nt - [nt])X_{[nt]+1,n}$ and define $\overline{S}_n(t) = \sup_{0\le s\le t}|S_n(s)|$. The following statements are equivalent:*

S1* *There exists a nonnegative $\mathcal{M}_{0,\inf}$-measurable random variable η such that, for any φ in $\mathcal{H}^*$ and any positive integer k,*

$$\mathbf{S1}^*(\varphi): \quad \lim_{n\to\infty}\Big\|\mathbb{E}\Big(\varphi(n^{-1/2}U_n) - \int \varphi(x\sqrt{\eta})W(dx)\ \Big|\mathcal{M}_{k,n}\Big)\Big\|_1 = 0$$

where W is the distribution of a standard Wiener process.

S2* *(b) and (c) of* **S2** *hold, and (a) is replaced by :*

(a^*) the sequence $(n^{-1}(\overline{S}_n(1))^2)_{n>0}$ is uniformly integrable, and

$$\lim_{t\to 0}\limsup_{n\to\infty}\mathbb{E}\left(\frac{(\overline{S}_n(t))^2}{nt}\Big(1\wedge\frac{\overline{S}_n(t)}{\sqrt{n}}\Big)\right) = 0\,.$$

Moreover the random variable η satisfies $\eta = \eta\circ T$ almost surely.

Remark 3. If we omit the assumption that $(n^{-1}(\overline{S}_n(1))^2)_{n>0}$ is uniformly integrable, we still obtain **S1**$^*(\varphi)$ for any bounded function φ of $\mathcal{H}^*$. Note that Theorem 3 remains valid if we replace (a^*) by the stronger condition

$$\lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty}\mathbb{E}\left(\frac{(\overline{S}_n(t))^2}{nt}\mathbb{I}_{\overline{S}_n(t)\ge M\sqrt{nt}}\right) = 0\,.$$

Remark 4. Let $D([0,1])$ be the space of caldlag functions equipped with the Skorohod distance d. Denote by $\mathcal{H}^*(D)$ the space of continuous functions φ from $(D([0,1]), d)$ to $\mathbb{R}$ such that the function $x \to |(1+\|x\|_\infty^2)^{-1}\varphi(x)|$ is bounded. Applying Theorem 15.5 in Billingsley (1968), we can also obtain that **S2*** holds if and only if, for any φ in $\mathcal{H}^*(D)$ and any positive integer k,

$$\lim_{n\to\infty}\Big\|\mathbb{E}\Big(\varphi(n^{-1/2}S_n) - \int \varphi(x\sqrt{\eta})W(dx)\ \Big|\mathcal{M}_{k,n}\Big)\Big\|_1 = 0\,.$$

The following result is an important consequence of Theorems 2 and 3:

Corollary 1 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$, $S_n(t)$ be as in Theorem 2 and $U_n(t)$ as in Theorem 3. Suppose that the sequence $(\mathcal{M}_{0,n})_{n\geq 1}$ is nondecreasing. If Condition* **S2** *(resp.* **S2*****) is satisfied, then, for any φ in $\mathcal{H}$ (resp. $\mathcal{H}^*$), the sequence $\varphi(n^{-1/2}S_n(t))$ (resp. $\varphi(n^{-1/2}U_n)$) converges weakly in $\mathbb{L}^1$ to $\int \varphi(x\sqrt{t\eta})g(x)dx$ (resp. $\int \varphi(x\sqrt{\eta})W(dx)$).*

Remark 5. Corollary 1 implies that the sequence $n^{-1/2}S_n(t)$ converges *stably* to a mixture of normal distributions. We refer to Aldous and Eagleson (1978) for a complete exposition of the concept of stability (introduced by Rényi (1963)) and its connection to weak $\mathbb{L}^1$-convergence. Note that stable convergence is a useful tool to establish weak convergence of joint distributions (see again Aldous and Eagleson, or Hall and Heyde (1980) Chapter 3, Section 3.2.(vi)). If furthermore η is constant, then the convergence is *mixing*. If $\mathbb{P}$ is ergodic, this result is a consequence of Theorem 4 in Eagleson (1976) (see Application 4.2 therein). For a review of mixing results see Csörgö and Fischler (1973).

3 Proof of Theorem 2

The fact that **S1** implies **S2** is obvious. In this section, we focus on the consequences of condition **S2**. We start with some preliminary results.

3.1 Definitions and preliminary lemmas

Definitions 1. Let μ be a signed measure on a metric space $(S, \mathcal{B}(S))$. Denote by $|\mu|$ the total variation measure of μ, and by $\|\mu\| = |\mu|(S)$ its norm. We say that a family Π of signed measures on $(S, \mathcal{B}(S))$ is tight if for every positive ϵ there exists a compact set K such that $|\mu|(K^c) < \epsilon$ for any μ in Π. Denote by $\mathcal{C}(S)$ the set of continuous and bounded functions from S to $\mathbb{R}$. We say that a sequence of signed measures $(\mu_n)_{n>0}$ converges weakly to a signed measure μ if for any φ in $\mathcal{C}(S)$, $\mu_n(\varphi)$ tends to $\mu(\varphi)$ as n tends to infinity.

Lemma 1 *Let $(\mu_n)_{n>0}$ be a sequence of signed measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and set $\hat{\mu}_n(t) = \mu_n(\exp(i < t, . >))$. Assume that the sequence $(\mu_n)_{n>0}$ is tight and that $\sup_{n>0}\|\mu_n\| < \infty$. The following statements are equivalent*

1. *the sequence $(\mu_n)_{n>0}$ converges weakly to the null measure.*

2. for any t in $\mathbb{R}^d$, $\hat{\mu}_n(t)$ tends to zero as n tends to infinity.

The proof of Lemma 1 will be done in Appendix.

Definitions 2. Define the set $R(\mathcal{M}_{k,n})$ of Rademacher $\mathcal{M}_{k,n}$-measurable random variables: $R(\mathcal{M}_{k,n}) = \{2\mathbb{I}_A - 1 : A \in \mathcal{M}_{k,n}\}$. Recall that g is the $\mathcal{N}(0,1)$-distribution and that W is the Wiener measure on $C([0,1])$. For the random variable η introduced in Theorem 2 and any bounded random variable Z, let

1. $\nu_n[Z]$ be the image measure of $Z.\mathbb{P}$ by the variable $n^{-1/2}S_n(t)$.
2. $\nu_n^*[Z]$ be the image measure of $Z.\mathbb{P}$ by the process $n^{-1/2}U_n$.
3. $\nu[Z]$ be the image measure of $g.\lambda \otimes Z.\mathbb{P}$ by the variable ϕ from $\mathbb{R} \otimes \Omega$ to $\mathbb{R}$ defined by $\phi(x, \omega) = x\sqrt{t\eta(\omega)}$.
4. $\nu^*[Z]$ be the image measure of $W \otimes Z.\mathbb{P}$ by the variable ϕ from $C([0,1]) \otimes \Omega$ to $C([0,1])$ defined by $\phi(x, \omega) = x\sqrt{\eta(\omega)}$.

Lemma 2 *Let $\mu_n[Z_n] = \nu_n[Z_n] - \nu[Z_n]$ and $\mu_n^*[Z_n] = \nu_n^*[Z_n] - \nu^*[Z_n]$. For any φ in $\mathcal{H}$ (resp $\mathcal{H}^*$), the statement* $\mathbf{S1}(\varphi)$ *(resp.* $\mathbf{S1}^*(\varphi)$*) is equivalent to:*

$\mathbf{S3}(\varphi)$ *(resp.* $\mathbf{S3}^*(\varphi)$*): for any Z_n in $R(\mathcal{M}_{k,n})$ the sequence $\mu_n[Z_n](\varphi)$ (resp. $\mu_n^*[Z_n](\varphi)$) tends to zero as n tends to infinity.*

Proof of Lemma 2. We prove that $\mathbf{S1}(\varphi) \Leftrightarrow \mathbf{S3}(\varphi)$, the $*$-case being unchanged. For Z_n in $R(\mathcal{M}_{k,n})$ and φ in $\mathcal{H}$, we have

$$\begin{aligned} |\mu_n[Z_n](\varphi)| &= \left| \mathbb{E}\Big(Z_n \Big(\varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x)dx \Big) \Big) \right| \\ &\leq \left\| \mathbb{E}\Big(\varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x)dx \,\Big|\mathcal{M}_{k,n} \Big) \right\|_1 . \end{aligned}$$

Consequently $\mathbf{S1}(\varphi)$ implies $\mathbf{S3}(\varphi)$. Now to prove that $\mathbf{S3}(\varphi)$ implies $\mathbf{S1}(\varphi)$, choose

$$A(n, \varphi) = \Big\{ \mathbb{E}\Big(\varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x)dx \,\Big|\mathcal{M}_{k,n} \Big) \geq 0 \Big\},$$

and $Z_n^\varphi = 2\mathbb{I}_{A(n,\varphi)} - 1$. Obviously

$$\mu_n[Z_n^\varphi](\varphi) = \left\| \mathbb{E}\Big(\varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x)dx \,\Big|\mathcal{M}_{k,n} \Big) \right\|_1 ,$$

and $\mathbf{S3}(\varphi)$ implies $\mathbf{S1}(\varphi)$.

3.2 Invariance of η

We first prove that if **S2** holds, the random variables η satisfies $\eta = \eta \circ T$ almost surely (or equivalently that η is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{I}$). From $\mathbf{S2}(c)$ and both the facts that $(X_{i,n})_{i\in\mathbb{Z}}$ is strictly stationary and $\mathcal{M}_{0,n} \subseteq \mathcal{M}_{1,n}$, we have for any t in $]0,1]$,

$$\lim_{n\to\infty} \left\| \mathbb{E}\left(\eta \circ T - \frac{S_n^2(t)\circ T}{nt} \Big| \mathcal{M}_{0,n} \right) \right\|_1 = 0 \,. \tag{3.1}$$

On the other hand, defining $\psi(x) = x^2(1-(1\wedge |x|))$ and using the fact that T preserves $\mathbb{P}$, we have

$$\begin{aligned} \left\| \frac{S_n^2(t)}{nt} - \frac{S_n^2(t)\circ T}{nt} \right\|_1 \leq 2 & \left\| \frac{S_n^2(t)}{nt} \Big(1 \wedge \frac{|S_n(t)|}{\sqrt{n}} \Big) \right\|_1 \\ & + \frac{1}{t} \left\| \psi\Big(\frac{S_n(t)}{\sqrt{n}}\Big) - \psi\Big(\frac{S_n(t)\circ T}{\sqrt{n}}\Big) \right\|_1 . \end{aligned} \tag{3.2}$$

To control the second term on right hand, note that the function ψ is 3-lipschitz and bounded by 1. It follows that for each positive ϵ,

$$\left\| \psi\Big(\frac{S_n(t)}{\sqrt{n}}\Big) - \psi\Big(\frac{S_n(t)\circ T}{\sqrt{n}}\Big) \right\|_1 \leq 3\epsilon + 2\mathbb{P}(|X_{0,n} - X_{[nt],n}| > \sqrt{n}\epsilon) \,.$$

Using that $n^{-1/2}X_{0,n}$ converges in probability to 0, we derive that

$$\lim_{n\to\infty} \left\| \psi\Big(\frac{S_n(t)}{\sqrt{n}}\Big) - \psi\Big(\frac{S_n(t)\circ T}{\sqrt{n}}\Big) \right\|_1 = 0 \,,$$

and the second term on right hand in (3.2) tends to 0 as n tends to infinity. This fact together with inequality (3.2) and Condition $\mathbf{S2}(a)$ yield

$$\lim_{t\to 0} \limsup_{n\to\infty} \left\| \frac{S_n^2(t)}{nt} - \frac{S_n^2(t)\circ T}{nt} \right\|_1 = 0 \,,$$

which together with $\mathbf{S2}(c)$ imply that

$$\lim_{t\to 0} \limsup_{n\to\infty} \left\| \mathbb{E}\left(\eta - \frac{S_n^2(t)\circ T}{nt} \Big| \mathcal{M}_{0,n} \right) \right\|_1 = 0 \,. \tag{3.3}$$

Combining (3.1) and (3.3), it follows that $\lim_{n\to\infty} \|\mathbb{E}(\eta - \eta\circ T|\mathcal{M}_{0,n})\|_1 = 0$, which implies that

$$\lim_{n\to\infty} \left\| \mathbb{E}\Big(\eta - \eta \circ T \Big| \bigcap_{k\geq n} \mathcal{M}_{0,k} \Big) \right\|_1 = 0 \,.$$

Applying the martingale convergence theorem, we obtain

$$\lim_{n\to\infty}\Big\|\mathbb{E}\Big(\eta-\eta\circ T\Big|\bigcap_{k\geq n}\mathcal{M}_{0,k}\Big)\Big\|_1=\|\mathbb{E}(\eta-\eta\circ T|\mathcal{M}_{0,\inf})\|_1=0\,. \tag{3.4}$$

According to $\mathbf{S2}(c)$, the random variable η is $\mathcal{M}_{0,\inf}$-measurable. Therefore, (3.4) implies that $\mathbb{E}(\eta\circ T|\mathcal{M}_{0,\inf})=\eta$. The fact that $\eta\circ T=\eta$ almost surely is a direct consequence of the following elementary result, whose proof will be done in Appendix.

Lemma 3 *Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space, X an integrable random variable, and $\mathcal{M}$ a σ-algebra of $\mathcal{A}$. If the random variable $\mathbb{E}(X|\mathcal{M})$ has the same law as X, then $\mathbb{E}(X|\mathcal{M})=X$ almost surely.*

3.3 S2 implies S1

We now turn to the main proof of the paper.

First, note that we can restrict ourselves to bounded functions of $\mathcal{H}$: if $\mathbf{S2}$ implies $\mathbf{S1}(h)$ for any continuous and bounded function h then we easily infer from $\mathbf{S2}(c)$ that $n^{-1}S_n^2(t)$ is uniformly integrable for any t in $[0,1]$, which implies that $\mathbf{S1}$ extends to the whole space $\mathcal{H}$.

Notations 2. Let $B_1^3(\mathbb{R})$ be the class of three-times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\max(\|h^{(i)}\|_\infty,\ i\in\{0,1,2,3\})\leq 1$.

Suppose now that $\mathbf{S1}(h)$ holds for any h in $B_1^3(\mathbb{R})$. Applying Lemma 2, this is equivalent to say that $\mathbf{S3}(h)$ holds for any h in $B_1^3(\mathbb{R})$, which obviously implies that $\mathbf{S3}(h)$ holds for $h_t=\exp(it.)$. Using that the probability $\nu_n[1]$ is tight (since it converges weakly to $\nu[1]$) and that $|\mu_n[Z_n]|\leq\nu_n[1]+\nu[1]$, we infer that $\mu_n[Z_n]$ is tight, and Lemma 1 implies that $\mathbf{S3}(h)$ (and therefore $\mathbf{S1}(h)$) holds for any continuous bounded function h.

On the other hand, from the asymptotic negligibility of $n^{-1/2}X_{0,n}$ we infer that, for any positive integer k, $n^{-1/2}(S_n(t)-S_n(t)\circ T^k)$ converges in probability to zero. Consequently, since any function h belonging to $B_1^3(\mathbb{R})$ is 1-lipschitz and bounded, we have

$$\lim_{n\to\infty}\big\|h(n^{-1/2}S_n(t))-h(n^{-1/2}S_n(t)\circ T^k)\big\|_1=0\,,$$

and $\mathbf{S1}(h)$ is equivalent to

$$\lim_{n\to\infty}\Big\|\mathbb{E}\Big(h(n^{-1/2}S_n(t)\circ T^k)-\int h(x\sqrt{t\eta})g(x)dx\ \Big|\mathcal{M}_{k,n}\Big)\Big\|_1=0\,.$$

Now, since both η and $\mathbb{P}$ are invariant by T, we infer that Theorem 2 is a straightforward consequence of Proposition 4 below:

Proposition 4 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be defined as in Theorem 2. If* **S2** *holds, then, for any h in $B_1^3(\mathbb{R})$ and any t in $[0,1]$,*

$$\lim_{n\to\infty}\left\|\mathbb{E}\Big(h(n^{-1/2}S_n(t)) - \int h(x\sqrt{t\eta})g(x)dx\,\Big|\mathcal{M}_{0,n}\Big)\right\|_1 = 0$$

where g is the distribution of a standard normal.

Proof of Proposition 4. We prove the result for $S_n(1)$, the proof of the general case being unchanged. Without loss of generality, suppose that there exists a sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ of $\mathcal{N}(0,1)$-distributed and independent random variables, independent of $\mathcal{M}_{\infty,\infty} = \sigma(\bigcup_{k,n}\mathcal{M}_{k,n})$.

Notations 3. Let i, p and n be three integers such that $1 \le i \le p \le n$. Set $q = [n/p]$ and define

$$\begin{aligned} U_{i,n} &= X_{iq-q+1,n} + \cdots + X_{iq,n}, & V_{i,n} &= \frac{1}{\sqrt{n}}(U_{1,n} + U_{2,n} + \cdots + U_{i,n}) \\ \Delta_i &= \varepsilon_{iq-q+1} + \cdots + \varepsilon_{iq}, & \Gamma_i &= \sqrt{\frac{\eta}{n}}(\Delta_i + \Delta_{i+1} + \cdots + \Delta_p)\,. \end{aligned}$$

Notations 4. Let g be any function from $\mathbb{R}$ to $\mathbb{R}$. For k and l in $[1,p]$ and any positive integer $n \ge p$, set $g_{k,l;n} = g(V_{k,n} + \Gamma_l)$, with the conventions $g_{k,p+1;n} = g(V_{k,n})$ and $g_{0,l;n} = g(\Gamma_l)$. Afterwards, we shall apply this notation to the successive derivatives of the function h. For brevity we shall omit the index n.

Let $s_n = \sqrt{\eta}(\varepsilon_1 + \cdots + \varepsilon_n)$. Since $(\varepsilon_i)_{i\in\mathbb{Z}}$ is independent of $\mathcal{M}_{\infty,\infty}$, we have, integrating with respect to $(\varepsilon_i)_{i\in\mathbb{Z}}$,

$$\begin{aligned} &\mathbb{E}\Big(h(n^{-1/2}S_n(1)) - \int h(x\sqrt{\eta})g(x)dx\,\Big|\mathcal{M}_{0,n}\Big) \\ &\quad= \mathbb{E}(h(n^{-1/2}S_n(1)) - h(V_{p,n})|\mathcal{M}_{0,n}) + \mathbb{E}(h(V_{p,n}) - h(\Gamma_1)|\mathcal{M}_{0,n}) \\ &\qquad+ \mathbb{E}(h(\Gamma_1) - h(n^{-1/2}s_n)|\mathcal{M}_{0,n})\,. \end{aligned} \tag{3.5}$$

Here, note that $|n^{-1/2}S_n(1) - V_{p,n}| \le n^{-1/2}(|X_{n-p+2,n}| + \cdots + |X_{n,n}|)$. Using the asymptotic negligibility of $n^{-1/2}X_{0,n}$, we infer that $n^{-1/2}S_n(1) - V_{p,n}$ converges in probability to zero. Since furthermore h is 1-lipschitz and bounded, we conclude that

$$\lim_{n\to\infty}\|h(n^{-1/2}S_n(1)) - h(V_{p,n})\|_1 = 0\,, \tag{3.6}$$

and the same arguments yield

$$\lim_{n\to\infty} \|h(\Gamma_1) - h(n^{-1/2}s_n)\|_1 = 0\,. \tag{3.7}$$

In view of (3.6) and (3.7), it remains to control the second term in the right hand side of (3.5). To this end, we use Lindeberg's decomposition, as done in Dedecker and Rio (2000).

$$h(V_{p,n}) - h(\Gamma_1) = \sum_{i=1}^{p} (h_{i,i+1} - h_{i-1,i+1}) + \sum_{i=1}^{p} (h_{i-1,i+1} - h_{i-1,i})\;. \tag{3.8}$$

Now, applying Taylor's integral formula we get that:

$$\begin{cases} h_{i,i+1} - h_{i-1,i+1} &= \dfrac{1}{\sqrt{n}} U_{i,n} h'_{i-1,i+1} &+ \dfrac{1}{2n} U_{i,n}^2 h''_{i-1,i+1} &+ R_i \\ h_{i-1,i+1} - h_{i-1,i} &= -\sqrt{\dfrac{\eta}{n}} \Delta_i h'_{i-1,i+1} &- \dfrac{\eta}{2n} \Delta_i^2 h''_{i-1,i+1} &+ r_i \end{cases}$$

where

$$|R_i| \le \frac{U_{i,n}^2}{n}\left(1 \wedge \frac{|U_{i,n}|}{\sqrt{n}}\right) \quad \text{and} \quad |r_i| \le \frac{\eta\Delta_i^2}{n}\left(1 \wedge \frac{\sqrt{\eta}|\Delta_i|}{\sqrt{n}}\right). \tag{3.9}$$

Since Δ_i is centered and independent of $\sigma\left(\mathcal{M}_{\infty,\infty} \cup \sigma(h'_{i-1,i+1})\right)$, we have $\mathbb{E}(\sqrt{\eta}\Delta_i h'_{i-1,i+1}|\mathcal{M}_{0,n}) = \mathbb{E}(\Delta_i)\mathbb{E}(\sqrt{\eta} h'_{i-1,i+1}|\mathcal{M}_{0,n}) = 0$. It follows that

$$\mathbb{E}(h(V_p) - h(\Gamma_1)|\mathcal{M}_{0,n}) = D_1 + D_2 + D_3, \tag{3.10}$$

where

$$\begin{aligned} D_1 &= \sum_{i=1}^{p} \mathbb{E}(n^{-1/2} U_{i,n} h'_{i-1,i+1}|\mathcal{M}_{0,n}), \\ D_2 &= \frac{1}{2}\sum_{i=1}^{p} \mathbb{E}(n^{-1}(U_{i,n}^2 - \eta\Delta_i^2) h''_{i-1,i+1}|\mathcal{M}_{0,n}), \\ D_3 &= \sum_{i=1}^{p} \mathbb{E}(R_i + r_i|\mathcal{M}_{0,n}). \end{aligned}$$

Control of D_3.

From (3.9) and the fact that T preserves $\mathbb{P}$, we get

$$\sum_{i=1}^{p} \|R_i\|_1 \le \mathbb{E}\left(\frac{S_n^2(1/p)}{n/p}\left(1 \wedge \frac{|S_n(1/p)|}{\sqrt{n}}\right)\right)\,,$$

and $\mathbf{S2}(a)$ implies that

$$\lim_{p\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{p}\|R_i\|_1 = 0\,. \tag{3.11}$$

Moreover, since for $t\in[0,1]$, the sequence $(\eta/n)^{-1/2}(\varepsilon_1+\cdots+\varepsilon_{[nt]})$ obviously satisfies $\mathbf{S2}(a)$, the same argument applies to $\sum_{i=1}^{p}\|r_i\|_1$. Finally

$$\lim_{p\to\infty}\limsup_{n\to\infty}\|D_3\|_1 = 0\,. \tag{3.12}$$

Control of D_1.

To prove that $\|D_1\|_1$ tends to zero as n tends to infinity, it suffices to show that, for any positive integer i less than p,

$$\lim_{n\to\infty}\|\mathbb{E}(n^{-1/2}U_{i,n}h'_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 = 0\,. \tag{3.13}$$

Denote by $\mathbb{E}_\varepsilon$ the integration with respect to the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Set $l(i,n) = (i-1)[n/p]$. Bearing in mind the definition of $h'_{i-1,i+1}$ and integrating with respect to $(\varepsilon_i)_{i\in\mathbb{Z}}$ we deduce that the random variable $\mathbb{E}_\varepsilon(h'_{i-1,i+1})$ is $\mathcal{M}_{l(i,n),n}$-measurable and bounded by one. Now, since the σ-algebra $\mathcal{M}_{0,n}$ is included into $\mathcal{M}_{l(i,n),n}$, we obtain

$$\|\mathbb{E}(n^{-1/2}U_{i,n}h'_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 \le \|\mathbb{E}(n^{-1/2}U_{i,n}|\mathcal{M}_{l(i,n),n})\|_1\,.$$

Using that T preserves $\mathbb{P}$, the latter equals $\|\mathbb{E}(n^{-1/2}S_n(1/p)|\mathcal{M}_{0,n})\|_1$ and $\mathbf{S2}(b)$ implies that (3.13) holds.

Control of D_2.

To prove that $\|D_2\|_1$ tends to zero as n tends to infinity, it suffices to show that, for any positive integer i less than p,

$$\lim_{n\to\infty}\|\mathbb{E}(n^{-1}(U_{i,n}^2-\eta\Delta_i^2)h''_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 = 0\,. \tag{3.14}$$

Integrating with respect to $(\varepsilon_i)_{i\in\mathbb{Z}}$, we have

$$\|\mathbb{E}((U_{i,n}^2-\eta\Delta_i^2)h''_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 = \|\mathbb{E}((U_{i,n}^2-\eta[np^{-1}])\mathbb{E}_\varepsilon(h''_{i-1,i+1})|\mathcal{M}_{0,n})\|_1\,.$$

Since $n^{-1}[np^{-1}]$ converges to p^{-1}, (3.14) will be proved if for any positive integer i less than p

$$\lim_{n\to\infty}\|\mathbb{E}((n^{-1}U_{i,n}^2-\eta p^{-1})\mathbb{E}_\varepsilon(h''_{i-1,i+1})|\mathcal{M}_{0,n})\|_1 = 0\,.$$

Arguing as for the control of D_1, we have

$$\|\mathbb{E}((n^{-1}U_{i,n}^2 - \eta p^{-1})\mathbb{E}_\varepsilon(h''_{i-1,i+1})|\mathcal{M}_{0,n})\|_1 \le \|\mathbb{E}(n^{-1}U_{i,n}^2 - \eta p^{-1}|\mathcal{M}_{l(i,n),n})\|_1 .$$

Since both η and $\mathbb{P}$ are invariant by the transformation T, the latter equals $\|\mathbb{E}(n^{-1}S_n^2(1/p) - \eta p^{-1}|\mathcal{M}_{0,n})\|_1$ and $\mathbf{S2}(c)$ implies that (3.14) holds.

End of the proof of Proposition 4.

From (3.12), (3.13) and (3.14) we infer that, for any h in $B_1^3(\mathbb{R})$,

$$\lim_{p\to\infty} \limsup_{n\to\infty} \|D_1 + D_2 + D_3\|_1 = 0 .$$

This fact together with (3.5), (3.6), (3.7) and (3.10) imply Proposition 4.

4 Proof of Theorem 3

Once again, it suffices to prove that $\mathbf{S2}^*$ implies $\mathbf{S1}^*$. Suppose that $\mathbf{S1}^*(\varphi)$ holds for any bounded function φ of $\mathcal{H}^*$. Since $(n^{-1}(S_n^*(1))^2)_{n>0}$ is uniformly integrable, $\mathbf{S1}^*(\varphi)$ obviously extends to the whole space $\mathcal{H}^*$.

Consequently, we can restrict ourselves to the space of continuous bounded functions from $C([0,1])$ to $\mathbb{R}$. According to Lemma 2, the proof of Theorem 2 will be complete if we show that, for any Z_n in $R(\mathcal{M}_{k,n})$, the sequence $\mu^*[Z_n]$ converges weakly to the null measure as n tends to infinity.

Definitions 3. For $0 \le t_1 < \cdots < t_d \le 1$, define the functions $\pi_{t_1\ldots t_d}$ and $Q_{t_1\ldots t_d}$ from $C([0,1])$ to $\mathbb{R}^d$ by the equalities $\pi_{t_1\ldots t_d}(x) = (x(t_1),\ldots,x(t_d))$ and $Q_{t_1\ldots t_d}(x) = (x(t_1), x(t_2)-x(t_1),\ldots,x(t_d)-x(t_{d-1}))$. For any signed measure μ on $(C([0,1]),\mathcal{B}(C([0,1])))$ and any function f from $C([0,1])$ to $\mathbb{R}^d$, denote by μf^{-1} the image measure of μ by f.

Let μ and ν be two signed measures on $(C([0,1]),\mathcal{B}(C([0,1])))$. Recall that if $\mu\pi_{t_1\ldots t_d}^{-1} = \nu\pi_{t_1\ldots t_d}^{-1}$ for any positive integer d and any d-tuple such that $0 \le t_1 < \cdots < t_d \le 1$, then $\mu = \nu$. Consequently Theorem 2 is a straightforward consequence of the two following items

1. relative compactness: for any Z_n in $R(\mathcal{M}_{k,n})$, the family $(\mu_n^*[Z_n])_{n>0}$ is relatively compact with respect to the topology of weak convergence.

2. finite dimensional convergence: for any positive integer d, any d-tuple $0 \le t_1 < \cdots < t_d \le 1$ and any Z_n in $R(\mathcal{M}_{k,n})$ the sequence $\mu^*[Z_n]\pi_{t_1\ldots t_d}^{-1}$ converges weakly to the null measure as n tends to infinity.

4.1 Finite dimensional convergence

Clearly it is equivalent to take $Q_{t_1\dots t_d}$ instead of $\pi_{t_1\dots t_d}$ in item 2. The following lemma shows that finite dimensional convergence is a consequence of Condition **S2**. The stronger condition **S2*** is only required for tightness.

Lemma 4 *For any a in $\mathbb{R}^d$ define f_a from $\mathbb{R}^d$ to $\mathbb{R}$ by $f_a(x) =< a, x >$. If* **S2** *holds then, for any a in $\mathbb{R}^d$, any d-tuple $0 \le t_1 < \cdots < t_d \le 1$ and any Z_n in $R(\mathcal{M}_{k,n})$, the sequence $\mu_n^*[Z_n](f_a \circ Q_{t_1\dots t_d})^{-1}$ converges weakly to the null measure.*

Write $\mu_n^*[Z_n](f_a \circ Q_{t_1\dots t_d})^{-1}(\exp(i.)) = \mu_n^*[Z_n]Q^{-1}_{t_1\dots t_d}(\exp(i < a, . >))$. According to Lemma 4, the latter converges to zero as n tends to infinity. Taking $Z_n = 1$, we infer that the probability measure $\nu_n^*[1]Q^{-1}_{t_1\dots t_d}$ converges weakly to the probability measure $\nu^*[1]Q^{-1}_{t_1\dots t_d}$ and hence is tight. Since $|\mu^*[Z_n]Q^{-1}_{t_1\dots t_d}| \le \nu_n^*[1]Q^{-1}_{t_1\dots t_d} + \nu^*[1]Q^{-1}_{t_1\dots t_d}$, the sequence $(\mu^*[Z_n]Q^{-1}_{t_1\dots t_d})_{n>0}$ is tight. Consequently we can apply Lemma 1 to conclude that $\mu^*[Z_n]Q^{-1}_{t_1\dots t_d}$ converges weakly to the null measure.

Proof of Lemma 4. According to Lemma 2, we have to prove the property **S1**$^*(\varphi \circ f_a \circ Q_{t_1\dots t_d})$ for any continuous bounded function φ. Arguing as in Section 3.3, we can restrict ourselves to the class of function $B_1^3(\mathbb{R})$. Let h be any element of $B_1^3(\mathbb{R})$ and write

$$\begin{aligned}
&h \circ f_a \circ Q_{t_1\dots t_d}(n^{-1/2}U_n) - \int h \circ f_a \circ Q_{t_1\dots t_d}(x\sqrt{\eta})W(dx) \\
&\quad= \sum_{\ell=1}^{d} h_\ell\Big(a_\ell\Big(\frac{U_n(t_\ell) - U_n(t_{\ell-1})}{\sqrt{n}}\Big)\Big) - \int h_\ell(a_\ell x\sqrt{(t_\ell - t_{\ell-1})\eta})\, g(x)\, dx\,,
\end{aligned}$$

where the random variable $h_\ell(x)$ is equal to

$$\int h\Big(\sum_{i=1}^{\ell-1} ai\Big(\frac{U_n(t_i) - U_n(t_{i-1})}{\sqrt{n}}\Big) + x + \sum_{i=\ell+1}^{d} a_i x_i\sqrt{(t_i - t_{i-1})\eta}\Big) \prod_{i=\ell+1}^{d} g(x_i)dx_i\,.$$

Note that for any ω in Ω, the random function h_ℓ belongs to $B_1^3(\mathbb{R})$. To complete the proof of Lemma 4, it suffices to see that, for any positive integers k and ℓ, the sequence

$$\Big\|\mathbb{E}\Big(h_\ell\Big(a_l\Big(\frac{U_n(t_\ell) - U_n(t_{\ell-1})}{\sqrt{n}}\Big)\Big) - \int h_\ell(a_\ell x\sqrt{(t_\ell - t_{\ell-1})\eta})\, g(x)\, dx\Big|\mathcal{M}_{k,n}\Big)\Big\|_1 \tag{4.1}$$

tends to zero as n tends to infinity. Since h_ℓ is 1-lipshitz and bounded, we infer from the asymptotic negligibility of $n^{-1/2}X_{0,n}$ that

$$\lim_{n\to\infty}\Big\|h_\ell\Big(a_\ell\Big(\frac{U_n(t_\ell)-U_n(t_{\ell-1})}{\sqrt{n}}\Big)\Big)-h_\ell\Big(a_\ell\Big(\frac{S_n(t_\ell-t_{\ell-1})\circ T^{[nt_{\ell-1}]+1}}{\sqrt{n}}\Big)\Big)\Big\|_1=0. \tag{4.2}$$

Denote by g_ℓ the random function $g_\ell = h_\ell \circ T^{-[nt_{\ell-1}]-1}$. Combining (4.1), (4.2) and the fact that $\mathcal{M}_{k-1-[nt_{\ell-1}],n} \subseteq \mathcal{M}_{k,n}$, we infer that it suffices to prove that

$$\lim_{n\to\infty}\Big\|\mathbb{E}\Big(g_\ell(a_\ell n^{-1/2}S_n(u))-\int g_\ell(a_\ell x\sqrt{u\eta})\,g(x)\,dx\Big|\mathcal{M}_{k,n}\Big)\Big\|_1=0\,. \tag{4.3}$$

Since the random functions g_ℓ is $\mathcal{M}_{0,n}$-measurable (4.3) can be prove exactly as property **S1** (see Section (3.3)). This completes the proof of Lemma 4.

4.2 Relative compactness

In this section, we shall prove that the sequence $(\mu_n^*[Z_n])_{n>0}$ is relatively compact with respect to the topology of weak convergence. That is, for any increasing function f from $\mathbb{N}$ to $\mathbb{N}$, there exists an increasing function g with value in $f(\mathbb{N})$ and a signed mesure μ on $(C([0,1]),\mathcal{B}(C([0,1])))$ such that $(\mu^*_{g(n)}[Z_{g(n)}])_{n>0}$ converges weakly to μ.

Let Z_n^+ (resp. Z_n^-) be the positive (resp. negative) part of Z_n, and write

$$\mu_n^*[Z_n]=\mu_n^*[Z_n^+]-\mu_n^*[Z_n^-]=\nu_n^*[Z_n^+]-\nu_n^*[Z_n^-]-\nu^*[Z_n^+]+\nu^*[Z_n^-]\,,$$

where $\nu_n^*[Z]$ and $\nu^*[Z]$ are defined in 2. and 4. of Definitions 2. Obviously, it is enough to prove that each sequence of finite positive measures $(\nu_n^*[Z_n^+])_{n>0}$, $(\nu_n^*[Z_n^-])_{n>0}$, $(\nu^*[Z_n^+])_{n>0}$ and $(\nu^*[Z_n^-])_{n>0}$ is relatively compact. We prove the result for the sequence $(\nu_n^*[Z_n^+])_{n>0}$, the other cases being similar.

Let f be any increasing function from $\mathbb{N}$ to $\mathbb{N}$. Choose an increasing function l with value in $f(\mathbb{N})$ such that

$$\lim_{n\to\infty}\mathbb{E}(Z^+_{l(n)})=\liminf_{n\to\infty}\mathbb{E}(Z^+_{f(n)})\,.$$

We must sort out two cases:

1. If $\mathbb{E}(Z^+_{l(n)})$ converges to zero as n tends to infinity, then, taking $g=l$, the sequence $(\nu^*_{g(n)}[Z^+_{g(n)}])_{n>0}$ converges weakly to the null measure.

2. If $\mathbb{E}(Z^+_{l(n)})$ converges to a positive real number as n tends to infinity, we introduce, for n large enough, the probability measure p_n defined by

$p_n = (\mathbb{E}(Z^+_{l(n)}))^{-1}\nu^*_{l(n)}[Z^+_{l(n)}]$. Obviously if $(p_n)_{n>0}$ is relatively compact with respect to the topology of weak convergence, then there exists an increasing function g with value in $l(\mathbb{N})$ (and hence in $f(\mathbb{N})$) and a measure ν such that $(\nu^*_{g(n)}[Z^+_{g(n)}])_{n>0}$ converges weakly to ν. Since $(p_n)_{n>0}$ is a family of probability measures, relative compactness is equivalent to tightness. Here we apply Theorem 8.2 in Billingsley (1968): to derive the tightness of the sequence $(p_n)_{n>0}$ it is enough to show that, for each positive ϵ,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} p_n(x : w(x,\delta) \geq \epsilon) = 0\,, \tag{4.4}$$

where $w(x,\delta)$ is the modulus of continuity of the function x. According to the definition of p_n, we have

$$p_n(x : w(x,\delta) \geq \epsilon) = \frac{1}{\mathbb{E}(Z^+_{l(n)})} Z^+_{l(n)}.\mathbb{P}\left(w\Big(\frac{U_{l(n)}}{\sqrt{l(n)}}, \delta\Big) \geq \epsilon \right)\,.$$

Since both $\mathbb{E}(Z^+_{l(n)})$ converges to a positive number and $Z^+_{l(n)}$ is bounded by one, we infer that (4.4) holds if

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{P}\left(w\Big(\frac{U_{l(n)}}{\sqrt{l(n)}}, \delta\Big) \geq \epsilon \right) = 0\,. \tag{4.5}$$

From Theorem 8.3 and inequality (8.16) in Billingsley (1968), it suffices to prove that, for any positive ϵ,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{1}{\delta}\mathbb{P}\left(\frac{\overline{S}_{l(n)}(\delta)}{\sqrt{l(n)\delta}} \geq \frac{\epsilon}{\sqrt{\delta}} \right) = 0\,. \tag{4.6}$$

We conclude by noting that (4.6) follows straightforwardly from $\mathbf{S2}(a^*)$ and Markov's inequality.

Conclusion: In both cases there exists an increasing function g with value in $f(\mathbb{N})$ and a measure ν such that $(\nu^*_{g(n)}[Z^+_{g(n)}])_{n>0}$ converges weakly to ν. Since this is true for any increasing function f with value in $\mathbb{N}$, we conclude that the sequence $(\nu^*_n[Z^+_n])_{n>0}$ is relatively compact with respect to the topology of weak convergence. Of course, the same arguments apply to the sequences $(\nu^*_n[Z^-_n])_{n>0}$, $(\nu^*[Z^+_n])_{n>0}$ and $(\nu^*[Z^-_n])_{n>0}$, which implies the relative compactness of the sequence $(\mu^*_n[Z_n])_{n>0}$.

5 Proof of Corollary 1

We have to prove that if **S2** holds, then, for any bounded random variable Z, any t in $[0,1]$ and any φ in $\mathcal{H}$,

$$\lim_{n\to\infty} \mathbb{E}\left(Z\varphi(n^{-1/2}S_n(t))\right) = \mathbb{E}\Big(Z\int \varphi(x\sqrt{t\eta})g(x)dx\Big)\,. \tag{5.1}$$

Since $n^{-1}S_n^2(t)$ is uniformly integrable, we need only prove (5.1) for continuous bounded functions. Recall that $\mathcal{M}_{\infty,\infty} = \sigma(\bigcup_{k,n}\mathcal{M}_{k,n})$. Since both $S_n(t)$ and η are $\mathcal{M}_{\infty,\infty}$-measurable, we can and do suppose that so is Z.

Set $Z_{k,n} = \mathbb{E}(Z|\mathcal{M}_{k,n})$, and use the decomposition

$$\mathbb{E}\left(Z\varphi(n^{-1/2}S_n(t))\right) - \mathbb{E}\Big(Z\int \varphi(x\sqrt{t\eta})g(x)dx\Big) = T_1 + T_2 + T_3\,,$$

where

$$\begin{aligned} T_1 &= \mathbb{E}\left((Z - Z_{k,n})\varphi(n^{-1/2}S_n(t))\right) \\ T_2 &= \mathbb{E}\Big(Z_{k,n}\Big(\varphi(n^{-1/2}S_n(t)) - \int \varphi(x\sqrt{t\eta})g(x)dx\Big)\Big) \\ T_3 &= \mathbb{E}\Big((Z_{k,n} - Z)\int \varphi(x\sqrt{t\eta})g(x)dx\Big)\,. \end{aligned}$$

By assumption, the array $\mathcal{M}_{k,n}$ is nondecreasing in k and n. Since the random variable Z is $\mathcal{M}_{\infty,\infty}$-measurable, the martingale convergence theorem implies that $\lim_{k\to\infty}\lim_{n\to\infty}\|Z_{k,n} - Z\|_1 = 0$. Consequently,

$$\lim_{k\to\infty}\limsup_{n\to\infty}|T_1| = \lim_{k\to\infty}\limsup_{n\to\infty}|T_3| = 0\,.$$

On the other hand, Theorem 2 implies that T_2 tends to zero as n tends to infinity, which completes the proof of Corollary 1.

6 Applications of Theorem 1

6.1 Proof of Proposition 1

From Theorem 1 in Volný (1993), we know that (1.1) is equivalent to the existence of a random variable m in $H_0 \ominus H_{-1}$ such that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\Big\|\sum_{i=1}^{n} X_0\circ T^i - m\circ T^i\Big\|_2 = 0\,. \tag{6.1}$$

Let $W_n = m\circ T + \cdots + m \circ T^n$. Since $(m\circ T^i)_{i\in\mathbb{Z}}$ is a stationary sequence of martingale differences with respect to the filtration $(\mathcal{M}_i)_{i\in\mathbb{Z}}$, it satisfies **s2**. More precisely, $n^{-1}\mathbb{E}(W_n^2|\mathcal{M}_0)$ converges to $\eta = \mathbb{E}(m^2|\mathcal{I})$ in $\mathbb{L}^1$. Now, we shall use (6.1) to see that the sequence $(X_i)_{i\in\mathbb{Z}}$ also satisfies **s2**.

Proof of s2(b). From (6.1) it is clear that $n^{-1/2}\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2$ tends to zero as n tends to infinity.

Proof of s2(c). To see that $n^{-1}\mathbb{E}(S_n^2|\mathcal{M}_0)$ converges to η in $\mathbb{L}^1$, write

$$\begin{aligned} \frac{1}{n}\left\|\mathbb{E}(S_n^2 - W_n^2|\mathcal{M}_0)\right\|_1 &\leq \frac{1}{n}\left\|S_n^2 - W_n^2\right\|_1 \\ &\leq \frac{\|S_n + W_n\|_2}{\sqrt{n}}\frac{\|S_n - W_n\|_2}{\sqrt{n}}\,. \end{aligned} \tag{6.2}$$

From (6.1) the latter tends to zero as n tends to infinity and therefore $(X_i)_{i\in\mathbb{Z}}$ satisfies **s2**(c) with $\eta = \mathbb{E}(m^2|\mathcal{I})$.

Proof of s2(a). Using both that $n^{-1}W_n^2$ is uniformly integrable and that the function $x \to (1\wedge |x|)$ is 1-lipschitz, we have, for any positive real M,

$$\lim_{n\to\infty}\left(\frac{W_n^2}{n}\left|\Big(1\wedge\frac{|S_n|}{M\sqrt{n}}\Big) - \Big(1\wedge\frac{|W_n|}{M\sqrt{n}}\Big)\right|\right) = 0\,. \tag{6.3}$$

Since $|x^2(1\wedge|y|) - z^2(1\wedge|t|)| \leq |x^2 - z^2| + z^2|(1\wedge|y|) - (1\wedge|t|)|$, we infer from (6.2) and (6.3) that

$$\lim_{n\to\infty}\left\|\frac{S_n^2}{n}\Big(1\wedge\frac{|S_n|}{M\sqrt{n}}\Big) - \frac{W_n^2}{n}\Big(1\wedge\frac{|W_n|}{M\sqrt{n}}\Big)\right\|_1 = 0\,.$$

Now, the uniform integrability of $n^{-1}W_n^2$ yields

$$\lim_{M\to\infty}\limsup_{n\to\infty}\mathbb{E}\left(\frac{S_n^2}{n}\Big(1\wedge\frac{|S_n|}{M\sqrt{n}}\Big)\right) = 0\,,$$

which means exactly that $n^{-1}S_n^2$ is uniformly integrable. This completes the proof of Proposition 1.

6.2 Proof of Proposition 2

Let P_i be the projection operator onto $H_i \ominus H_{i-1}$: for any function f in $\mathbb{L}^2(\mathbb{P})$, $P_i(f) = \mathbb{E}(f|\mathcal{M}_i) - \mathbb{E}(f|\mathcal{M}_{i-1})$. We first recall a result due to Volný (1993), Theorem 6 (see Theorem 5 of the same paper or Annexe A Corollary 2 in Dedecker (1998) for weaker conditions).

Proposition 5 *Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. Define the σ-algebra $\mathcal{M}_{-\infty} = \bigcap_{i\in\mathbb{Z}} \mathcal{M}_i$ and consider the condition*

$$\mathbb{E}(X_0|\mathcal{M}_{-\infty}) = 0 \quad \textit{and} \quad \sum_{i>0} \|P_0(X_i)\|_2 < \infty\,. \tag{6.4}$$

Condition (6.4) implies (1.2).

We now prove that (1.3) implies (6.4) and hence (1.2). First, we have the orthogonal decomposition

$$X_k = \mathbb{E}(X_k|\mathcal{M}_{-\infty}) + \sum_{i=0}^{\infty} P_{k-i}(X_k)\,. \tag{6.5}$$

Since (1.3) implies that $\mathbb{E}(X_k|\mathcal{M}_{-\infty}) = 0$, we infer from (6.5) and the stationarity of $(X_i)_{i\in\mathbb{Z}}$ that

$$\sum_{k>0} L_k \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2^2 = \sum_{k>0} L_k \sum_{i\le 0} \|P_i(X_k)\|_2^2 = \sum_{i>0} \Big(\sum_{k=1}^{i} L_k\Big) \|P_0(X_i)\|_2^2\,.$$

Setting $a_i = L_1 + \cdots + L_i$, we infer that (1.3) is equivalent to

$$\mathbb{E}(X_0|\mathcal{M}_{-\infty}) = 0\,, \quad \sum_{i>0} a_i \|P_0(X_i)\|_2^2 < \infty \quad \text{and} \quad \sum_{i>0} \frac{1}{a_i} < \infty\,. \tag{6.6}$$

Now, Hölder's inequality in ℓ^2 gives

$$\sum_{i>0} \|P_0(X_i)\|_2 \le \Big(\sum_{i>0} \frac{1}{a_i}\Big)^{1/2} \Big(\sum_{i>0} a_i \|P_0(X_i)\|_2^2\Big)^{1/2} < \infty\,,$$

and (6.4) (hence (1.2)) follows from (1.3).

Now, to complete the proof of Proposition 2, it remains to show that (1.3) implies $\mathbf{s2}(a^*)$. This is a direct consequence of Proposition 8 Section 7, whose proof will be done by applying the following maximal inequality:

Proposition 6 *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of square-integrable and centered random variables, adapted to a nondecreasing filtration $(\mathcal{M}_i)_{i\in\mathbb{Z}}$. Define the σ-algebra $\mathcal{M}_{-\infty} = \bigcap_{i\in\mathbb{Z}} \mathcal{M}_i$ and the random variables $S_n = X_1 + \cdots + X_n$ and $S_n^* = \max\{0, S_1, \ldots, S_n\}$. For any positive integer i, let $(Y_{i,j})_{j\ge 1}$ be the martingale $Y_{i,j} = \sum_{k=1}^{j} P_{k-i}(X_k)$ and $Y_{i,n}^* = \max\{0, Y_{i,1}, \ldots, Y_{i,n}\}$. Let λ be any nonnegative real number and $\Gamma(i,k,\lambda) = \{Y_{i,k}^* > \lambda\}$. Assume that the sequence is regular: for any integer k, $\mathbb{E}(X_k|\mathcal{M}_{-\infty}) = 0$. For any two* *sequences of nonnegative numbers* $(a_i)_{i\geq 0}$ *and* $(b_i)_{i\geq 0}$ *such that* $K = \sum a_i^{-1}$ *is finite and* $\sum b_i = 1$ *we have*

$$\mathbb{E}\left((S_n^* - \lambda)_+^2\right) \leq 4K \sum_{i=0}^{\infty} a_i \Big(\sum_{k=1}^{n} \mathbb{E}(P_{k-i}^2(X_k)\mathbb{1}_{\Gamma(i,k,b_i\lambda)})\Big).$$

Proof. The proof is adapted from McLeish (1975b). From decomposition (6.5) with $\mathbb{E}(X_k|\mathcal{M}_{-\infty}) = 0$, we have $S_j = \sum_{i\geq 0} Y_{i,j}$ and therefore

$$(S_j - \lambda)_+ \leq \sum_{i\geq 0}(Y_{i,j} - b_i\lambda)_+ .$$

Applying Hölder's inequality and taking the maximum on both side, we get

$$(S_n^* - \lambda)_+^2 \leq K \sum_{i\geq 0} a_i (Y_{i,n}^* - b_i\lambda)_+^2 .$$

Taking the expectation and applying Proposition 1(a) of Dedecker and Rio (2000) to the martingale $(Y_{i,n})_{n\geq 1}$, we obtain Proposition 6.

To be complete, we would like to mention that condition (1.3) is close to optimality, as shown by Proposition 7 below.

Proposition 7 *There exists a sequence* $(X_i)_{i\in\mathbb{Z}}$ *satisfying the assumptions of Theorem 1, such that*

$$n^{-1}\mathbb{E}(S_n^2) \quad \text{converges to } \sigma^2 \text{ and } \quad \sum_{k>0} \|\mathbb{E}(X_k|\mathcal{M}_0)\|_2^2 < \infty ,$$

but the random variables $n^{-1/2}S_n$ *do not converges in distribution.*

See Dedecker (1998) Annexe A.3 for a proof.

6.3 Proof of Proposition 3

Proof of s2(a^*). Let $\overline{S}_n = \max\{|S_1|, \ldots, |S_n|\}$. From Proposition 1 in Dedecker and Rio (2000), we infer that $(n^{-1}(\overline{S}_n)^2)_{n>0}$ is uniformly integrable as soon as (1.4) holds.

Proof of s2(c). The fact that $\mathbb{E}(n^{-1}S_n^2|\mathcal{M}_0)$ converges in $\mathbb{L}^1$ to η has been already proved in Dedecker and Rio (2000), Section 4, Control of D_2.

Proof of s2(b). Using first the stationarity of $(X_i)_{i\in\mathbb{Z}}$ and next the orthogonal decomposition (6.5), we obtain

$$\sum_{i=0}^{\infty} \|P_0(X_i)\|_2^2 = \sum_{i=0}^{\infty} \|P_{-i}(X_0)\|_2^2 \leq \|X_0\|_2^2 . \tag{6.7}$$

Now, from the decomposition

$$\frac{X_{-1}}{\sqrt{n}}\mathbb{E}(S_n|\mathcal{M}_0) = \frac{X_{-1}}{\sqrt{n}}\mathbb{E}(S_n|\mathcal{M}_{-1}) + \frac{X_{-1}}{\sqrt{n}}\sum_{i=1}^{n} P_0(X_i)\,,$$

we infer that

$$\frac{1}{\sqrt{n}}\|X_{-1}\mathbb{E}(S_n|\mathcal{M}_0)\|_1 \le \frac{1}{\sqrt{n}}\|X_0\mathbb{E}(S_n\circ T|\mathcal{M}_0)\|_1 + \frac{\|X_0\|_2}{\sqrt{n}}\sum_{i=1}^{n}\|P_0(X_i)\|_2\,. \tag{6.8}$$

By (1.4), the first term on right hand tends to zero as n tends to infinity. On the other hand, we infer from (6.7) and Cauchy-Shwarz's inequality that $n^{-1/2}\sum_{i=1}^{n}\|P_0(X_i)\|_2$ vanishes as n goes to infinity, and so does the left hand term in (6.8). By induction, we can prove that for any positive integer k,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\|X_{-k}\mathbb{E}(S_n|\mathcal{M}_0)\|_1 = 0\,. \tag{6.9}$$

Now

$$\begin{aligned}\frac{1}{\sqrt{n}}\|\mathbb{E}(|X_0||\mathcal{I})\mathbb{E}(S_n|\mathcal{M}_0)\|_1 \le \frac{1}{\sqrt{n}}&\left\|\mathbb{E}(S_n|\mathcal{M}_0)\Big(\mathbb{E}(|X_0||\mathcal{I}) - \frac{1}{k}\sum_{i=1}^{k}|X_{-i}|\Big)\right\|_1\\ &+\frac{1}{\sqrt{n}}\left\|\mathbb{E}(S_n|\mathcal{M}_0)\frac{1}{k}\sum_{i=1}^{k}|X_{-i}|\right\|_1\,.\end{aligned} \tag{6.10}$$

From (6.9), the second term on right hand tends to zero as n tends to infinity. Applying first Cauchy-Schwarz's inequality and next the $\mathbb{L}^2$-ergodic theorem, we easily deduce that the first term on right hand is as small as we wish by choosing k large enough. Therefore

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\|\mathbb{E}(|X_0||\mathcal{I})\mathbb{E}(S_n|\mathcal{M}_0)\|_1 = 0\,. \tag{6.11}$$

Set $A = \{\mathbb{1}_{\mathbb{E}(|X_0||\mathcal{I})>0}\}$ and $B = A^c = \{\mathbb{1}_{\mathbb{E}(|X_0||\mathcal{I})=0}\}$. For any positive real m, we have

$$\begin{aligned}\frac{1}{\sqrt{n}}\|\mathbb{1}_A\mathbb{E}(S_n|\mathcal{M}_0)\|_1 \le \frac{1}{m\sqrt{n}}&\|\mathbb{E}(|X_0||\mathcal{I})\mathbb{E}(S_n|\mathcal{M}_0)\|_1\\ &+\frac{1}{\sqrt{n}}\|\mathbb{1}_{0<\mathbb{E}(|X_0||\mathcal{I})<m}\mathbb{E}(S_n|\mathcal{M}_0)\|_1\,.\end{aligned} \tag{6.12}$$

From (6.11), the first term on right hand tends to zero as n tends to infinity. Letting m goes to zero we infer that the second term on right hand of (6.12) is as small as we wish. Consequently

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\|\mathbb{1}_A\mathbb{E}(S_n|\mathcal{M}_0)\|_1 = 0\,. \tag{6.13}$$

On the other hand, noting that $\mathbb{E}(|X_0|\mathbb{1}_B) = 0$, we infer that X_0 is zero on the set B. Since B is invariant by T, X_k is zero on B for any k in $\mathbb{Z}$. Now arguing as in Claim 1(b) in Dedecker and Rio (2000), we obtain $\mathbb{E}(\mathbb{E}(|S_n||\mathcal{M}_0)|\mathcal{I}) = \mathbb{E}(|S_n||\mathcal{I})$. These two facts lead to

$$\|\mathbb{1}_B\mathbb{E}(S_n|\mathcal{M}_0)\|_1 \le \mathbb{E}(\mathbb{1}_B\mathbb{E}(\mathbb{E}(|S_n||\mathcal{M}_0)|\mathcal{I})) \le \mathbb{E}(|S_n|\mathbb{1}_B) \le 0 \tag{6.14}$$

Collecting (6.13) and (6.14), we conclude that $n^{-1/2}\|\mathbb{E}(S_n|\mathcal{M}_0)\|_1$ tends to zero as n tends to infinity. This completes the proof.

7 Applications of Theorem 3

In this section we extend Propositions 2 and 3 to the case of triangular arrays. In the next section, we shall see how to apply these results to Kernel density estimators.

Consider first the following condition, close to (1.3): there exists a sequence $(L_k)_{k>0}$ of positive numbers such that

$$\sum_{i>0}\Big(\sum_{k=1}^{i} L_k\Big)^{-1} < \infty \quad \text{and} \quad \lim_{N\to\infty}\limsup_{n\to\infty}\sum_{k=N}^{\infty} L_k\|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2 = 0\,. \tag{7.1}$$

If (7.1) is satisfied, define $Q(N,X)$ and $N_1(X)$ as follows:

$$Q(N,X) = \limsup_{n\to\infty}\sum_{k=N}^{\infty} L_k\|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2$$

and $N_1(X) = \inf\{N > 0 : Q(N) = 0\}$. If $N_1(X)$ is finite, we say that the array $(X_{i,n})$ is asymptotically $(N_1(X)-1)$-conditionally centered of type 1 (as usual, it is m-conditionally centered if $\mathbb{E}(X_{m+1,n}|\mathcal{M}_{0,n}) = 0$).

Proposition 8 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be as in Theorem 2. Define the random variables $V_n(t) = X_{1,n}^2 + \cdots + X_{n,[nt]}^2$. Assume that (7.1) is satisfied, and that Lindeberg's condition holds: for any positive ϵ, $\lim_{n\to\infty}\mathbb{E}(X_{0,n}^2\mathbb{1}_{|X_{0,n}|>\epsilon\sqrt{n}}) = 0$. Assume furthermore that*

$$\lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty}\mathbb{E}\Big(\frac{V_n(t)}{nt}\mathbb{1}_{V_n(t)\ge Mnt}\Big) = 0\,. \tag{7.2}$$

Then **S2**(a^*) *is satisfied.*

Remark 6. Let us compare Proposition 8, with Theorem (2.4) in McLeish (1977). In the particular case of triangular arrays, condition (2.2)(b) in McLeish with $\sigma_{n,i}^2 = n^{-1}$ (which seems to be the usual case) is equivalent with our notation to the assumption that $X_{0,n}$ is uniformly integrable. It is easy to verify that this assumption ensures that both Lindeberg's condition and (7.2) hold. However, in many interesting cases, the sequence $X_{0,n}$ is not uniformly integrable while both Lindeberg's condition and (7.2) are satisfied (it is the case, for instance, when considering Kernel estimators).

As a consequence of Proposition 8, we obtain the invariance principle:

Corollary 2 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be as in Theorem 2. Assume that both (7.1) and Lindeberg's condition are satisfied. Assume furhtermore that, for each $0 \leq k < N_1(X)$, there exists an $\mathcal{M}_{0,\inf}$-measurable random variable λ_k such that for any t in $]0,1]$,*

$$\frac{1}{nt}\sum_{i=1}^{[nt]} X_{i+k,n}X_{i,n} \text{ converges in } \mathbb{L}^1 \text{ to } \lambda_k. \tag{7.3}$$

Then Condition $\mathbf{S2}^*$ *holds with* $\eta = \lambda_0 + 2\sum_{k=1}^{N_1(X)-1}\lambda_k$.

Remark 7. Let us discuss the measurability assumption on λ_k. Starting from (7.3), one can easily show first that λ_k is invariant by T, and next that it is a limit of $\mathcal{M}_{0,n}$-measurables random variables. Suppose furthermore that the sequence $(\mathcal{M}_{0,n})_{n\geq 1}$ is nondecreasing, then λ_k is a limit of $\mathcal{M}_{0,\inf}$-measurable random variables. In that case, the assumption that λ_k is $\mathcal{M}_{0,\inf}$-measurable is useless, being automatically satisfied.

Remark 8. Note that if $X_{i,n} = X_i$ then both Lindeberg's condition and assumption (7.3) are satisfied, so that Corollary 2 extends Proposition 2 to the case of triangular arrays.

The next condition is the natural extension of Condition (1.4)

$$\lim_{N\to\infty}\limsup_{n\to\infty}\sup_{N\leq m\leq n}\Big\|X_{0,n}\sum_{k=N}^{m}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\Big\|_1 = 0\,. \tag{7.4}$$

If (7.4) is satisfied, define $R(N,X)$ and $N_2(X)$ as follows:

$$R(N,X) = \limsup_{n\to\infty}\sup_{N\leq m\leq n}\Big\|X_{0,n}\sum_{k=N}^{m}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\Big\|_1,$$

and $N_2(X) = \inf\{N > 0 \,:\, R(N, X) = 0\}$. If $N_2(X)$ is finite, we say that the array $(X_{i,n})$ is asymptotically $(N_2(X) - 1)$-conditionally centered of type 2.

Proposition 9 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ be as in Theorem 2 and $V_n(t)$ as in Proposition 8. If conditions (7.2) and (7.4) are satisfied then* **S2***(a^*) holds.*

As a consequence we obtain the following invariance principle:

Corollary 3 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be as in Theorem 2. Assume that (7.4) and* **S2***(b) are satisfied. Assume furhtermore that, for each $0 \le k < N_2(X)$, there exists an $\mathcal{M}_{0,\inf}$-measurable random variable λ_k such that (7.3) holds. Then Condition* **S2*** *holds with*

$$\eta = \lambda_0 + 2 \sum_{k=1}^{N_2(X)-1} \lambda_k \,.$$

Remark 9. Let us have a look to a particular case, for which $N_1(X) = 1$ (resp. $N_2(X) = 1$). Conditions (7.1) (resp. (7.4)) and (7.3) are satisfied if condition R1. (resp. R1') and R2. below are fulfilled

- R1. $\lim_{n\to\infty} \sum_{k=1}^{n} L_k \|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2 = 0$.
- R1' $\lim_{n\to\infty} \sum_{k=1}^{n} \|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_1 = 0$.
- R2. For any t in $]0, 1]$, $\frac{1}{nt}\sum_{i=1}^{[nt]} X_{i,n}^2$ converges in $\mathbb{L}^1$ to λ.

In the stationary case, these results extend on classical results for triangular arrays of martingale differences (see for instance Hall and Heyde (1980), Theorem 3.2), for which Condition R1. (resp. R1') is automatically satisfied. We shall see Section 8.3 that this particular case is sufficient to improve on many results in the context of Kernel estimators. In the same way, Corollary 2 (or 3) provides sufficient conditions for asymptotically m-conditionally centered arrays of type 1 (or 2) to satisfy the functional CCLT.

7.1 Proofs of Proposition 8 and Corollary 2

Proof of Proposition 8. With the same notations as in Proposition 6, define, for $b_i = 2^{-i-1}$,

$$T_{i,n}(t) = \sum_{k=1}^{[nt]} P_{k-i}^2(X_{k,n}) \quad \text{and} \quad c_{i,n}(t, M) = \mathbb{E}\Big(\frac{T_{i,n}(t)}{nt} \mathbb{1}_{\Gamma(i,[nt],b_i M\sqrt{nt})}\Big) \,.$$

Define $S_n^*(t) = \sup_{0\le s\le t}\{0, S_n(s)\}$. From Proposition 6, we have

$$\frac{1}{nt}\mathbb{E}\left((S_n^*(t) - M\sqrt{nt})_+^2\right) \le 4K\sum_{i=0}^{\infty} a_i c_{i,n}(t,M)\,. \tag{7.5}$$

Note that, by stationarity, $c_{i,n}(t,M) \le c_{i,n}(t,0) \le \|P_0(X_{i,n})\|_2^2$. Now, taking $a_i = L_1 + \cdots + L_i$, we have

$$\sum_{i=N}^{\infty} a_i\|P_0(X_{i,n})\|_2^2 = \sum_{k=N}^{\infty} L_k\|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2 + \sum_{k=1}^{N-1} L_k\|\mathbb{E}(X_{N,n}|\mathcal{M}_{0,n})\|_2^2\,, \tag{7.6}$$

and from (7.1) and the definition of $N_1(X)$, we infer that

$$\lim_{N\to N_1(X)}\limsup_{n\to\infty}\sum_{i=N}^{\infty} a_i c_{i,n}(t,0) = 0\,. \tag{7.7}$$

Suppose we can prove that, for any $0 \le i < N_1(X)$,

$$\lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty}\mathbb{E}\Big(\frac{T_{i,n}(t)}{nt}\mathbb{I}_{T_{i,n}(t)\ge Mnt}\Big) = 0\,. \tag{7.8}$$

Since by stationarity $\mathbb{E}(T_{i,n}(t)) = [nt]\|P_0(X_{i,n})\|_2^2$, (7.8) implies first that $\sup_{n>0}\|P_0(X_{i,n})\|_2^2 \le B_i < \infty$, which together with Doob's inequality yield the upper bounds

$$\text{for } 0 \le i < N_1(X), \qquad \sup_{t\in[0,1]}\sup_{n>0}\mathbb{P}\left(\Gamma(i,[nt],b_iM\sqrt{nt})\right) \le \frac{4B_i}{b_i^2M^2}\,. \tag{7.9}$$

According to (7.7), we can find a finite integer $N(\epsilon) \le N_1(X)$ such that

$$\limsup_{n\to\infty} 4K\sum_{i=N(\epsilon)}^{\infty} a_i c_{i,n}(t,0) \le \epsilon\,.$$

Now, since $c_{i,n}(t,M) \le c_{i,n}(t,0)$ we obtain from (7.5) that

$$\limsup_{n\to\infty}\frac{1}{nt}\mathbb{E}\left((S_n^*(t) - M\sqrt{nt})_+^2\right) \le \epsilon + \limsup_{n\to\infty}\sum_{i=0}^{N(\epsilon)-1} a_i c_{i,n}(t,M)\,. \tag{7.10}$$

Here note that (7.8) together with (7.9) yield

$$\text{for } 0 \le i < N_1(X), \qquad \lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty} c_{i,n}(t,M) = 0$$

so that, from (7.10)

$$\lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty}\frac{1}{nt}\mathbb{E}\left((S_n^*(t) - M\sqrt{nt})_+^2\right) \le \epsilon\,. \tag{7.11}$$

Of course the same arguments apply to the sequence $(-X_i)_{i\in\mathbb{Z}}$ so that (7.11) holds for $\overline{S}_n(t)$. This being true for any positive ϵ, $\mathbf{S2}(a^*)$ follows.

To complete the proof, it remains to show that Lindeberg's condition together with assumption (7.2) imply (7.8). Since

$$\begin{aligned} P_{k-i}^2(X_{k,n}) &\leq 2\left(\mathbb{E}^2(X_{k,n}|\mathcal{M}_{k-i,n}) + \mathbb{E}^2(X_{k,n}|\mathcal{M}_{k-i-1,n})\right) \\ &\leq 2\left((\mathbb{E}(X_{k,n}^2|\mathcal{M}_{k-i,n}) + \mathbb{E}(X_{k,n}^2|\mathcal{M}_{k-i-1,n})\right) , \end{aligned}$$

we infer that (7.8) holds if, setting $U_{i,n}(t) = \sum_{k=1}^{[nt]} \mathbb{E}(X_{k,n}^2|\mathcal{M}_{k-i,n})$,

$$\text{for any } i \geq 0, \quad \lim_{M\to\infty} \sup_{t\in[0,1]} \limsup_{n\to\infty} \mathbb{E}\Big(\frac{U_{i,n}(t)}{nt} \mathbb{I}_{U_{i,n}(t)\geq Mnt}\Big) = 0 \,. \tag{7.12}$$

Now if Lindeberg's condition holds, classical arguments ensure that

$$\lim_{n\to\infty} \frac{1}{nt} \|U_{i,n}(t) - U_{i+1,n}(t)\|_1 = 0 \,,$$

so that if (7.12) holds for i, it holds for $i+1$. For $i=0$, note that (7.12) is exactly (7.2) since $U_{0,n}(t) = V_n(t)$. This ends the proof of Proposition 8.

Proof of Corollary 2. Since Condition (7.3) implies (7.2), $\mathbf{S2}(a^*)$ follows from Proposition 8. It remains to prove $\mathbf{S2}(b)$ and (c).

From (7.6) we infer that (7.1) is equivalent to

$$\lim_{N\to N_1(X)} \limsup_{n\to\infty} \sum_{i=N}^{\infty} a_i \|P_0(X_{i,n})\|_2^2 = 0 \,. \tag{7.13}$$

Since the sequence a_i^{-1} is nonincreasing and such that $\sum a_i^{-1} < \infty$, it follows that $a_i^{-1} = o(i^{-1})$, which implies that $i = O(a_i)$. Consequently (7.13) holds for $a_i = i$ and from (7.6) again we obtain

$$\lim_{N\to N_1(X)} \limsup_{n\to\infty} \sum_{k=N}^{\infty} \|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2 = 0 \,. \tag{7.14}$$

Starting from (7.14), we first prove $\mathbf{S2}(b)$. From (7.3) with $k=0$ we easily infer that, for each positive N, $n^{-1/2}\|\mathbb{E}(X_{1,n} + \cdots + X_{N-1,n}|\mathcal{M}_{0,n})\|_2$ tends to zero as $n\to\infty$. Therefore, to prove $\mathbf{S2}(b)$ it suffices to see that

$$\lim_{N\to N_1(X)} \limsup_{n\to\infty} \frac{1}{n} \Big\| \sum_{k=N}^{nt} \mathbb{E}(X_{k,n}|\mathcal{M}_{0,n}) \Big\|_2^2 = 0 \,. \tag{7.15}$$

Since $2\mathbb{E}(X_{i,n}|\mathcal{M}_{0,n})\mathbb{E}(X_{j,n}|\mathcal{M}_{0,n}) \leq \mathbb{E}^2(X_{i,n}|\mathcal{M}_{0,n}) + \mathbb{E}^2(X_{j,n}|\mathcal{M}_{0,n})$,

$$\frac{1}{n} \Big\| \sum_{k=N}^{n} \mathbb{E}(X_{k,n}|\mathcal{M}_{0,n}) \Big\|_2^2 \leq \sum_{k=N}^{nt} \|\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_2^2 \,,$$

and (7.15) follows from (7.14).

We now prove **S2**(c). For any finite integer $0 \leq N \leq N_1(X)$, define the variable $\eta_N = \lambda_0 + 2(\lambda_1 + \cdots + \lambda_{N-1})$ and the two sets

$$\begin{aligned} \Lambda_N &= [1,[nt]]^2 \cap \{(i,j) \in \mathbb{Z}^2 : |i-j| < N\} \quad \text{and} \\ \overline{\Lambda}_N &= [1,[nt]]^2 \cap \{(i,j) \in \mathbb{Z}^2 : j-i \geq N\}, \quad \text{so that,} \end{aligned}$$

$$\Big\|\mathbb{E}\Big(\frac{S_n^2(t)}{nt} - \eta_N\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 \leq \Big\|\eta_N - \frac{1}{nt}\sum_{\Lambda_N} X_{i,n}X_{j,n}\Big\|_1 + \frac{2}{nt}\Big\|\sum_{\overline{\Lambda}_N}\mathbb{E}(X_{i,n}X_{j,n}|\mathcal{M}_{0,n})\Big\|_1 . \quad (7.16)$$

From (7.3), we can easily prove that the first term on right hand goes to zero as n tends to infinity. It remains to control the second term on right hand. Set $Y_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n}|\mathcal{M}_{0,n})$ and write

$$\frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N}\mathbb{E}(X_{i,n}X_{j,n}|\mathcal{M}_{0,n})\Big\|_1 \leq \frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N}\mathbb{E}(Y_{i,n}Y_{j,n}|\mathcal{M}_{0,n})\Big\|_1 + \frac{1}{nt}\sum_{\overline{\Lambda}_N}\|\mathbb{E}(X_{i,n}|\mathcal{M}_{0,n})\mathbb{E}(X_{j,n}|\mathcal{M}_{0,n})\|_1 . \quad (7.17)$$

Arguing as for (7.15), we infer from (7.14) that the second term on right hand in (7.17) is as small as we wish by choosing N large enough. Next, by using the operators P_l, we have:

$$\begin{aligned} \frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N}\mathbb{E}(Y_{i,n}Y_{j,n}|\mathcal{M}_{0,n})\Big\|_1 &\leq \frac{1}{nt}\sum_{i=1}^{nt}\sum_{k=N}^{\infty}\|\mathbb{E}(Y_{i,n}Y_{i+k,n}|\mathcal{M}_{0,n})\|_1 \\ &\leq \frac{1}{nt}\sum_{i=1}^{nt}\sum_{k=N}^{\infty}\sum_{l=1}^{i}\|P_l(X_{i,n})P_l(X_{i+k,n})\|_1 \end{aligned}$$

Using Cauchy-Schwarz's inequality, we obtain that the last term is less than

$$\frac{1}{nt}\sum_{i=1}^{nt}\sum_{l=-\infty}^{i}\|P_l(X_{i,n})\|_2\Big(\sum_{k=N}^{\infty}\|P_l(X_{i+k,n}\|_2\Big),$$

and by stationarity, we conclude that

$$\frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N}\mathbb{E}(Y_{i,n}Y_{j,n}|\mathcal{M}_{0,n})\Big\|_1 \leq \Big(\sum_{i=0}^{\infty}\|P_0(X_{i,n})\|_2\Big)\Big(\sum_{k=N}^{\infty}\|P_0(X_{k,n})\|_2\Big).$$

This last inequality together with (7.13) and Cauchy-Schwarz's inequality yield

$$\lim_{N\to N_1(X)} \limsup_{n\to\infty} \frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N} \mathbb{E}(Y_{i,n}Y_{j,n}|\mathcal{M}_{0,n})\Big\|_1 = 0\,,$$

and $\mathbf{S2}(c)$ follows. This completes the proof of Corollary 2.

7.2 Proofs of Proposition 9 and Corollary 3

Proof of Proposition 9. Let $S_n^*(t)$ be as in (7.5) and define the set $G(t,M,n)$ by $G(t,M,n)=\{S_n^*(t)>M\sqrt{nt}\}$. From Proposition 1 in Dedecker and Rio (2000), we have, for any positive integer N,

$$\frac{1}{nt}\mathbb{E}\left((S_n^*(t)-M\sqrt{nt})_+^2\right) \le 8\mathbb{E}\Big(\mathbb{1}_{G(t,M,n)}\frac{1}{nt}\sum_{k=1}^{nt}\sum_{i=0}^{N-1}|X_{k,n}X_{k+i,n}|\Big) \\ + 8\sup_{N\le m\le nt}\Big\|X_{0,n}\sum_{k=N+1}^{m}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\Big\|_1 \quad (7.18)$$

From (7.4) we can choose $N(\epsilon)$ large enough so that the second term on right hand is less than ϵ. Taking $M=0$ in (7.18) and using Assumption (7.2), we infer that there exists a finite real B such that

$$\limsup_{n\to\infty}\frac{1}{nt}\mathbb{E}\left((S_n^*(t))^2\right)\le B \quad \text{so that} \quad \limsup_{n\to\infty}\mathbb{P}(G(t,M,n))\le \frac{B}{M^2}\,.$$

This last bound together with (7.2) imply that

$$\lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty}\mathbb{E}\Big(\mathbb{1}_{G(t,M,n)}\frac{1}{nt}\sum_{k=1}^{nt}\sum_{i=0}^{N-1}|X_{k,n}X_{k+i,n}|\Big)=0\,,$$

so that, from (7.18),

$$\lim_{M\to\infty}\sup_{t\in[0,1]}\limsup_{n\to\infty}\frac{1}{nt}\mathbb{E}\left((S_n^*(t)-M\sqrt{nt})_+^2\right)\le\epsilon\,. \quad (7.19)$$

Of course the same arguments apply to the sequence $(-X_i)_{i\in\mathbb{Z}}$ so that (7.19) holds for $\overline{S}_n(t)$. This being true for any positive ϵ, $\mathbf{S2}(a^*)$ follows.

Proof of Corollary 3. Since Condition (7.3) implies (7.2), $\mathbf{S2}(a^*)$ follows from Proposition 9. It remains to prove $\mathbf{S2}(c)$. Starting from inequality (7.16) for any finite integer $N\le N_2(X)$, we have to show that

$$\lim_{N\to N_2(X)}\limsup_{n\to\infty}\frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N}\mathbb{E}(X_{i,n}X_{j,n}|\mathcal{M}_{0,n})\Big\|_1=0\,. \quad (7.20)$$

Using first the inclusion $\mathcal{M}_{0,n} \subseteq \mathcal{M}_{i,n}$ for any positive i and second the stationarity of the sequence, we obtain succesively

$$\begin{aligned} \frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N} \mathbb{E}(X_{i,n}X_{j,n}|\mathcal{M}_{0,n})\Big\|_1 &\leq \frac{1}{nt}\Big\|\sum_{\overline{\Lambda}_N} X_{i,n}\mathbb{E}(X_{j,n}|\mathcal{M}_{i,n})\Big\|_1 \\ &\leq \sup_{N\leq m\leq n}\Big\|X_{0,n}\sum_{k=N}^{m}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\Big\|_1 . \end{aligned}$$

and (7.20) follows from (7.4). This completes the proof of Corollary 3.

8 Kernel estimators

Let Y be a real-valued random variable with unknown density f, and define the stationary sequence $(Y_i)_{i\in\mathbb{Z}} = (Y \circ T^i)_{i\in\mathbb{Z}}$. We wish to estimate f at point x from the data $Y_1, \ldots, Y_n$. To this aim, we shall consider as usual (cf. Rosenblatt (1956b)) the kernel-type estimator of f

$$f_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\Big(\frac{x-Y_i}{h_n}\Big) .$$

It is well-known that the study of the bias $\mathbb{E}(f_n(x))-f(x)$ does not depend on the dependence properties of the process $(Y_i)_{i\in\mathbb{Z}}$, but on the regularity of f. Consequently, we only deal with the asymptotic behavior of $f_n(x)-\mathbb{E}(f_n(x))$.

8.1 A general result

Definitions 4. We say that a Borel measurable function K from $\mathbb{R}$ to $\mathbb{R}$ is a kernel if:

$$\|K\|_\infty < \infty, \quad \lim_{|u|\to\infty} |u|K(u) = 0, \quad \text{and} \quad \int K(u)\,du = 1 . \tag{8.1}$$

Definitions 5. For any x in $\mathbb{R}^d$ and any positive real M, denote by $\mathbb{L}^1_{M,x}$ the space of integrable function from $\mathbb{R}^d$ to $\mathbb{R}$ with support in the cube $[x_1 - M, x_1 + M] \times \cdots \times [x_d - M, x_d + M]$. If f belongs to $\mathbb{L}^1_{M,x}$, denote by $\|f\|_{1,M}(x)$ its norm. For any positive measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, define the two quantities $\|\mu\|_{\infty,M}(x) = \sup\{\mu(|f|) : \|f\|_{1,M}(x) \leq 1\}$ and $\|\mu\|_\infty(x) = \lim_{M\to 0} \|\mu\|_{\infty,M}(x)$. If $\|\mu\|_\infty(x) < \infty$, we say that μ belongs to $\mathbb{L}^\infty_x$.

For each i in $\mathbb{Z}$, denote by μ_i the law of (Y_0, Y_i). We make the following assumptions:

A1 Y has density f which is continuous at x.

A2 for each i in $\mathbb{Z}$, μ_i belongs to $\mathbb{L}^\infty_{(x,x)}$.

Proposition 10 *Let K be a kernel, and h_n be a sequence of positive numbers such that h_n tends to zero and nh_n tends to infinity as n tends to infinity. Let Y_i be a strictly stationary sequence satisfying* **A1** *and* **A2**. *Define succesively $\mathcal{M}_{i,n} = \mathcal{M}_i = \sigma(Y_j, j \le i)$,*

$$X_{i,n} = \frac{1}{\sqrt{h_n}}\Big\{K\Big(\frac{x-Y_i}{h_n}\Big) - \mathbb{E}\Big(K\Big(\frac{x-Y_i}{h_n}\Big)\Big)\Big\}\,, \tag{8.2}$$

and U_n as in Theorem 3. Assume that either (7.1) or (7.4) and **S2***(b) holds. If furthermore*

$$\lim_{n\to\infty} \sup_{k\le n} \frac{1}{n}\sum_{i=1}^{k} \mathrm{Cov}(X_{0,n}^2, X_{i,n}^2) = 0\,, \tag{8.3}$$

then the process U_n satisfies **S1*** *with $\eta = f(x)\|K\|_2^2$.*

Proof. It is a consequence of either Corollary 2 or Corollary 3. In order to apply Corollary 2, we need to prove that $X_{0,n}$ satisfies Lindeberg's condition. Define $K_n(x) = h_n^{-1}K(h_n^{-1}x)$. Since **A1** holds, classical arguments ensure that $\mathbb{E}(K_n(x-Y))$ converges to $f(x)$, and consequently $\sqrt{h_n}\mathbb{E}(K_n(x-Y))$ tends to 0 as n tends to infinity. Now, recall that

$$X_{0,n} = \sqrt{h_n}\,(K_n(x-Y) - \mathbb{E}(K_n(x-Y)))\ .$$

Therefore it suffices to show that the sequence $\sqrt{h_n}K_n(x-Y)$ satisfies Lindeberg's condition. The density f being continuous at x, there exist two positive reals M and C such that for any y in $[x-M, x+M]$, $f(y)$ is less than C. Setting $K[M] = K\mathbb{1}_{[-M,M]^c}$, we have

$$\begin{aligned}\frac{1}{h_n}\mathbb{E}\left(K^2\Big(\frac{x-Y}{h_n}\Big)\mathbb{1}_{\sqrt{h_n}|K_n(x-Y)|>\epsilon\sqrt{n}}\right) &\le \frac{1}{h_n}\|K[M/h_n]\|_\infty^2 \\ &+ C\int K^2(z)\mathbb{1}_{|K(z)|>\epsilon\sqrt{nh_n}}\,dz\,.\end{aligned}$$

Note that $uK^2(u)$ tends to zero as $|u|$ tends to infinity, which implies that the first term on right hand vanishes as h_n tends to zero. On the other hand, since $\|K\|_2$ is finite, the second term tends to zero as nh_n tends to infinity.

To complete the proof, it remains to see that (8.3) implies (7.3) with $\lambda_0 = f(x)\|K\|_2^2$ and $\lambda_k = 0$ for any positive integer k. To prove the second point, note that for any $k > 0$ ($\star$ denoting the convolution),

$$\|X_{0,n}X_{k,n}\|_1 \le h_n(\mu_k \star |K_n \otimes K_n|(x,x) + 3(f \star |K_n|(x))^2)\,. \tag{8.4}$$

Again, choose M such that for any y in $[x-M, x+M]$, $f(y)$ is less than C. Splitting the integral in four parts over the four sets $[x-M, x+M]^2$, $[x-M, x+M] \times [x-M, x+M]^c$, $[x-M, x+M]^c \times [x-M, x+M]$ and $([x-M, x+M]^c)^2$, we get the upper bound

$$\begin{aligned}\mu_k \star |K_n \otimes K_n|(x,x) \leq \|K\|_1^2 \, \|\mu_k\|_{\infty,M}(x) \\ + \frac{2C}{h_n}\|K[M/h_n]\|_\infty \|K\|_1 + \frac{1}{h_n^2}\|K[M/h_n]\|_\infty^2 \,. \quad (8.5)\end{aligned}$$

Now assumption **A2** ensures that for M small enough, the first term on right hand is finite. The two other terms on right hand tending to zero as n tends to infinity, we infer that $\sup_{n>0} \mu_k \star |K_n \otimes K_n|(x,x) < \infty$. Both this fact and (8.4) imply that, for any positive k, $\|X_{0,n}X_{k,n}\|_1$ tends to 0 as n tends to infinity, so that (7.3) holds with $\lambda_k = 0$.

It remains to see that (7.3) holds for $k = 0$ with $\lambda_0 = f(x)\|K\|_2^2$. Since the function $\|K\|_2^{-2}K^2$ is a kernel, classical arguments ensure that $\mathbb{E}(X_{0,n}^2)$ converges to $f(x)\|K\|_2^2$. Now

$$\frac{1}{(nt)^2}\Big\|\sum_{i=1}^{[nt]} X_{i,n} - \mathbb{E}(X_{0,n})\Big\|_2^2 = \frac{[nt]}{(nt)^2}\mathrm{Var}(X_{0,n}^2) + \frac{2}{nt}\sum_{k=1}^{[nt]} \frac{1}{nt}\sum_{i=1}^{k-1} \mathrm{Cov}(X_{0,n}^2, X_{i,n}^2)\,.$$

Using that $\|K\|_4^{-4}K^4$ is a kernel, we infer that $\mathrm{Var}(X_{0,n}^2) = O(h_n^{-1})$, so that $n^{-1}\mathrm{Var}(X_{0,n}^2)$ tends to zero as nh_n tends to infinity. Since furthermore (8.3) implies that the second term on right hand tends to zero as n tends to infinity, the result follows.

8.2 Application to mixing sequences

In this section, we give three differents applications of Proposition 10 to the case of mixing sequences.

Definitions 6. Let $\mathcal{U}$ and $\mathcal{V}$ be two σ-algebras of $\mathcal{A}$. The strong mixing coefficient of Rosenblatt (1956a) is defined by

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)| : U \in \mathcal{U},\ V \in \mathcal{V}\}. \quad (8.6)$$

The ϕ-mixing coefficient introduced by Ibragimov (1962) can be defined by

$$\phi(\mathcal{U}, \mathcal{V}) = \sup\{\|\mathbb{P}(V|\mathcal{U}) - \mathbb{P}(V)\|_\infty, V \in \mathcal{V}\}\,. \quad (8.7)$$

Between those coefficients, the following relation holds : $2\alpha(\mathcal{U}, \mathcal{V}) \leq \phi(\mathcal{U}, \mathcal{V})$.

Definitions 7. Let Y and ε be two real valued random variables, and define $(Y_i, \varepsilon_i)_{i\in\mathbb{Z}} = (Y \circ T^i, \varepsilon \circ T^i)_{i\in\mathbb{Z}}$. We say that $(Y_i, \varepsilon_i)_{i\in\mathbb{Z}}$ is a functional autoregressive process if there exists a Borel-measurable function ψ such that $Y_i = \psi(Y_{i-1})+\varepsilon_i$, where ε_1 is independent of the σ-algebra $\mathcal{M}_0 = \sigma(Y_i, i \leq 0)$.

Consider the two following assumptions, which are stronger than **A2**:

A3 assumption **A2** holds and moreover $\lim_{M\to 0} \sup_{i\in\mathbb{Z}} \|\mu_i\|_{\infty,M}(x, x) < \infty$.

A4 $(Y_i, \varepsilon_i)_{i\in\mathbb{Z}}$ is a functional autoregressive process and ε has density h which is continuous and bounded.

Corollary 4 *Let $(Y_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence. Let h_n, $(X_{i,n})_{i\in\mathbb{Z}}$, and $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be as in Proposition 10, and define the process U_n as in Theorem 3. Write $\phi_{\infty,1}(k) = \phi(\mathcal{M}_0, \sigma(Y_k))$ and $\alpha_{\infty,1}(k) = \alpha(\mathcal{M}_0, \sigma(Y_k))$. Consider the three following conditions:*

$$\textbf{(i)} \sum_{k>0} \phi_{\infty,1}(k) < \infty, \quad \textbf{(ii)} \sum_{k>n} \alpha_{\infty,1}(k) = o\Big(\frac{1}{n}\Big), \quad \textit{and } \textbf{(iii)} \sum_{k>0} \alpha_{\infty,1}(k) < \infty.$$

If either $(\mathbf{A1}, \mathbf{A2}, \textbf{(i)})$, $(\mathbf{A1}, \mathbf{A3}, \textbf{(ii)})$ *or* $(\mathbf{A4}, \textbf{(iii)})$ *holds, then the process U_n satisfies* $\mathbf{S1}^*$ *with $\eta = f(x)\|K\|_2^2$.*

Remark 10. The mixing rate **(ii)** was first required by Robinson (1983) for the more stringent coefficients $\alpha_{\infty,\infty}(k) = \alpha(\mathcal{M}_0, \sigma(Y_i, i \geq k))$. To understand the difference between $\alpha_{\infty,\infty}$ and $\alpha_{\infty,1}$, note that the convergence of $\alpha_{\infty,\infty}(n)$ to zero implies that $\mathcal{M}_\infty$ is independent of $\mathcal{I}$, while the same property for $\alpha_{\infty,1}(n)$ does not even implies that $\sigma(Y_0, Y_1)$ is independent of $\mathcal{I}$. This means in particular that a large class of nonergodic processes are concerned by Corollary 4. For such processes, it may be surprising that the variance term η is degenerate. In fact, this is due to Assumption **A2** which implies that, for any positive integer i, the covariance between $X_{0,n}$ and $X_{i,n}$ tends to zero as n tends to infinity.

Remark 11. Let us recall some recent results about mixing rates for autoregressive processes (cf. Tuominen and Tweedie (1994) and Ango Nzé (1998)). Suppose that $(Y_i, \varepsilon_i)_{i\in\mathbb{Z}}$ is a functional autoregressive process and that

1. the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ is i.i.d, the density h of ε satisfies $\|h\|_\infty(0) > 0$, and there exists $S \geq 1$ such that $\mathbb{E}|\varepsilon|^S < \infty$.

2. ψ is continuous and there exist a positive real R and δ in $]0,1[$ such that for any real x more than R, $|\psi(x)| \le |x|(1-|x|^{-\delta})$.

Then the sequence $(Y_i)_{i\in\mathbb{Z}}$ is arithmetically absolutely regular with rate $n^{1-S/\delta}$. In particular, condition **(iii)** is satisfied as soon as $S > 2\delta$.

8.3 Proof of Corollary 4.

We shall prove that if either **(A1,A2,(i))**, **(A1,A3,(ii))** or **(A4,(iii))** holds, then Condition (8.3), (7.4) and $\mathbf{S2}(b)$ hold, so that Proposition 10 applies.

We begin by a preliminary lemma concerning the autoregressive model, whose proof is omitted.

Lemma 5 *Let $(Y_i, \varepsilon_i)_{i\in\mathbb{Z}}$ be a functional autoregressive process, as defined in Definitions 7, and suppose that ε satisfies* **A4**. *Then the sequence $(Y_i)_{i\in\mathbb{Z}}$ satisfies* **A1** *and* **A2**.

To check (8.3), (7.4) and $\mathbf{S2}(b)$, we need to control covariances. This can be done with the help of the two following inequalities: if X and Y are two real random variables respectively $\mathcal{U}$ and $\mathcal{V}$-measurable, then we have

$$|\mathrm{Cov}(X,Y)| \le 4\alpha(\mathcal{U},\mathcal{V})\|X\|_\infty\|Y\|_\infty\,, \tag{8.8}$$

and for any conjugate exponents p,q

$$|\mathrm{Cov}(X,Y)| \le 2\phi^{1/p}(\mathcal{U},\mathcal{V})\phi^{1/q}(\mathcal{V},\mathcal{U})\|X\|_p\|Y\|_q\,. \tag{8.9}$$

(8.8) is due to Ibragimov (1962) and (8.9) to Peligrad (1983).

Proof of Condition (8.3)

a) Condition **(A1,A2,(i))**. From inequality (8.9), we have

$$|\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| \le 2\phi_{\infty,1}(k)\|X_{0,n}^2\|_1\|X_{k,n}\|_\infty^2 \le \frac{8}{h_n}\phi_{\infty,1}(k)\mathbb{E}(X_{0,n}^2)\|K\|_\infty^2\,,$$

and (8.3) follows from the inequality

$$\frac{1}{n}\sum_{k=1}^{n}|\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| \le 2\mathbb{E}(X_{0,n}^2)\|K\|_\infty\Big(\frac{1}{nh_n}\sum_{k=1}^{\infty}\phi_{\infty,1}(k)\Big)\,.$$

b) Condition **(A4,(iii))**. Recall that $Y_i = \psi(Y_{i-1}) + \varepsilon_i$, and define

$$Z_{k-1,n} = \int \frac{1}{h_n}\Big\{K\Big(\frac{x-\psi(Y_{k-1})-z}{h_n}\Big) - \mathbb{E}\Big(K\Big(\frac{x-Y_k}{h_n}\Big)\Big)\Big\}^2 h(z)dz \tag{8.10}$$

Since ε_k is independant of $\mathcal{M}_{k-1}$, $\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2) = \mathrm{Cov}(X_{0,n}^2, Z_{k-1,n})$. Applying inequality (8.8), we obtain

$$|\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq \frac{16}{h_n}\alpha_{\infty,1}(k-1)\|K\|_\infty^2 \|Z_{k-1,n}\|_\infty \,,$$

Since h is bounded, we easily deduce from (8.10) that there exists a constant C such that $\|Z_{k-1,n}\|_\infty \leq C$. We conclude the proof as in **a)**.

c) Condition **(A1,A3,(ii))**. Write first

$$\begin{aligned} |\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| &\leq \|X_{0,n}^2 X_{k,n}^2\|_1 + \|X_{0,n}^2\|_1^2 \\ &\leq \frac{4\|K\|_\infty}{h_n}\|X_{0,n}X_{k,n}\|_1 + \|X_{0,n}^2\|_1^2 \,. \end{aligned}$$

From (8.4), (8.5) and **A3**, we infer that $\sup_{k>0}\|X_{0,n}X_{k,n}\|_1 = O(h_n)$, so that there exists a constant C such that

$$\sup_{n>0,k>0} |\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq C \,. \tag{8.11}$$

On the other hand, applying once again inequality (8.8), we have

$$|\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq \frac{64}{h_n^2}\alpha_{\infty,1}(k)\|K\|_\infty^4 \,. \tag{8.12}$$

Now (8.11) and (8.12) together yield

$$\frac{1}{n}\sum_{k=1}^{n} |\mathrm{Cov}(X_{0,n}^2, X_{k,n}^2)| \leq \frac{C}{nh_n} + \frac{64\|K\|_\infty^4}{nh_n^2}\sum_{k=[1/h_n]}^{\infty} \alpha_{1,\infty}(k) \,,$$

and (8.3) follows from assumption **(ii)** and the fact that nh_n tends to infinity.

Proof of Condition (7.4)

We shall prove that the sequence is asymptotically 0-conditionally centered of type 2, and more precisely that Condition R1' of Remark 9 holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n} \|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})\|_1 = 0 \,. \tag{8.13}$$

a) Condition **(A1,A2,(i))**. Set $s(k,n) = \mathbb{I}_{\mathbb{E}(X_{k,n}|\mathcal{M}_0)>0} - \mathbb{I}_{\mathbb{E}(X_{k,n}|\mathcal{M}_0)\leq 0}$. Since

$$\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| = \mathrm{Cov}(|X_{0,n}|s(k,n), X_{k,n}) \,, \tag{8.14}$$

we obtain from (8.9) the upper bound

$$\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq 4\phi_{\infty,1}(k)\frac{1}{\sqrt{h_n}}\|X_{0,n}\|_1\|K\|_\infty .$$

Since **(i)** holds and $\sup_{n>0} h_n^{-1/2}\|X_{0,n}\|_1$ is finite, the last inequality yields

$$\sum_{k=1}^{\infty}\sup_{n>0}\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| < \infty . \tag{8.15}$$

On the other hand, from inequality (8.4) we infer that

$$\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq h_n(\mu_k \star |K_n \otimes K_n|(x,x) + 3(f \star |K_n|(x))^2) , \tag{8.16}$$

and the last bound is a $O(h_n)$ because of **A2** and (8.5). Since **(i)** is satisfied, (8.13) follows from (8.15), (8.16) and the dominated convergence theorem.

b) Condition **(A4,(iii))**. Since $Y_i = \psi(Y_{i-1}) + \varepsilon_i$, we obtain from (8.14)

$$\begin{aligned}&\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \\ &\qquad = \sqrt{h_n}\mathrm{Cov}\left(|X_{0,n}|s(k,n), \int K(u)h(x - h_n u - \psi(Y_{k-1}))du\right)\end{aligned}$$

and we conclude as in **a)** by using (8.16) and the inequality

$$\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq 8\alpha_{\infty,1}(k-1)\|K\|_\infty\|h\|_\infty\|K\|_1 .$$

c) Condition **(A1,A3,(ii))**. From (8.16) and **A3**, we infer that there exists a constant C such that

$$\sup_{k>0}\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq Ch_n . \tag{8.17}$$

On the other hand, from (8.14) and inequality (8.8) we have

$$\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq 4\alpha_{\infty,1}(k)\|X_{0,n}\|_\infty^2 \leq \frac{16}{h_n}\alpha_{\infty,1}(k)\|K\|_\infty^2 . \tag{8.18}$$

Now (8.17) and (8.18) together yield

$$\sum_{k=1}^{n}\mathbb{E}|X_{0,n}\mathbb{E}(X_{k,n}|\mathcal{M}_0)| \leq Ch_n\left[\frac{\epsilon}{h_n}\right] + \frac{16\|K\|_\infty^2}{h_n}\sum_{k=[\epsilon/h_n]}^{\infty}\alpha_{1,\infty}(k) ,$$

for any positive ϵ. Hence (8.13) follows straightforwardly from **(ii)**.

Proof of S2(b)

Assume that **(iii)** is satisfied. Using again $s(k,n)$, we have

$$\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 \leq \sum_{k=1}^{[nt]} \mathrm{Cov}(X_{k,n}, s(k,n))\,.$$

Since $s(k,n)$ is $\mathcal{M}_0$-measurable and bounded by 1, inequality (8.8) leads to

$$\mathrm{Cov}(X_{k,n}, s(k,n)) \leq 4\alpha_{\infty,1}(k)\|X_{0,n}\|_\infty \leq \frac{8}{\sqrt{h_n}}\alpha_{\infty,1}(k)\|K\|_\infty\,.$$

Finally

$$\frac{1}{\sqrt{n}}\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 \leq \frac{8\|K\|_\infty}{\sqrt{nh_n}}\sum_{k=1}^{\infty}\alpha_{\infty,1}(k)\,,$$

which tends to zero as nh_n tends to infinity. To conclude, note that **(i)** and **(ii)** are both stronger than (**iii**) and hence imply **S2**(b).

9 Appendix

9.1 Proof of Lemma 1

$1 \Rightarrow 2$ is obvious. It remains to prove that $2 \Rightarrow 1$. We proceed in 3 steps.

Step 1. Let $\mathcal{D}(\mathbb{R}^d)$ be the space of functions from $\mathbb{R}^d$ to $\mathbb{C}$ which are infinetely derivable with compact support. Let φ be any element of $\mathcal{D}(\mathbb{R}^d)$ and set $\bar{\varphi}(t) = \hat{\varphi}(-t)$. From Plancherel equality, we have $\mu_n(\varphi) = (2\pi)^{-d}\hat{\mu}_n(\bar{\varphi})$. The function $\bar{\varphi}$ being infinitely derivable and fast decreasing, it belongs to $\mathbb{L}^1(\lambda)$. Since $|\hat{\mu}_n|$ converges to zero everywhere and is bounded by $\sup_{n>0}\|\mu_n\|$, the dominated convergence theorem implies that $\hat{\mu}_n(\bar{\varphi})$ tends to zero as n tends to infinity. Consequently, for any φ in $\mathcal{D}(\mathbb{R}^d)$, $\mu_n(\varphi)$ converges to zero as n tends to infinity.

Step 2. Let φ be any function from $\mathbb{R}^d$ to $\mathbb{R}$, continuous and with compact support. For any positive ϵ, there exists φ_ϵ in $\mathcal{D}(\mathbb{R}^d)$ such that $\|\varphi - \varphi_\epsilon\|_\infty \leq \epsilon$. Since furthermore $\sup_{n>0}\|\mu_n\|$ is finite, we infer from Step 1 that $\mu_n(\varphi)$ tends to zero as n tends to infinity.

Step 3. For any positive integer k, let f_k be a positive and continuous function from $\mathbb{R}^d$ to $\mathbb{R}$ satisfying: $\|f_k\|_\infty \leq 1$, $f(x) = 1$ for any x in $[-k,k]^d$, $f(x) = 0$ for any x in $([-k-1,k+1]^d)^c$.

For any continuous bounded function φ, write

$$|\mu_n(\varphi)| \leq |\mu_n(\varphi f_k)| + \|\varphi\|_\infty|\mu_n|(([-k,k]^d)^c)\,.$$

From Step 2 the first term on right hand tends to zero as n tends to infinity. Since the sequence $(\mu_n)_{n>0}$ is tight, the second term on right hand is as small as we wish by choosing k large enough. This completes the proof of 1.

9.2 Proof of Lemma 3

For any real number m, consider $A_1 = \{X \leq m\}$, $A_2 = \{\mathbb{E}(X|\mathcal{M}) \leq m\}$, $C_1 = A_1 \cap A_2^c$ and $C_2 = A_2 \cap A_1^c$. Since by assumption the random variables X and $\mathbb{E}(X|\mathcal{M})$ are identically distributed, it follows that $\mathbb{P}(A_1) = \mathbb{P}(A_2)$, $\mathbb{P}(C_1) = \mathbb{P}(C_2)$ and $\mathbb{E}(X\mathbb{1}_{A_1}) = \mathbb{E}(X\mathbb{1}_{A_2})$. This implies in particular that $\mathbb{E}((X-m)\mathbb{1}_{C_1}) = \mathbb{E}((X-m)\mathbb{1}_{C_2})$. These terms having opposite signs, they are zero. Since $X - m$ is positive on C_2, it follows that C_2 and consequently $A_1 \Delta A_2$ have probability zero (Δ denoting the symmetric difference). Now, it is easily seen that

$$\mathbb{E}((\mathbb{E}(X|\mathcal{M}))^2\mathbb{1}_{A_1}) = \mathbb{E}((\mathbb{E}(X|\mathcal{M}))^2\mathbb{1}_{A_2}) = \mathbb{E}(X^2\mathbb{1}_{A_1}) \quad \text{and} \tag{9.19}$$

$$\mathbb{E}(X\mathbb{E}(X|\mathcal{M})\mathbb{1}_{A_1}) = \mathbb{E}(X\mathbb{E}(X|\mathcal{M})\mathbb{1}_{A_2}) = \mathbb{E}((\mathbb{E}(X|\mathcal{M}))^2\mathbb{1}_{A_2}) \tag{9.20}$$

According to (8.12) and (8.13), we obtain

$$\begin{aligned} \|(X - \mathbb{E}(X|\mathcal{M}))\mathbb{1}_{A_1}\|_2^2 &= \|X\mathbb{1}_{A_1}\|_2^2 + \|\mathbb{E}(X|\mathcal{M})\mathbb{1}_{A_1}\|_2^2 - 2\mathbb{E}(X\mathbb{E}(X|\mathcal{M})\mathbb{1}_{A_1}) \\ &= 0 \end{aligned} \tag{9.21}$$

Since (8.14) is true for any real m, it follows that $X = \mathbb{E}(X|\mathcal{M})$ almost surely, and Lemma 3 is proved.

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L.S.T.A. Université Paris VI
4, place jussieu, Boîte 158
75252 Paris cedex 05
E-mail: dedecker@ccr.jussieu.fr, merleve@ccr.jussieu.fr