Conditional convergence to infinitely divisible distributions with finite variance

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Abstract

We obtain new conditions for partial sums of an array with stationary rows to converge to a mixture of infinitely divisible distributions with finite variance. More precisely, we show that these conditions are necessary and sufficient to obtain *conditional convergence*. If the underlying σ-algebras are nested, conditional convergence implies stable convergence in the sense of Rényi. From this general result we derive new criteria expressed in terms of conditional expectations, which can be checked for many processes such as m-conditionally centered arrays or mixing arrays. When it is relevant, we establish the weak convergence of partial sum processes to a mixture of Lévy processes in the space of cadlag functions equipped with Skorohod's topology. The cases of Wiener processes, Poisson processes and Bernoulli distributed variables are studied in detail.

Key words: infinitely divisible distributions, Lévy processes, stable convergence, triangular arrays, mixing processes.

1 Introduction

For any distribution function F of a finite measure and any real γ, denote by $\mu^1_{\gamma,F}$ the probability measure with characteristic function

$$\phi_{\gamma,F}(z) = \exp\Big(iz\gamma + \int (e^{izx} - 1 - izx)\frac{1}{x^2}dF(x)\Big)\,, \tag{1.1}$$

and define for any positive real t the probability $\mu^t_{\gamma,F} = \mu^1_{t\gamma,tF}$. The distribution $\mu^t_{\gamma,F}$ has mean $t\gamma$ and variance $tF(\infty)$ and satisfies the equation $\mu^t_{\gamma,F} * \mu^s_{\gamma,F} = \mu^{t+s}_{\gamma,F}$. One says that $\mu^1_{\gamma,F}$ is an infinitely divisible distribution with finite variance.

Suppose that for each n, $(X_{i,n})_{1\leq i\leq n}$ are i.i.d. random variables such that $\mathbb{E}(X^2_{0,n})$ tends to 0 as n tends to infinity. From Theorem 2 of Chapter 4 in Gnedenko and Kolmogorov (1954), we

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know that $S_n(t) = X_{1,n} + \cdots + X_{[nt],n}$ converges in distribution to $\mu^t_{\gamma,F}$ if and only if $n\mathbb{E}(X_{0,n})$ converges to γ, and $\lim_{n\to\infty} n\mathbb{E}(X^2_{0,n}\mathbb{I}_{X_{0,n}\leq x}) = F(x)$ for any continuity point x of F.

Brown and Eagleson (1971) extended this result to (non necessarily stationary) arrays whose rows are martingale differences sequences. If $\mathcal{M}_{i,n} = \sigma(X_{k,n}, 1 \leq k \leq i)$ and $\mathbb{E}(X_{k,n}|\mathcal{M}_{k-1,n}) = 0$, the main condition for the convergence to $\mu^t_{0,F}$ is: for any continuity point x of F

$$\sum_{k=1}^{[nt]} \mathbb{E}(X^2_{k,n}\mathbb{I}_{X_{k,n}\leq x}|\mathcal{M}_{k-1,n}) \quad \text{converges in probability to} \quad tF(x)\,. \tag{1.2}$$

As noticed by Eagleson (1975), there is no reason why the function F appearing in (1.2) should be nonrandom. In fact it is easy to build simple examples for which F is random (see the example of Section 2.5), so that the limiting distribution is a mixture of infinitely divisible distributions. If $X_{i,n} = n^{-1/2}X_i$, the limit is a mixture of centered Gaussian distributions (i.e. $\gamma = 0$ and $F = \lambda\mathbb{I}_{[0,\infty[}$, λ possibly random). In that case, Aldous and Eagleson (1978) proved that $S_n(t)$ converges *stably* in the sense of Rényi (1963) to a random variable with characteristic function $\mathbb{E}(\phi_{0,tF}(z))$. If $\mathcal{M}_{i,n} \subseteq \mathcal{M}_{i,n+1}$, Jeganathan (1982, part I) proved the stable convergence to infinitely divisible distributions under Brown and Eagleson's conditions. Stable convergence is more precise than convergence in distribution and may be useful in several contexts, especially in connection with randomly normalized or randomly indexed sums (see Aldous and Eagleson (1978) and Chapters 2, 3 and 9 of Castaing *et al.* (2004)).

For arrays $(X_{i,n})_{i\in\mathbb{Z}}$ with stationary rows and $\mathcal{M}_{i,n} = \sigma(X_{k,n}, k \leq i)$, Dedecker and Merlevède (2002) proposed necessary and sufficient conditions for $S_n(t)$ to satisfy the *conditional central limit theorem*, which implies stable convergence to a mixture of Gaussian distributions provided that $\mathcal{M}_{i,n} \subseteq \mathcal{M}_{i,n+1}$. The conditions may be written as:

$$\limsup_{n\to\infty} \|\mathbb{E}(S_n(t)|\mathcal{M}_{0,n})\|_1 = 0 \ \text{ and } \ \lim_{t\to 0}\limsup_{n\to\infty} \left\|\mathbb{E}\Big(\frac{S_n^2(t)}{t}\mathbb{I}_{S_n(t)\leq x} - \lambda\mathbb{I}_{x\geq 0}\Big|\mathcal{M}_{0,n}\Big)\right\|_1 = 0\,. \tag{1.3}$$

where the second limit holds for some nonnegative random variable λ and any $x \neq 0$.

The natural question is now: what happens when $\lim_{n\to\infty}\|\mathbb{E}(S_n(t) - \gamma t|\mathcal{M}_{0,n})\|_1 = 0$ for some random variable γ, and we replace $\lambda\mathbb{I}_{[0,\infty[}$ by any (random) distribution function F in (1.3)? Such conditions would be necessary and sufficient, since we can easily prove that $\lim_{t\to 0}\limsup_{n\to\infty}\|t^{-1}\int_{-\infty}^{x} y^2\mu^t_{\gamma,F}(dy) - F(x)\|_1 = 0$ for any continuity point of $x \to \mathbb{E}(F(x))$. Two other questions are: can we obtain from (1.3) (with any F) sufficient conditions in terms of individual variables $X_{i,n}$ for $S_n(t)$ to converge to a mixture of infinitely divisible distributions? Can we say something about the convergence of the process $\{S_n(t), t \in [0,1]\}$ in the space of cadlag functions equipped with Skorokhod's distance?

In Section 2, we shall give positive answers to these questions. We first show in Theorem 1 that the result of Dedecker and Merlevède (2002) remains valid when replacing 0 and $\lambda\mathbb{I}_{[0,\infty[}$ in (1.3) by any square integrable random variable γ and any random distribution function F such that $\mathbb{E}(F(\infty))$ is finite, and we present the application of this result to stable convergence. Next, we give in Section 2.1 sufficient conditions for (1.3) to hold for a large class of dependent arrays. The dependence conditions are expressed in terms of conditional expectations and may be checked for many processes, such as m-conditionally centered arrays or nonuniform mixing arrays. In some important cases, we show that our conditions are optimal (see Corollary 3). Furthermore, in the particular case of m-dependent Bernoulli-distributed arrays, we infer from Hudson *et al.* (1989) and Kobus (1995) that the conditions we impose are necessary and sufficient. In Section 2.2 we give sufficient conditions for the process $\{S_n(t), t \in [0,1]\}$ to converge stably to a mixture of Lévy processes in the space of cadlag functions equipped with Skorokhod's distance. The additional condition we impose is related to the topoligical stucture of that space, and may be shown to be necessary in some particular cases (see Remark 6). The cases of Wiener and Poisson processes are studied in detail in Sections 2.3 and 2.4 respectively. In Section 2.5 we give the application of our results to the important case of Bernoulli-distributed random variables. In that case the limiting distribution is a mixture of integer-valued compound Poisson distributions.

To prove Theorem 1 of Section 2, we adapt Lindeberg's method with increasing blocks in place of individual variables. The idea is to split $S_n(1)$ into p blocks distributed as $S_n(1/p)$ and to replace them by blocks of i.i.d variables with law $\mu_{\gamma,F}^{1/n}$. To go back to individual variables, we use a second adaptation of Lindeberg's method (see the proof of Lemma 3) and a maximal inequality established in Dedecker and Rio (2000). The latter is used once again to prove the tightness of $\{S_n(t), t \in [0,1]\}$ in the space of cadlag functions (see the proof of Lemma 5).

2 Convergence to infinitely divisible distributions

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving $\mathbb{P}$. An element A is said to be invariant if $T(A) = A$. Let $\mathcal{I}$ be the σ-algebra of all invariant sets. The probability $\mathbb{P}$ is ergodic if each element of $\mathcal{I}$ has measure 0 or 1.

We say that a function F from $\mathbb{R} \times \Omega$ to $\mathbb{R}_+$ is a $\mathcal{M}$-measurable distribution function if for every ω the function $F(.,\omega)$ is a distribution function of a finite measure, and for any x in $\mathbb{R} \cup \{\infty\}$ the random variable $F(x)$ is $\mathcal{M}$-measurable with $\mathbb{E}(F(\infty)) < \infty$.

Let $\mathcal{H}$ be the space of continuous real functions φ such that $x \to |(1+x^2)^{-1}\varphi(x)|$ is bounded. Given a $\mathcal{M}$-measurable random variable γ and a $\mathcal{M}$-measurable distribution function F, we introduce for each ω the probability measure $\mu^t_{\gamma(\omega),F(.,\omega)}$ defined via (1.1). Since $\mathbb{E}(F(\infty))$ is finite, the random measure $\mu^t_{\gamma,F}$ maps $\mathcal{H}$ into $\mathbb{L}^1(\mathcal{M})$.

Theorem 1 *For each positive integer n, let $\mathcal{M}_{0,n}$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{M}_{0,n} \subseteq T^{-1}(\mathcal{M}_{0,n})$. Define the nondecreasing filtration $(\mathcal{M}_{i,n})_{i\in\mathbb{Z}}$ by $\mathcal{M}_{i,n} = T^{-i}(\mathcal{M}_{0,n})$ and $\mathcal{M}_{i,\inf} = \sigma\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\mathcal{M}_{i,k}\right)$. Let $X_{0,n}$ be a $\mathcal{M}_{0,n}$-measurable and square integrable random variable. Define $(X_{i,n})_{i\in\mathbb{Z}}$ by $X_{i,n} = X_{0,n} \circ T^i$, and let $S_n(t) = X_{1,n} + \cdots + X_{[nt],n}$, for t in $[0,1]$. Suppose that $\mathbb{E}(X_{0,n}^2)$ tends to zero as n tends infinity. The following statements are equivalent:*

S1 *There exists an $\mathcal{M}_{0,\inf}$-measurable square integrable random variable γ and an $\mathcal{M}_{0,\inf}$-measurable distribution function F, such that for any φ in $\mathcal{H}$, any t in $[0,1]$ and any* *positive integer k,*

$$\mathbf{S1}(\varphi) \qquad \lim_{n\to\infty} \Big\|\mathbb{E}\Big(\varphi(S_n(t)) - \mu^t_{\gamma,F}(\varphi)\ \Big|\mathcal{M}_{k,n}\Big)\Big\|_1 = 0\,.$$

S2 *(a) There exists an $\mathcal{M}_{0,\inf}$-measurable square integrable random variable γ such that*

$$\lim_{t\to 0}\limsup_{n\to\infty} \Big\|\mathbb{E}\Big(\frac{S_n(t)}{t} - \gamma\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 = 0\,.$$

(b) There exists a $\mathcal{M}_{0,\inf}$-measurable distribution function F such that, for any continuity point x (including $+\infty$) of the function $x \to \mathbb{E}(F(x))$,

$$\lim_{t\to 0}\limsup_{n\to\infty} \Big\|\mathbb{E}\Big(\frac{S_n^2(t)}{t}\mathbb{1}_{S_n(t)\le x} - F(x)\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 = 0\,. \tag{2.1}$$

Moreover, $\gamma = \gamma\circ T$ almost surely, and $F = F\circ T$ almost surely.

Remark 1. Let $Z_n(t)$ be such that $\lim_{t\to 0}\limsup_{n\to\infty} t^{-1}\|Z_n(t)\|_2^2 = 0$. It is easy to see that $\mathbf{S2}(b)$ holds if and only if, for any continuity point x (including $+\infty$) of the function $x\to\mathbb{E}(F(x))$, (2.1) holds with $S_n(t) - Z_n(t)$ instead of $S_n(t)$. We shall use this simple remark in Section 2.1, with $Z_n(t) = [nt]\mathbb{E}(X_{0,n})$ or $Z_n(t) = [nt]\mathbb{E}(X_{0,n}|\mathcal{I})$.

Let us look to the case where $F = \lambda\mathbb{1}_{[a,\infty[}$. If $a = 0$, $\mu^t_{\gamma,F}$ is the normal distribution $\mathcal{N}(t\gamma, t\lambda)$. If $\lambda = 0$, $\mu^t_{\gamma,F}$ is the unit mass at $t\gamma$. For any (a,λ) in $\mathbb{R}^*\times\mathbb{R}^*_+$, $\mu^t_{\gamma,F}$ is the law of $a(X(a,t\lambda) - t\lambda/a^2) + t\gamma$, $X(a,\lambda)$ having Poisson distribution $\mathcal{P}(\lambda/a^2)$. As an immediate consequence of Theorem 1, we obtain the following corollary:

Corollary 1 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$, $S_n(t)$ be as in Theorem 1. The statements* **P1**, **P2** *are equivalent:*

P1 *There exist a $\mathcal{M}_{0,\inf}$-measurable random variable a, a $\mathcal{M}_{0,\inf}$-measurable square integrable random variable γ and a nonnegative $\mathcal{M}_{0,\inf}$-measurable random variable λ such that,* **S1** *holds for the couple $(\gamma, F = \lambda\mathbb{1}_{[a,\infty[})$.*

P2 *Condition* **S2***(a) holds and*

S2*(b1) There exist a $\mathcal{M}_{0,\inf}$-measurable random variable a such that*

$$\lim_{t\to 0}\limsup_{n\to\infty}\mathbb{E}\Big(\frac{S_n^2(t)}{t}(1\wedge|S_n(t)-a|)\Big) = 0\,.$$

S2*(b2) There exist a nonnegative $\mathcal{M}_{0,\inf}$-measurable random variable λ such that (2.1) holds for $x=\infty$ and $F(\infty)=\lambda$.*

It is clear that **S2** imply much more than convergence in distribution. Arguing as in Corollary 1 in Dedecker and Merlevède (2002), one can prove that, for any bounded $\sigma(\cup_{i\ge 1}\mathcal{M}_{i,\inf})$-measurable variable Z and any φ in $\mathcal{H}$, the sequence $\mathbb{E}(Z\varphi(S_n(t))$ converges to $\mathbb{E}(Z\mu^t_F(\varphi))$. In particular, the following corollary holds:

Corollary 2 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1. Suppose that the sequence of σ-algebras $(\mathcal{M}_{0,n})_{n\geq 1}$ is nondecreasing. If Condition* **S2** *is satisfied, then, for any φ in $\mathcal{H}$, the sequence $\varphi(S_n(t))$ converges weakly in $\mathbb{L}^1$ to $\mu_F^t(\varphi)$.*

Remark 2. Corollary 2 implies that, if the sequence $(\mathcal{M}_{0,n})_{n\geq 1}$ is nondecreasing, then $S_n(t)$ converges *stably* to a random variable Y_t whose conditional distribution with respect to $\mathcal{I}$ is μ_F^t. We refer to Aldous and Eagleson (1978) and to Chapters 2, 3 and 9 of the book by Castaing *et al* (2004) for a complete exposition of the concept of stability (introduced by Rényi (1963)) and its connection to weak $\mathbb{L}^1$-convergence. Note that stable convergence is a useful tool to establish weak convergence of joint distributions as well as randomly indexed sums (see again Aldous and Eagleson (1978) and the references therein, or the book by Hall and Heyde (1980)). Note also that the condition on $(\mathcal{M}_{0,n})_{n\geq 1}$ is exactly the "nesting condition" (3.21) in Theorem 3.2 of Hall and Heyde (1980), which is known to be related to the stable convergence (see the discussion on page 59 of the latter). If furthermore F is constant, then the convergence is *mixing*. If $\mathbb{P}$ is ergodic, this result is a consequence of Theorem 4 in Eagleson (1976a) (see Application 4.2 therein). For a review of mixing results see Csörgő and Fischler (1973).

2.1 Sufficient conditions

In this section, we give sufficient conditions in terms of the individual variables $X_{i,n}$ for Property **S1** to hold. In the sequel, $\mathcal{B}$ is either the σ-field $\mathcal{I}$ of all invariant sets or the trivial σ-field $\{\emptyset, \Omega\}$. We then define the array with stationary rows $X'_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n}|\mathcal{B})$. The kind of dependence we consider is described in the two following definitions.

Definition 1. Let $(X_{i,n})$ and $\mathcal{M}_{i,n}$ be as in Theorem 1, and define for any positive integer N

$$R_1(N,X) = \lim_{t\to 0} \limsup_{n\to\infty} \sup_{N\leq m\leq [nt]} n\Big\|X'_{0,n}\sum_{k=N}^{m}\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\Big\|_1 , \tag{2.2}$$

and $N_1(X) = \inf\{N>0 \,:\, R_1(N)=0\}$ ($N_1(X)$ may be infinite). We say that $(X_{i,n})$ satisfies the weak-dependence condition **WD** if **S2**(a) holds and $R_1(N,X)$ tends to zero as N tends to infinity. If $N_1(X)$ is finite, we say that the array $(X'_{i,n})$ is asymptotically $(N_1(X)-1)$-conditionally centered (as usual, it is m-condionally centered if $\mathbb{E}(X'_{m+1,n}|\mathcal{M}_{0,n})=0$).

In addition to the weak dependence condition **WD**, we need to control some residual terms.

Definition 2. Let $(X_{i,n})$ be as in Theorem 1. For any (k,n) in $(\mathbb{N}\times\mathbb{Z})$, define $S'_{k,n}$ by: $S'_{k,n}=0$ if $k\leq 0$ and $S'_{k,n} = X'_{1,n}+\cdots+X'_{k,n}$ otherwise. For any positive integer N define

$$R_2(N,X) = \limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\Big(\sum_{k=1}^{[nt]}\left(X'^2_{k,n}+2|X'_{k,n}(S'_{k-1,n}-S'_{k-N,n})|\right)(1\wedge|S'_{k-N}|)\Big) , \tag{2.3}$$

and $N_2(X) = \inf\{N>0 \,:\, R_2(N)=0\}$ ($N_2(X)$ may be infinite). We say that the array $(X_{i,n})$ is **EQ** if $n\mathbb{E}(X'^2_{0,n})$ is bounded and $R_2(N,X)$ tends to zero as N tends to infinity.

We now give sufficient conditions in terms of finite blocks $X'_{k-N,n}+\cdots+X'_{k,n}$ for **S1** to hold. We say that a function F from $\mathbb{R}\times\Omega$ is a $\mathcal{M}$-measurable BV function if it can be written as the difference of two $\mathcal{M}$-measurable distribution functions. We say that a sequence of random BV functions F_N converges weakly to a random BV function F if for any continuous bounded function f, $\lim_{N\to\infty}\|\int f dF_N - \int f dF\|_1 = 0$.

Proposition 1 *Take $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and* **EQ** *and set $N_0(X) = N_1(X)\vee N_2(X)$. For any N and any f, let $V_{N,n}(k) = \sum_{i=k-N+1}^{k-1} X'_{i,n}$ and $\Delta f(N,n,k) = f(V_{N,n}(k)+X'_{k,n})-f(V_{N,n}(k))$. Let $f_x(y) = y^2\mathbb{1}_{y\le x}$ and consider the assumption:*

$A(N)$ There exists a $\mathcal{M}_{0,\inf}$-measurable BV function F_N, such that, for any continuity point x (including $+\infty$) of $x\to\mathbb{E}(F_N(x))$,

$$\lim_{t\to 0}\limsup_{n\to\infty}\left\|\mathbb{E}\Big(\frac{1}{t}\sum_{k=1}^{[nt]}\Delta f_x(N,n,k) - F_N(x)\Big|\mathcal{M}_{0,n}\Big)\right\|_1 = 0\,.$$

Assume that there exists a nondecreasing sequence of integers $(N_i)_{i\in\mathbb{N}^*}$ converging to $N_0(X)$ such that $A(N_i)$ holds for any i in $\mathbb{N}^*$. Then F_{N_i} converges weakly to a $\mathcal{M}_{0,\inf}$-measurable distribution function F, and **S1** *holds for F. In particular, if $N_0(X)$ is finite,* **S1** *holds for F_{N_0}.*

If $N_0(X) = 1$ (recall that $N_0(X) = N_1(X)\vee N_2(X)$), we have the more precise result:

Corollary 3 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and* **EQ** *with $N_0(X) = 1$. Then* **S1** *holds for a couple (γ, F) if and only if: for any continuity point x (including $+\infty$) of the function $x\to\mathbb{E}(F(x))$,*

$$\lim_{t\to 0}\limsup_{n\to\infty}\left\|\mathbb{E}\Big(\frac{1}{t}\sum_{k=1}^{[nt]}X'^2_{k,n}\mathbb{1}_{X'_{k,n}\le x} - F(x)\Big|\mathcal{M}_{0,n}\Big)\right\|_1 = 0\,. \tag{2.4}$$

In particular, the following result holds: let $\mathcal{B} = \{\emptyset,\Omega\}$, assume that $(X_{i,n})$ has i.i.d. rows, that $n\mathbb{E}(X_{0,n})$ tends to γ, and that $n\mathbb{E}(X'^2_{0,n})$ is bounded. Then **S1** *holds with respect to $\mathcal{M}_{0,n} = \sigma(X_{i,n}, i\le 0)$ if and only if there exists a distribution function F such that, for any continuity point x (including $+\infty$) of the function F, the sequence $n\mathbb{E}(X'^2_{0,n}\mathbb{1}_{X'_{0,n}\le x})$ converges to $F(x)$.*

Remark 3. Corollary 3 applies to arrays of martingales differences (with $\mathcal{B} = \{\emptyset,\Omega\}$, $X'_{i,n} = X_{i,n}$ and $\gamma = 0$) for which $N_2(X) = 1$. If $\overline{S'}_n(t) = \sup_{0\le s\le t}|S'_n(s)|$, define

$$R_3(N,X) = \limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\Big((1\wedge\overline{S'}_n(t))\Big(\sum_{i=1}^{[nt]}\mathbb{E}(X'^2_{i,n}|\mathcal{M}_{i-N,n})\Big)\Big)\,,$$

and $N_3(X) = \inf\{N\ge 0 : R_3(N) = 0\}$ ($N_3(X)$ may be infinite). We shall see in Proposition 3 that $N_2(X) = 1$ as soon as $N_3(X) = 1$. From Proposition 8 of Section 2.2, it is easy to see that both (2.4) holds and $N_3(X) = 1$ as soon as, for any continuity point of $x\to\mathbb{E}(F(x))$,

$$\lim_{t\to 0}\limsup_{n\to\infty}\Big\|\frac{1}{t}\sum_{k=1}^{[nt]}\mathbb{E}(X^2_{k,n}\mathbb{1}_{X_{k,n}\le x}|\mathcal{M}_{k-1,n}) - F(x)\Big\|_1 = 0\,. \tag{2.5}$$

For arrays of martingale differences, (2.5) is close to Condition (3') of Theorem 2 in Eagleson (1975). Note that, in Condition (3'), the array needs not be stationary, and the convergence to F holds for $t = 1$ and almost surely (in fact, it needs only hold in probability, see Jeganathan (1982) part I). Here, assuming the stationarity and the slightly different Condition (2.5), we also obtain the convergence to a Lévy Process in the space of cadlag functions equipped with the Skorohod's topology (see Section 2.2). In the stationary case, this result is close to that given in Remark 1 of Jeganathan (1983). To conclude this remark, note that Jeganathan (1982 part II, 1983) also give sufficient conditions involving the conditional probabilities of $X_{i,n}$ given $\mathcal{M}_{i-1,n}$ for the convergence to any infinitely divisible distributions and any Lévy processes.

Finally, we give sufficient conditions for stationary arrays of nonuniformly ϕ and ρ-mixing variables to be **WD** and **EQ** and for **S1** to hold. Let us recall the definition of the ϕ-mixing coefficients: for two σ-algebras $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{A}$, set $\phi(\mathcal{U},\mathcal{V}) = \sup\{\|\mathbb{P}(V|\mathcal{U}) - \mathbb{P}(V)\|_\infty, V \in \mathcal{V}\}$. If $\mathbb{L}^2(\mathcal{U})$ is the space of all square integrable and $\mathcal{U}$-measurable random variables, the ρ-mixing coefficient is defined by $\rho(\mathcal{U},\mathcal{V}) = \sup\{|\mathrm{Cov}(X,Y)|/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)},\ X \in \mathbb{L}^2(\mathcal{U}), Y \in \mathbb{L}^2(\mathcal{V})\}$.

We define the ϕ-mixing coefficients of the array $(X_{i,n})_{i,n}$ by

$$\phi_{\infty,N}(k,n) = \sup\{\phi(\mathcal{M}_{0,n}, \sigma(X_{i_1,n},\dots,X_{i_N,n})), 0 \le k \le i_1 \le \cdots \le i_N\}, \tag{2.6}$$

and $\rho_{\infty,N}$ is defined in the same way. We call these coefficients nonuniform, because they control the dependence between $\mathcal{M}_{0,n}$ and any N-tuple $(X_{i_1,n},\dots X_{i_N,n})$, while the uniform $\phi_{\infty,\infty}$ and $\rho_{\infty,\infty}$-mixing coefficients control the dependence between the past and the whole future.

Corollary 4 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1. Assume that $n\mathbb{E}(X_{0,n})$ converges to γ and let $\mathcal{B} = \{\emptyset, \Omega\}$, so that $X'_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n})$. For any two conjugate exponents $p \le q$ and any positive integer N, consider the conditions*

$$C_\phi(p,N) \quad (a): \sup_{n>0} n\|X'_{0,n}\|_p\|X'_{0,n}\|_q < \infty \quad \textit{and} \quad (b): \lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=m}^{n}\phi^{1/p}_{\infty,N}(k,n) = 0.$$

$$C_\rho(N) \quad (a): \sup_{n>0} n\mathbb{E}(X'^2_{0,n}) < \infty \quad \textit{and} \quad (b): \lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=m}^{n}\rho_{\infty,N}(k,n) = 0\,.$$

$$C_0 \qquad \lim_{K\to\infty}\limsup_{n\to\infty} n\mathbb{E}(X'^2_{0,n}\mathbb{1}_{|X'_{0,n}|\ge K}) = 0.$$

1. *If C_0 and $C_\phi(p,1)$ (resp. $C_\rho(1)$) hold then $(X_{i,n})$ is* **WD** *and* **EQ**.

2. *Consider the condition $B(N)$: There exists a BV function F_N such that, for any continuity point x (including $+\infty$) of the function F_N,*

$$\lim_{n\to\infty} n\mathbb{E}(S'^2_{N,n}\mathbb{1}_{S'_{N,n}\le x}) - n\mathbb{E}(S'^2_{N-1,n}\mathbb{1}_{S'_{N-1,n}\le x}) = F_N(x)\,.$$

   If $C_\phi(p,N)$ (resp. $C_\rho(N)$) holds for any finite integer $N \le N_0(X)$, then **S1** *holds as soon as $B(N_i)$ holds for a sequence $(N_i)_{i\in\mathbb{N}^*}$ converging to $N_0(X)$.*

Remark 4. If $\mathbb{E}(X_{0,n})$ tends to γ, $n\mathbb{E}(X'^2_{0,n})$ is bounded and $(X_{i,n})$ is m-dependent, then it is **WD** and **EQ** with $N_0(X) \le m+1$. Hence **S1** holds for (γ, F) as soon as $B(N_0)$ holds for F.

2.2 Convergence to Lévy processes

Let $D([0,1])$ be the space of caldlag functions equipped with the Skorohod distance d. Let $\mathcal{C}(D)$ be the space of continuous bounded functions from $(D([0,1]),d)$ to $\mathbb{R}$. Let π_t be the projection from $D([0,1])$ to $\mathbb{R}$ defined by: $\pi_t(x) = x(t)$. For any real number γ and any distribution function F such that $F(\infty)$ is finite, define the Lévy distribution $\mu_{\gamma,F}$ as the unique measure on $D([0,1])$ such that: for any t in $[0,1]$, π_t has law $\mu^t_{\gamma,F}$ and for any k-tuple $t_0 = 0 \le t_1 \le \cdots t_k \le 1$, the variables $(\pi_{t_i} - \pi_{t_{i-1}})_{1\le i\le k}$ are independent. Take $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ as in Theorem 1. We say that $\{S_n(t), t \in [0,1]\}$ converges conditionally to a mixture of Lévy processes if:

LP There exists an $\mathcal{M}_{0,\inf}$-measurable random variable γ and an $\mathcal{M}_{0,\inf}$-measurable distribution function F, such that for any φ in $\mathcal{C}(D)$ and any positive integer k,

$$\lim_{n\to\infty} \left\| \mathbb{E}\Big(\varphi(S_n) - \mu_F(\varphi) \,\Big|\mathcal{M}_{k,n}\Big) \right\|_1 = 0\,.$$

Remark 5. Assume that the sequence $(\mathcal{M}_{0,n})_{n\ge1}$ is nondecreasing. As for Corollary 2 (with the same proof), Property **LP** implies that, for any φ in $\mathcal{C}(D)$, $\varphi(S_n)$ converges weakly in $\mathbb{L}^1$ to $\mu_F(\varphi)$. As a consequence, we obtain that $\lim_{n\to\infty}\mathbb{P}((S_n \in A)\cap B) = \mathbb{P}(\mu_F(A)\cap B)$, for any set A with boundary ∂A satisfying $\mathbb{E}(\mu_F(\partial A)) = 0$. According to Rényi's definition (extended to separable metric spaces), this means exactly that $\{S_n(t), t\in[0,1]\}$ converges *stably* in $D([0,1])$. More precisely, following Aldous and Eagleson (1978), we see that $\{S_n(t), t\in[0,1]\}$ converges *stably* to a random variable Y whose conditional distribution given $\mathcal{I}$ is μ_F.

Convergence in the Skorohod topology is somewhat restrictive. To obtain the relative compactness of the law of $\{S_n(t), t\in[0,1]\}$ we impose a more restrictive condition than **EQ**.

Definition 3 Let $(X_{i,n})$ be as in Theorem 1. Recall that $\mathcal{B}$ is either $\mathcal{I}$ or $\{\emptyset,\Omega\}$, and that $X'_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n}|\mathcal{B})$ and $S'_{k,n} = X'_{1,n}+\cdots+X'_{k,n}$. Define $\overline{S'}_{k,n} = \max\{|S'_{1,n}|,\ldots,|S'_{k,n}|\}$. We say that $(X_{i,n})$ is 1-**EQ** if $n\mathbb{E}(X'^2_{0,n})$ is bounded and $\lim_{t\to0}\limsup_{n\to\infty} t^{-1}\mathbb{E}(\sum_{k=1}^{[nt]} X'^2_{k,n}(1\wedge \overline{S'}_{k-1,n})) = 0$. Note that if $(X_{i,n})$ is 1-**EQ**, then it is **EQ** with $N_2(X) = 1$.

The following Proposition provides sufficient conditions for Property **LP** to hold.

Proposition 2 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and* 1-**EQ**. *If* **S1** *holds for some couple (γ,F), then* **LP** *holds for the same couple (γ,F).*

The following Proposition gives a suficient condition for property 1-**EQ**. Recall that $R_3(N)$ and $N_3(X)$ have been defined in Remark 3.

Proposition 3 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$, $S_n(t)$ be as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and that $n\mathbb{E}(X'^2_{0,n})$ is bounded. Let $L_1(N)$: for any $1\le i<N, \lim_{n\to\infty} n\mathbb{E}(X'^2_{i,n}(1\wedge|X'_{0,n}|)) = 0$. If $R_3(N)$ tends to 0 as N tends to infinity and $L_1(N_3)$ holds, then $(X_{i,n})$ is 1-***EQ**.

Remark 6. A simple bound for $R_3(N)$ is given in item 2 of Proposition 8. If $n\mathbb{E}(X'^2_{0,n})$ is bounded, Condition L_1 is equivalent to the Lindeberg-type condition: for any positive ϵ and any $1 \leq i < N$, $n\mathbb{E}(X'^2_{i,n}\mathbb{1}_{|X'_{0,n}|>\epsilon})$ converges to zero. If C_0 holds, Condition L_1 is equivalent to: $L'_1(N)$ For any positive ϵ and any $1 \leq i < N$, $\lim_{n\to\infty} n\mathbb{P}(|X'_{i,n}| > \epsilon, |X'_{0,n}| > \epsilon) = 0$. For stationary ϕ-mixing processes, Samur (1987) proved that $L'_1(\infty)$ is necessary for the weak convergence of $\{S_n(t), t \in [0,1]\}$ to a Lévy measure on $D([0,1])$ (see also Corollary 5.9 in Kobus (1995) for a more precise result in the case of stable limits).

Remark 7. The classical Lindeberg's condition is L_0 : $\lim_{n\to\infty} n\mathbb{E}(X'^2_{0,n}(1 \wedge |X'_{0,n}|)) = 0$. If L_0 holds, then L_1 holds for any positive integer i and moreover $R_3(0) = 0$ (see the proof of Proposition 8 in Dedecker and Merlevède (2002)). We shall see in the next section that, if $R_3(0) = 0$, then the limiting process is necessarily a mixture of Gaussian processes.

It follows from Proposition 2 that for **WD** and 1-**EQ** arrays, conditional convergence to Lévy processes with law $\mu_{\gamma,F}$ follows from conditional convergence to $\mu^t_{\gamma,F}$ for any t in $[0,1]$. In particular, the conclusion of Proposition 1 remains valid if we replace **S1** by **LP**. Note that, since for such arrays $N_2(X) = 1$, we have that $N_0(X) = N_1(X)$. The class of **WD**-arrays for which $N_1(X) = 1$ is much larger than martingale differences arrays. A first example is given by Kernel density estimators (see Dedecker and Merlevède (2002), Section 8). The following Proposition provides useful conditions ensuring that $N_1(X) = 1$.

Proposition 4 *Let $X_{i,n}$, and $\mathcal{M}_{i,n}$ be as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and that $n\mathbb{E}(X'^2_{0,n})$ is bounded. Consider the condition C_1 : $\lim_{\epsilon\to 0}\limsup_{n\to\infty} n\mathbb{E}(X'^2_{0,n}\mathbb{1}_{|X'_{0,n}|} \leq \epsilon) = 0$. If C_1 holds and $L_1(N)$ holds for some N in $\overline{\mathbb{N}}^*$ such that $R_1(N,X) = 0$, then $N_1(X) = 1$.*

Remark 8. If $n\mathbb{E}(X'^2_{0,n}\mathbb{1}_{X'_{0,n}\leq x})$ converges to $F(x)$, Condition C_1 means that F is continuous at zero. For such a F, one says that the Lévy distribution $\mu^t_{\gamma,F}$ is purely non-Gaussian.

2.3 Convergence to Wiener processes

Let (γ, a, λ) be three $\mathcal{M}_{0,\inf}$-measurable random variables as in Corollary 1. We say that **P1**(γ, a, λ) holds if **P1** is realized for the parameter (γ, a, λ).

Definition 4. Let $X_{i,n}$ be as in Theorem 1. We say that the array $(X_{i,n})$ is 0-**EQ** if $n\mathbb{E}(X'^2_{0,n})$ is bounded and $R_3(0) = 0$. Note that if $(X_{i,n})$ is 0-**EQ**, then it is 1-**EQ**.

The next Proposition shows that if $(X_{i,n})$ is **WD** and 0-**EQ**, then the limiting distribution is necessarily a mixture of Gaussian distributions (i.e. **P1**$(\gamma, 0, \lambda)$ holds). From Proposition 2, this implies the functional property **LP** for γ and the distribution function $F = \lambda\mathbb{1}_{[0,\infty[}$.

Proposition 5 *Take $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and* 0-**EQ**. *Then* **S2***(b1) holds with $a = 0$ and* **P1**$(\gamma, 0, \lambda)$ *holds if and only if* **S2***(b2) holds. Moreover*

1. *Setting $U_{N,n}(k) = \big(V_{N,n}(k) + X'_{k,n}\big)^2 - \big(V_{N,n}(k)\big)^2$, Condition* **S2***(b2) is equivalent to*

$$\liminf_{N\to N_1(X)}\limsup_{t\to 0}\limsup_{n\to\infty}\Big\|\mathbb{E}\Big(\frac{1}{t}\sum_{k=1}^{[nt]} U_{N,n}(k) - \lambda\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 = 0\,. \tag{2.7}$$

2. *Assume that there exists a nondecreasing sequence $(N_i)_{i\in\mathbb{N}^*}$ converging to $N_1(X)$, and a sequence λ_{N_i} of $\mathcal{M}_{0,\inf}$-measurable random variables such that, for any i in $\mathbb{N}^*$,*

$$\lim_{t\to 0}\limsup_{n\to\infty}\Big\|\mathbb{E}\Big(\frac{1}{t}\sum_{k=1}^{[nt]}\Big(X'^2_{k,n}+2\sum_{j=1}^{N_i-1}X'_{k,n}X'_{k-j,n}\Big)-\lambda_{N_i}\Big|\mathcal{M}_{0,n}\Big)\Big\|_1=0\,. \tag{2.8}$$

Then λ_{N_i} converges in $\mathbb{L}^1$ to some nonnegative variable λ, and **S2**$(b2)$ *holds for this λ.*

Note that if $X_{i,n}=n^{-1/2}X_i$ for some centered square integrable random variable X_i, and $\mathcal{M}_{i,n}=\mathcal{M}_i$, then condition **WD** with $\mathcal{B}=\{\emptyset,\Omega\}$ is equivalent to:

$$\text{the sequence } X_0\sum_{k=1}^{n}\mathbb{E}(X_k|\mathcal{M}_0) \text{ converges in } \mathbb{L}^1\,, \tag{2.9}$$

and Condition (2.9) also implies **S2**$(b2)$. Applying the $\mathbb{L}^1$-ergodic theorem to the sequence $(X^2_{i,n})$, we see that $(X_{i,n})$ is 0-**EQ**. From Proposition 5, we obtain the following conditional invariance principle, which was first proved in Dedecker and Merlevède (2002):

Corollary 5 *Let the random variables $X_{i,n}$ and the σ-algebras $\mathcal{M}_{i,n}$ of Theorem 1 be such that $X_{i,n}=n^{-1/2}X_i$ for some centered square-integrable random variable X_i, and $\mathcal{M}_{i,n}=\mathcal{M}_i$. If (2.9) holds then the Donsker process $\{S_n(t),t\in[0,1]\}$ satisfies* **LP** *for $\gamma=0$ and some distribution function $F=\lambda\mathbb{1}_{[0,\infty[}$. More precisely, for any φ in $\mathcal{C}(D)$ and any positive integer k*

$$\lim_{n\to\infty}\Big\|\mathbb{E}\Big(\varphi(S_n)-\int\varphi(x\sqrt{\lambda})W(dx)\ \Big|\mathcal{M}_k\Big)\Big\|_1=0$$

where W is the standard Wiener distribution and $\lambda=\mathbb{E}(X_0^2|\mathcal{I})+2\sum_{k>0}\mathbb{E}(X_0X_k|\mathcal{I})$.

2.4 Convergence to Poisson processes

In Proposition 6 below, we give sufficient conditions on a **WD** and 1-**EQ** array for **P1**(γ,a,λ) to hold. Via Proposition 2, this implies the functional property **LP** for γ and $F=\lambda\mathbb{1}_{[a,\infty[}$. Recall that the quantities $R_1(N,X)$ and $N_1(X)$ have been defined in Definition 1.

Proposition 6 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *with $N_1(X)=1$ and 1-***EQ**. *Then* **P1**(γ,a,λ) *holds if and only if Conditions C_2 and C_3 hold*

$$C_2:\quad \lim_{n\to\infty}n\mathbb{E}(X'^2_{0,n}(1\wedge|a-X'_{0,n}|))=0 \quad \text{and } C_3:\quad \lim_{t\to 0}\limsup_{n\to\infty}\Big\|\mathbb{E}\Big(\frac{1}{t}\sum_{i=1}^{[nt]}X'^2_{i,n}-\lambda\Big|\mathcal{M}_{0,n}\Big)\Big\|_1=0.$$

If $\mathbb{P}(a=0)=0$, Condition C_2 implies Condition C_1 of Proposition 4. Combining Propositions 4 and 6, we obtain the following Corollary:

Corollary 6 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1. Assume that $(X_{i,n})$ is* **WD** *and that $n\mathbb{E}(X'^2_{0,n})$ is bounded. If $L_1(N)$ is satisfied for some N in $\overline{\mathbb{N}}$ such that $R_1(N,X)=0$ and C_2 holds for some a such that $\mathbb{P}(a=0)=0$, then $N_1(X)=1$. If furthermore $R_3(N)=0$, then* **P1**(γ,a,λ) *holds if and only if C_3 holds.*

Remark 9. We shall see in Section 3.3 that both $R_3(N) = 0$ for N in $\overline{\mathbb{N}}^*$ and C_3 holds as soon as $C_4(N):\ \lim_{P\to N}\limsup_{t\to 0}\limsup_{n\to\infty}\|t^{-1}\sum_{k=1}^{[nt]}\mathbb{E}(X'^2_{k,n}|\mathcal{M}_{k-P,n}) - \lambda\|_1 = 0$.

The first application of Corollary 6 is to m-conditionally centered arrays.

Corollary 7 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$, $S_n(t)$ be as in Theorem 1. Assume that $(X'_{i,n})$ is m-conditionally centered and that $\|n\mathbb{E}(X_{0,n}|\mathcal{B}) - \gamma\|_1$ tends to zero. Then* $\mathbf{P1}(\gamma, a, \lambda)$ *holds for some a such that $\mathbb{P}(a = 0) = 0$ as soon as C_2 holds, and $L_1(N)$, $C_4(N)$ are satisfied for $N = m + 1$. Assume furthermore that $(X_{i,n})_{i,n}$ is m-dependent and take $\mathcal{M}_{0,n} = \sigma(X_{i,n}, i \le 0)$ and $\mathcal{B} = \{\emptyset, \Omega\}$. Condition C_4 reduces to: the sequence $n\mathbb{E}(X'^2_{0,n})$ converges to λ.*

Remark 10. Eagleson (1976b) gave a criterion for (non necessarily stationary) martingale differences arrays (i.e. $m = 0$). In the stationary case, his result is the same as ours (with $\mathcal{B} = \{\emptyset, \Omega\}$ and $\gamma = 0$), except that he only requires convergence in probability for $t = 1$ in $C_4(1)$. Here, on the one hand, we need to impose $\mathbb{L}^1$-convergence in C_4 to obtain the conditional version of the Poisson convergence. On the other hand, the fact that it holds for any t implies the convergence of the process $\{S_n(t), t \in [0, 1]\}$ to a Poisson process. Note also that if $(X_{i,n})_{i,n}$ is a m-dependent array of Bernoulli random variables, then the conditions of Corollary 7 are optimal (see Theorem 2 of Hudson *et al.*(1989)). Therefore Corollary 7 seems to be a reasonable extension of both martingale and m-dependent cases.

Corollary 6 contains more information than Corollary 7. As a consequence of Corollaries 4 and 6, we obtain sufficient conditions for stationary arrays of nonuniformly mixing variables.

Corollary 8 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1 and let $\mathcal{B} = \{\emptyset, \Omega\}$. Assume that $n\mathbb{E}(X_{0,n})$ converges to γ and that Condition $C_\phi(p, 1)$ (resp. $C_\rho(1)$) holds. If furthermore $L_1(\infty)$ is satisfied and C_2 holds for some a such that $\mathbb{P}(a = 0) = 0$, then $N_1(X) = 1$ and* $\mathbf{P1}(\gamma, a, \lambda)$ *holds as soon as $n\mathbb{E}(X'^2_{0,n})$ converges to λ as n tends to infinity.*

2.5 The case of Bernoulli distributed variables

Let $(X_{i,n})$ be an array of Bernoulli distributed variables with parameter p_n such that np_n is bounded. We are interested in the process $S_n(t) = X_{1,n} + \cdots + X_{[nt],n}$. An interesting example is $X_{0,n} = \mathbb{1}_{X_0 > u_n}$ for some numerical sequence u_n (see Hsing *et al.* (1988) for the importance of the exceedance process $S_n(t)$ in extreme value theory). If for each n the sequence $(X_{i,n})$ is i.i.d, it is well known that $S_n(1)$ converges in distribution if and only if np_n converges to a nonnegative number λ and that the limiting distribution is Poisson with parameter λ. For m-dependent sequences, necessary and sufficients conditions for the convergence of $S_n(1)$ are given in Hudson *et al.* (1989). Using our notations, these conditions are equivalent to $B(m+1)$ of Corollary 4 : there is a distribution function F_{m+1} such that

$$\lim_{n\to\infty} n\mathbb{E}(S^2_{m+1,n}(1)\mathbb{1}_{S_{m+1,n}(1)\le x}) - n\mathbb{E}(S^2_{m,n}(1)\mathbb{1}_{S_{m,n}(1)\le x}) = F_{m+1}(x)\,. \qquad (2.10)$$

Since $X_{i,n}$ is either 0 or 1, it is clear that F_{m+1} is piecewise constant with jumps at points $1, \ldots, m + 1$. In fact the limiting distribution is integer-valued coumpound Poisson. More precisely it is the law of the sum $V_1 + \cdots + V_N$ where N is Poisson distributed with parameter $\lambda = \sum_{i=1}^{m+1}(F_{m+1}(i) - F_{m+1}(i-1))/i^2$ and is independent of the sequence $(V_k)_{k\geq 1}$ which is i.i.d. with marginal distribution $\mathbb{P}(V_1 = i) = (F_{m+1}(i) - F_{m+1}(i-1))/(\lambda i^2)$. See Theorem 1.1 in Kobus (1995) for more details on this important question.

Now, let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be as in Theorem 1. From Theorem 1, Property **S1** holds if and only if **S2** holds. Hence the distribution function F in $\mathbf{S2}(b)$ is piecewise constant with jumps at integer points, and the limiting distribution of $S_n(t)$ is a mixture of integer-valued coumpound Poisson distributions. If $(X_{i,n})$ is **WD** and **EQ**, then **S1** holds as soon as $A(N_i)$ of Proposition 1 holds with N_i converging to N_0. For instance, suppose that $C_\phi(1,N)$ (resp. $C_\rho(N)$) holds for any $N \leq N_0$. In that case **S1** holds as soon as $n\mathbb{E}(X_{0,n})$ converges to γ and Condition $B(N_i)$ of Corollary 4 holds for a sequence N_i converging to N_0. Both γ and the distribution function F are nonrandom and the limiting distribution of $S_n(t)$ is integer-valued coumpound Poisson. This extends the result of Hudson *et al.* (1989), since (2.10) is exactly $B(m+1)$. Note also that if **S2** holds, we can establish the convergence of $(S_n(t_1), \ldots, S_n(t_k))$ for any k-tuple $(t_1, \ldots, t_k)$ (cf. Dedecker and Merlevède (2002), Section 4). Therefore, there is convergence for the point process $S_n(A) = \sum_{i\in nA} X_{i,n}$ indexed by subsets of $[0,1]$ (see Kallenberg (1975) Theorem 4.2).

To be complete, let us give a simple example of an array of Bernoulli random variables for which γ and F are random. Let $(Z_{i,n})$ be an i.i.d. array of Bernoulli-distributed variables with parameter α/n and ε be a Bernoulli-distributed variable with parameter $1/2$ independent of $(Z_{i,n})$. Set $\mathcal{M}_{i,n} = \sigma(\varepsilon, Z_{k,n}, k \leq i)$. Then $X_{i,n} = \varepsilon Z_{i,n}$ is Bernoulli-distributed with parameter $\alpha/2n$ and $\mathbb{E}(X_{i+1,n}|\mathcal{M}_{i,n}) = \varepsilon\alpha/n$. $X'_{i,n} = X_{i,n} - \varepsilon\alpha/n$ is a martingale difference array, so that $(X_{i,n})$ is **WD** with $N_1(X) = 1$ and $\gamma = \alpha\varepsilon$. Furthermore, it is clearly **EQ** with $N_2(X) = 1$. Consequently, $N_0(X) = 1$ and Corollary 3 applies: **S1** holds if and only if (2.4) holds. Now, (2.4) is satisfied with $F = \alpha\varepsilon \mathbb{1}_{[1,\infty[}$, and $S_n(t)$ converges in distribution to a variable whose conditional distribution with respect to ε is Poisson with parameter $t\alpha\varepsilon$. From Proposition 6 $\{S_n(t), t \in [0,1]\}$ converges in distribution to a mixture of Poisson processes in $D([0,1])$.

3 Proofs

In the two following sections, we prove Theorem 1. The fact that γ and F are invariant by T can be proved as in Section 3.2 in Dedecker and Merlevède (2002).

3.1 S1 implies S2

Since $\mu_{\gamma,F}$ has mean γ, it is clear that **S1** implies $\mathbf{S2}(a)$. We now prove that **S1** implies $\mathbf{S2}(b)$.

Lemma 1 *Let $\mathcal{C}^b$ be the set of continuous bounded functions and $\mathcal{F} = \{x \to x^2 g(x), g \in \mathcal{C}^b\}$. Let $\nu(dx) = x^{-2}dF(x)$. For any f in $\mathcal{F}$, we have* $\lim_{t\to 0} \|t^{-1}\mu^t_{\gamma,F}(f) - \nu(f)\|_1 = 0\,.$

The proof of this Lemma will be done at the end of this section. Let $f_x(y) = y^2 \mathbb{1}_{y \le x}$, and define $f_{x,\epsilon_+}(y) = f_{x+\epsilon}(y) + \epsilon^{-1} y^2 (x-y) \mathbb{1}_{x \le y \le x+\epsilon}$ and $f_{x,\epsilon_-}(y) = f_{x-\epsilon,\epsilon_+}$. We have the inequality

$$\Big\| \mathbb{E}\Big(\frac{1}{t} f_x(S_n(t)) - F(x) \Big| \mathcal{M}_{0,n} \Big) \Big\|_1 \le \Big\| \mathbb{E}\Big(\frac{1}{t} f_{x,\epsilon_+}(S_n(t)) - \nu(f_{x,\epsilon_+}) | \mathcal{M}_{0,n} \Big) \Big\|_1 + \Big\| \nu(f_{x,\epsilon_+}) - \nu(f_{x,\epsilon_-}) \Big\|_1 + \frac{1}{t} \Big\| \mathbb{E}\Big(f_{x,\epsilon_+}(S_n(t)) - f_{x,\epsilon_-}(S_n(t)) \Big| \mathcal{M}_{0,n} \Big) \Big\|_1 . \quad (3.1)$$

Note first that, setting $G(x) = \mathbb{E}(F(x))$, $\lim_{\epsilon \to 0} \ \|\nu(f_{x,\epsilon_+}) - \nu(f_{x,\epsilon_-})\|_1 = G(x) - G(x-)$, and since x is a continuity point of G, the latter is zero. Next, we infer from **S1** that

$$\limsup_{n\to\infty} \Big\| \mathbb{E}\Big(\frac{1}{t} f_{x,\epsilon_+}(S_n(t)) - \nu(f_{x,\epsilon_+}) \Big| \mathcal{M}_{0,n} \Big) \Big\|_1 \le \Big\| \frac{1}{t} \mu^t_{\gamma,F}(f_{x,\epsilon_+}) - \nu(f_{x,\epsilon_+}) \Big\|_1 ,$$

and Lemma 1 implies that the latter tends to zero as t goes to zero. It remains to control the last term on right hand in (3.1). Applying first **S1** and then Lemma 1, we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t} \Big\| \mathbb{E}\Big(f_{x,\epsilon_+}(S_n(t)) - f_{x,\epsilon_-}(S_n(t)) \Big| \mathcal{M}_{0,n} \Big) \Big\|_1 \le \Big\| \nu(f_{x,\epsilon_+}) - \nu(f_{x,\epsilon_-}) \Big\|_1 ,$$

and we know that the latter tends to zero with ϵ. Hence, for all continuity point of G, $\lim_{t\to 0} \limsup_{n\to\infty} \|\mathbb{E}(t^{-1} f_x(S_n(t)) - F(x) | \mathcal{M}_{0,n})\|_1 = 0$, which is exactly **S2**(b).

Proof of Lemma 1. Arguing as in Corollary 8.9 in Sato (1999) and using Theorem 8.7 in Sato (1999), we obtain that for any bounded function f of $\mathcal{F}$ and any fixed ω,

$$\lim_{t\to 0} \frac{1}{t} \mu^t_{\gamma,F}(f) = \nu(f) . \quad (3.2)$$

Since $t^{-1} \int x^2 \mu^t_{\gamma,F}(dx) = t\gamma^2 + \int x^2 \nu(dx)$, we infer that (3.2) extends to the class $\mathcal{F}$. Now, every f of $\mathcal{F}$ satisfies $|f(x)| \le Mx^2$, so that $|t^{-1}\mu_{\gamma,F}(f) - \nu(f)| \le 2MF(\infty) + t\gamma^2$. Since $F(\infty)$ and γ^2 are integrable, Lemma 1 follows from (3.2) and the dominated convergence theorem.

3.2 S2 implies S1

Let $B^3_1(\mathbb{R})$ be the class of three-times continuously differentiable real functions such that $\|h''\|_\infty \le 1$ and $\|h'''\|_\infty \le 1$. Assume for a while that **S1**(h) holds for any h of $B^3_1(\mathbb{R})$. In such a case, we know from Dedecker and Merlevède (2002) that **S1** extends to any continuous bounded function. Since $x \to x^2/2$ belongs to $B^3_1(\mathbb{R})$, we infer that $S^2_n(t)$ is uniformly integrable for any t in $[0,1]$, which implies that **S1** extends to $\mathcal{H}$. Hence, it suffices to prove that **S2** implies **S1**(h) for any h of $B^3_1(\mathbb{R})$. If h belongs to $B^3_1(\mathbb{R})$, then $|h(x+a) - h(x)| \le a|h'(0)| + a|x| + a^2/2$. Hence $\|h(S_n(t)) - h(S_n(t) \circ T^k)\|_1 \le u_n(|h'(0)| + \|S_n(t)\|_2 + u_n/2)$ with $u_n = \|S_n(t) - S_n(t) \circ T^k\|_2$. From **S2**$(b)$, the stationarity of $(X_{i,n})$ and the fact that $\mathbb{E}(X^2_{0,n})$ tends to zero, we infer that for any t in $[0,1]$ the sequence $\|S_n(t)\|_2$ is bounded. The asymptotic negligibility of $X_{0,n}$ also implies that u_n goes to zero as n increases. Combining the two preceding arguments, we obtain

$$\lim_{n\to\infty} \|h(S_n(t)) - h(S_n(t) \circ T^k)\|_1 = 0. \quad (3.3)$$

(3.3) implies that **S1**(h) holds if and only if $\lim_{n\to\infty} \|\mathbb{E}(h(S_n(t) \circ T^k) - \mu^t_F(h) | \mathcal{M}_{k,n})\|_1 = 0$. Now, since both F and $\mathbb{P}$ are T-invariant, the fact that **S2** implies **S1** follows from:

Proposition 7 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be defined as in Theorem 1. If* **S2** *holds, then, for any h in $B_1^3(\mathbb{R})$ and any t in $[0,1]$, $\lim_{n\to\infty} \|\mathbb{E}(h(S_n(t)) - \mu^t_{\gamma,F}(h)\ |\mathcal{M}_{0,n})\|_1 = 0$.*

Proof of Proposition 7. We prove the result for $S_n(1)$, the proof of the general case being unchanged. Let $\mathcal{M}_\infty = \sigma(\bigcup_{k,n} \mathcal{M}_{k,n})$. Without loss of generality, suppose that there exists an array $(\varepsilon_{i,n})_{i\in\mathbb{Z}}$ of i.i.d random variables conditionally to $\mathcal{M}_\infty$, with conditional marginal distribution $\mu^{1/n}_{\gamma,F}$.

Notations 1. Let i, p and n be three integers such that $1 \le i \le p \le n$. Set $q = [n/p]$ and define

$$\begin{aligned} U_{i,n} &= X_{iq-q+1,n} + \cdots + X_{iq,n}, & V_{i,n} &= U_{1,n} + U_{2,n} + \cdots + U_{i,n} \\ \Delta_{i,n} &= \varepsilon_{iq-q+1,n} + \cdots + \varepsilon_{iq,n}, & \Gamma_{i,n} &= \Delta_{i,n} + \Delta_{i+1,n} + \cdots + \Delta_{p,n}\,. \end{aligned}$$

Notations 2. Let g be any function from $\mathbb{R}$ to $\mathbb{R}$. For k and l in $[1,p]$ and any positive integer $n \ge p$, set $g_{k,l;n}(x) = g(V_{k,n} + x + \Gamma_{l,n})$, with the conventions $g_{k,p+1;n}(x) = g(V_{k,n} + x)$ and $g_{0,l;n}(x) = g(\Gamma_{l,n} + x)$. Afterwards, we shall apply this notation to the successive derivatives of the function h. For brevity we shall omit the index n and write $g_{k,l}$ for $g_{k,l}(0)$.

Let $s_n = \varepsilon_{1,n} + \cdots + \varepsilon_{n,n}$. Since $(\varepsilon_{i,n})_{i\in\mathbb{Z}}$ is i.i.d conditionally to $\mathcal{M}_\infty$ with conditional marginal distribution $\mu^{1/n}_{\gamma,F}$, we have,

$$\begin{aligned} \mathbb{E}\Big(h(S_n(1)) - \mu_F(h)\,\Big|\mathcal{M}_{0,n}\Big) = \mathbb{E}(h(S_n(1)) - h(V_{p,n})|\mathcal{M}_{0,n}) + \mathbb{E}(h(V_{p,n}) - h(\Gamma_{1,n})|\mathcal{M}_{0,n}) \\ + \mathbb{E}(h(\Gamma_{1,n}) - h(s_n)|\mathcal{M}_{0,n})\,. \end{aligned} \tag{3.4}$$

Here, note that $|S_n(1) - V_{p,n}| \le (|X_{n-p+2,n}| + \cdots + |X_{n,n}|)$. Arguing as in (3.3), we infer that

$$\lim_{n\to\infty} \|h(S_n(1)) - h(V_{p,n})\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|h(\Gamma_{1,n}) - h(s_n)\|_1 = 0\,. \tag{3.5}$$

In view of (3.5), it remains to control the second term in the right hand side of (3.4). To this end, we use Lindeberg's decomposition.

$$h(V_{p,n}) - h(\Gamma_{1,n}) = \sum_{i=1}^{p} (h_{i,i+1} - h_{i-1,i+1}) + \sum_{i=1}^{p} (h_{i-1,i+1} - h_{i-1,i})\,. \tag{3.6}$$

Now, applying Taylor's integral formula we get that:

$$\begin{cases} h_{i,i+1} - h_{i-1,i+1} = U_{i,n} h'_{i-1,i+1} + U_{i,n}^2 \displaystyle\int_0^1 (1-t) h''_{i-1,i+1}(tU_{i,n})dt \\ h_{i-1,i+1} - h_{i-1,i} = -\Delta_{i,n} h'_{i-1,i+1} - \Delta_{i,n}^2 \displaystyle\int_0^1 (1-t) h''_{i-1,i+1}(t\Delta_{i,n})dt \end{cases}$$

Set $G(x) = \mathbb{E}(F(x))$. Let $\epsilon > 0$ and choose a finite grid $x_0 \le x_1 \le \ldots \le x_N$ of continuity points of G such that $|x_j - x_{j+1}| \le \epsilon$, $G(x_0) \le \epsilon$ and $G(\infty) - G(x_N) \le \epsilon$. Let $g_i(x) =$

$\int_0^1 (1-t)h''_{i-1,i+1}(tx)dt$. Since $h \in B_1^3(\mathbb{R})$, g_i is bounded by 1/2 and 1/6-lipschitz. Hence

$$\begin{cases} 6\left|g_i(x_j) - \int_0^1 (1-t)h''_{i-1,i+1}(tU_{i,n})dt\right| \mathbb{1}_{x_j<U_{i,n}\le x_{j+1}} & \le & \epsilon \\ 6\left|g_i(x_j) - \int_0^1 (1-t)h''_{i-1,i+1}(t\Delta_{i,n})dt\right| \mathbb{1}_{x_j<\Delta_{i,n}\le x_{j+1}} & \le & \epsilon\,. \end{cases}$$

It follows that

$$|\mathbb{E}(h(V_{p,n}) - h(\Gamma_1)|\mathcal{M}_{0,n})| \le D_1 + D_2 + D_3 + D_4\,, \quad \text{where} \tag{3.7}$$

$$\begin{aligned} D_1 &= \Big|\sum_{i=1}^p \mathbb{E}((U_{i,n}-\Delta_{i,n})h'_{i-1,i+1}|\mathcal{M}_{0,n})\Big|, \\ D_2 &= \Big|\sum_{i=1}^p \sum_{j=0}^{N-1} \mathbb{E}((U_{i,n}^2 \mathbb{1}_{x_j<U_{i,n}\le x_{j+1}} - \Delta_{i,n}^2 \mathbb{1}_{x_j<\Delta_{i,n}\le x_{j+1}})g_i(x_j)|\mathcal{M}_{0,n})\Big|, \\ D_3 &= \frac{1}{2}\sum_{i=1}^p \mathbb{E}(U_{i,n}^2 \mathbb{1}_{U_{i,n}\notin]x_0,x_N]} + \Delta_{i,n}^2 \mathbb{1}_{\Delta_{i,n}\notin]x_0,x_N]}|\mathcal{M}_{0,n}) \\ D_4 &= \frac{\epsilon}{6}\sum_{i=1}^p \mathbb{E}(U_{i,n}^2 + \Delta_{i,n}^2|\mathcal{M}_{0,n}). \end{aligned}$$

Control of D_3 and D_4. The probability $\mathbb{P}$ being invariant by T, we have

$$\sum_{i=1}^p \|U_{i,n}^2 \mathbb{1}_{U_{i,n}\notin]x_0,x_N]}\|_1 = \mathbb{E}\Big(\frac{S_n^2(1/p)}{1/p}\mathbb{1}_{S_n(1/p)\notin]x_0,x_N]}\Big) \quad \text{and} \quad \epsilon\sum_{i=1}^p \|U_{i,n}^2\|_1 = \epsilon\mathbb{E}\Big(\frac{S_n^2(1/p)}{1/p}\Big)\,.$$

Consequently, $\mathbf{S2}(b)$ implies both $\lim_{p\to\infty}\limsup_{n\to\infty}\ \epsilon\sum_{i=1}^p \|U_{i,n}^2\|_1 = \epsilon G(\infty)$ and

$$\lim_{p\to\infty}\limsup_{n\to\infty}\ \sum_{i=1}^p \|U_{i,n}^2 \mathbb{1}_{U_{i,n}\notin]x_0,x_N]}\|_1 = G(x_0) + G(\infty) - G(x_N) \le 2\epsilon\,.$$

From Lemma 1, we infer that the same arguments apply to $\Delta_{i,n}$ and finally

$$\limsup_{p\to\infty}\limsup_{n\to\infty}\|D_3 + D_4\|_1 \le 2\epsilon + \frac{\epsilon G(\infty)}{3}\,. \tag{3.8}$$

Control of D_1. Clearly $\|D_1\|_1 \le \sum_{i=1}^p \|\mathbb{E}((U_{i,n}-\Delta_{i,n})h'_{i-1,i+1}|\mathcal{M}_{0,n})\|_1$. Define the index $l(i,n) = (i-1)[n/p]$ and recall that $q = [n/p]$. By definition of $\Delta_{i,n}$,

$$\mathbb{E}(\Delta_{i,n}h'_{i-1,i+1}|\mathcal{M}_\infty) = \mathbb{E}(h'_{i-1,i+1}|\mathcal{M}_\infty)\int x\mu_{\gamma,F}^{q/n}(dx) = \mathbb{E}(h'_{i-1,i+1}|\mathcal{M}_\infty)\frac{q\gamma}{n}\,,$$

where the random variable $\mathbb{E}(h'_{i-1,i+1}|\mathcal{M}_\infty)$ is $\mathcal{M}_{l(i,n),n}$-measurable and bounded by one. Using that $\mathcal{M}_{0,n} \subseteq \mathcal{M}_{l(i,n),n} \subseteq \mathcal{M}_\infty$, we obtain

$$\begin{aligned}\|\mathbb{E}((U_{i,n} - \Delta_{i,n})h'_{i-1,i+1}|\mathcal{M}_{0,n})\|_1 &= \|\mathbb{E}((U_{i,n} - n^{-1}q\gamma)\mathbb{E}(h'_{i-1,i+1}|\mathcal{M}_\infty)|\mathcal{M}_{0,n})\|_1 \\ &\leq \|\mathbb{E}(U_{i,n} - n^{-1}q\gamma|\mathcal{M}_{l(i,n),n})\|_1 \,.\end{aligned}$$

Since both $\mathbb{P}$ and γ are invariant by T, the latter equals $\|\mathbb{E}(S_n(1/p) - n^{-1}q\gamma|\mathcal{M}_{0,n})\|_1$. Consequently, we infer that $\|D_1\|_1 \leq p\|\mathbb{E}(S_n(1/p) - n^{-1}q\gamma|\mathcal{M}_{0,n})\|_1$ and $\mathbf{S2}(a)$ implies that

$$\lim_{p\to\infty} \limsup_{n\to\infty} \|D_1\|_1 = 0 \,. \tag{3.9}$$

Control of D_2. We shall prove that, for any nonnegative integer j less than $N-1$,

$$\lim_{p\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{p} \Big\|\mathbb{E}((U_{i,n}^2 \mathbb{1}_{x_j<U_{i,n}\leq x_{j+1}} - \Delta_{i,n}^2 \mathbb{1}_{x_j<\Delta_{i,n}\leq x_{j+1}})g_i(x_j)|\mathcal{M}_{0,n})\Big\|_1 = 0 \,. \tag{3.10}$$

Define $f_j(t) = t^2 \mathbb{1}_{x_j<t\leq x_{j+1}}$ and recall that $q = [n/p]$. By definition of $\Delta_{i,n}$,

$$\mathbb{E}(\Delta_{i,n}^2 \mathbb{1}_{x_j<\Delta_{i,n}\leq x_{j+1}} g_i(x_j)|\mathcal{M}_\infty) = \mu_{\gamma,F}^{q/n}(f_j)\mathbb{E}(g_i(x_j)|\mathcal{M}_\infty) \,,$$

where the random variable $\mathbb{E}(g_i(x_j)|\mathcal{M}_\infty)$ is $\mathcal{M}_{l(i,n),n}$-measurable and bounded by one. Since $\mathcal{M}_{0,n} \subseteq \mathcal{M}_{l(i,n),n} \subseteq \mathcal{M}_\infty$, we obtain

$$\begin{aligned}&\|\mathbb{E}((U_{i,n}^2 \mathbb{1}_{x_j<U_{i,n}\leq x_{j+1}} - \Delta_{i,n}^2 \mathbb{1}_{x_j<\Delta_{i,n}\leq x_{j+1}})g_i(x_j)|\mathcal{M}_{0,n})\|_1 \\ &\qquad = \|\mathbb{E}((U_{i,n}^2 \mathbb{1}_{x_j<U_{i,n}\leq x_{j+1}} - \mu_{\gamma,F}^{q/n}(f_j))\mathbb{E}(g_i(x_j)|\mathcal{M}_\infty)|\mathcal{M}_{0,n})\|_1 \\ &\qquad\qquad \leq \|\mathbb{E}(U_{i,n}^2 \mathbb{1}_{x_j<U_{i,n}\leq x_{j+1}} - \mu_{\gamma,F}^{q/n}(f_j)|\mathcal{M}_{l(i,n),n})\|_1 \,.\end{aligned}$$

Since both $\mu_F(f_j)$ and $\mathbb{P}$ are invariant by the transformation T, (3.10) follows from

$$\lim_{p\to\infty} \limsup_{n\to\infty} p\|\mathbb{E}(f_j(S_n(1/p)) - \mu_{\gamma,F}^{q/n}(f_j)|\mathcal{M}_{0,n})\|_1 = 0 \,. \tag{3.11}$$

We infer from Lemma 1 that $\lim_{p\to\infty} \limsup_{n\to\infty} \|p\mu_{\gamma,F}^{q/n}(f_j) - F(x_{j+1}) + F(x_j)\|_1 = 0$, and (3.11) follows from $\mathbf{S2}(b)$.

End of the proof of Proposition 7. From (3.8), (3.9), (3.10) we infer that, for h in $B_1^3(\mathbb{R})$, $\lim_{\epsilon\to 0} \limsup_{p\to\infty} \limsup_{n\to\infty} \|D_1 + D_2 + D_3 + D_4\|_1 = 0$. This fact together with (3.4), (3.5) and (3.7) imply Proposition 7.

3.3 Sufficient conditions for EQ

In Proposition 8, we give conditions for a **WD**-array to be **EQ**. We need a maximal inequality.

Lemma 2 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be as in Theorem 1, and recall that $\mathcal{B}$ is either $\mathcal{I}$ or $\{\emptyset, \Omega\}$. Let $X'_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n}|\mathcal{B})$, $S'_n(t) = \sum_{i=1}^{[nt]} X'_{i,n}$ and $\overline{S'}_n(t) = \sup_{0\leq s\leq t} |S'_n(s)|$. Assume that $(X_{i,n})$ is* **WD** *and that $\limsup_{n\to\infty} n\mathbb{E}(X'^2_{0,n}) \leq C$ for some positive constant C. Then*

$$\limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t}\mathbb{E}((\overline{S'}_n(t))^2) < \infty \,. \tag{3.12}$$

Proof of Lemma 2. Let $S_n'^*(t) = \sup_{0\le s\le t}(S_n'(s))_+$. Applying Proposition 1(a) of Dedecker and Rio (2000) to the array $(X'_{i,n})$ with $\lambda = 0$, we get

$$\frac{1}{t}\mathbb{E}\left((S_n'^*(t))^2\right) \le 8\mathbb{E}\Big(\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=0}^{N-1}|X'_{k,n}X'_{k+i,n}|\Big) + 8n\sup_{N\le m\le [nt]}\Big\|X'_{0,n}\sum_{k=N}^{m}\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\Big\|_1. \quad (3.13)$$

Since $(X_{i,n})$ is **WD**, we can choose $N(\epsilon)$ large enough so that $R_1(N(\epsilon), X) \le \epsilon$. Now, using the elementary inequality $2|X'_{k,n}X'_{l,n}| \le X'^2_{k,n} + X'^2_{l,n}$ together with the stationarity of the sequence, we obtain $\limsup_{t\to 0}\limsup_{n\to\infty} t^{-1}\mathbb{E}((S_n'^*(t))^2) \le 4N(\epsilon)C + 8\epsilon$. Of course the same arguments applies to the array $(-X'_{i,n})$ and (3.12) follows.

Definition 5. Let $(X_{i,n})$ and $\mathcal{M}_{i,n}$ be as in Theorem 1. For any nonnegative integer N, define the variable $U_{N,n}(t)$ by $U_{M,N,n}(t) = \sum_{i=M}^{N-1}\sum_{k=1}^{[nt]}|X'_{k-i,n}|\mathbb{E}(|X'_{k,n}|\,|\mathcal{M}_{k-i,n})$. For any $1 \le M \le N$, define $R_4(M,N,X) = \limsup_{t\to 0}\limsup_{n\to\infty} t^{-1}\mathbb{E}(U_{M,N,n}(t)(1\wedge \overline{S'}_n(t)))$.

Proposition 8 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as Theorem 1. Recall that $R_3(N,X)$ has been defined in Remark 3. Assume that $\limsup_{n\to\infty} n\mathbb{E}(X'^2_{0,n}) \le C$ for some positive constant C.*

1. *If the array $(X_{i,n})$ is **WD**, then $R_2(N) \le R_3(N) + 2(M-1)\sqrt{CR_3(N)} + 2R_4(M,N)$, for any $1 \le M \le N$. Consequently $N_2(X) \le N_3(X) \vee 1$ and $(X_{i,n})$ is **EQ** as soon as $R_3(N)$ tends to zero as N tends to infinity and $\lim_{M\to\infty}\limsup_{N\to\infty} R_4(M,N) = 0$.*

2. *For any sequence u_n of random variables such that nu_n is equiintegrable,*

$$R_3(N) \le \limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\Big\|(1\wedge\overline{S'}_n(t))\sum_{k=1}^{[nt]}\left(\mathbb{E}(X'^2_{k,n}|\mathcal{M}_{k-N,n}) - u_n\right)\Big\|_1.$$

   In particular $R_3(N) \le \limsup_{n\to\infty} n\|\mathbb{E}(X'^2_{N,n}|\mathcal{M}_{0,n}) - \mathbb{E}(X'^2_{0,n})\|_1$.

3. *For any sequences $(u_{i,n})_{M\le i<N}$ of random variables such that $nu^2_{i,n}$ is equiintegrable,*

$$R_4(M,N) \le \limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\Big\|(1\wedge\overline{S'}_n(t))\sum_{i=M}^{N-1}\sum_{k=1}^{[nt]}|X'_{k-i,n}|(\mathbb{E}(|X'_{k,n}|\,|\mathcal{M}_{k-i,n}) - u_{i,n})\Big\|_1.$$

   In particular $R_4(M,N) \le \limsup_{n\to\infty}\sum_{i=M}^{N-1} n\|X'_{0,n}(\mathbb{E}(|X'_{i,n}|\,|\mathcal{M}_{0,n}) - \mathbb{E}(|X'_{0,n}|))\|_1$.

Proof of 1. For $0 \le i < N$ define $V_{i,N,n}(t) = \mathbb{E}(\sum_{k=1}^{[nt]}|X'_{k,n}X'_{k-i,n}|(1\wedge|S'_{k-N}|))$ and also $Q(i,N) = \limsup_{t\to 0}\limsup_{n\to\infty} t^{-1}V_{i,N,n}(t)$. By definition of $R_2(N,X)$, we have

$$R_2(N,X) \le Q(0,N) + 2\sum_{i=1}^{M-1}Q(i,N) + 2\limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\sum_{i=M}^{N-1}V_{i,N,n}(t). \quad (3.14)$$

Clearly we have the two inequalities $V_{0,N,n}(t) \leq \mathbb{E}((1 \wedge |\overline{S'}_n(t)|) \sum_{i=1}^{[nt]} \mathbb{E}(X'^2_{i,n}|\mathcal{M}_{i-N,n}))$ and $\sum_{i=M}^{N-1} V_{i,N,n}(t) \leq \mathbb{E}((1 \wedge |\overline{S'}_n(t)|) U_{M,N,n}(t))$. Consequently $Q(0,N) \leq R_3(N)$. In the same way, we obtain that

$$\limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t} \sum_{i=M}^{N-1} V_{i,N,n}(t) \leq R_4(M,N) \,. \tag{3.15}$$

Now, applying Cauchy-Schwarz twice, we have

$$V_{i,N,n}(t) \leq \mathbb{E}\Big(\Big(\sum_{k=1}^{[nt]} X'^2_{k-i,n}\Big)^{1/2}\Big(\sum_{k=1}^{[nt]} X'^2_{k,n}(1 \wedge |S'_{k-N}|)\Big)^{1/2}\Big) \leq \sqrt{nt\mathbb{E}(X'^2_{0,n})}\sqrt{V_{0,N,n}(t)}\,,$$

and we derive from (**??**) that $Q(i,N) \leq \sqrt{CR_3(N)}$. This completes the proof.

Proof of 2. By the triangle inequality, we have

$$R_3(N) \leq \limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t}\Big\|(1 \wedge \overline{S'}_n(t)) \sum_{k=1}^{[nt]} \big(\mathbb{E}(X'^2_{k,n}|\mathcal{M}_{k-N,n}) - u_n\big)\Big\|_1 \\ + \limsup_{t\to 0} \limsup_{n\to\infty} \mathbb{E}(nu_n(1 \wedge \overline{S'}_n(t)))\,.$$

By Lemma 2, $\limsup_{n\to 0} \mathbb{E}((1 \wedge \overline{S'}_n(t)) = O(\sqrt{t})$, so that the second term in right hand is 0.

Proof of 3. By the triangle inequality, we have

$$R_4(M,N) \leq \limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t}\Big\|(1 \wedge \overline{S'}_n(t)) \sum_{i=M}^{N-1} \sum_{k=1}^{[nt]} |X'_{k-i,n}| \big(\mathbb{E}(|X'_{k,n}||\mathcal{M}_{k-i,n}) - u_{i,n}\big)\Big\|_1 \\ + \limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t} \sum_{i=M}^{N-1} \mathbb{E}\Big(u_{i,n}(1 \wedge \overline{S'}_n(t)) \sum_{k=1}^{[nt]} |X'_{k-i,n}|\Big)\,. \tag{3.16}$$

Next, applying Cauchy-Schwarz inequality, we obtain that

$$\frac{1}{t}\mathbb{E}\Big(u_{i,n}(1 \wedge \overline{S'}_n(t)) \sum_{k=1}^{[nt]} |X'_{k-i,n}|\Big) \leq \sqrt{\mathbb{E}(nu^2_{i,n}(1 \wedge \overline{S'}_n(t)))}\sqrt{n\mathbb{E}(X'^2_{0,n})}\,. \tag{3.17}$$

and we conclude as for 2.

3.4 Proof of Proposition 1 and Corollary 3

We first prove the following Lemma, which is the main result of this section:

Lemma 3 *Let $X_{i,n}$ and $\mathcal{M}_{i,n}$ be as in Theorem 1. Let $X'_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n}|\mathcal{B})$ and $S'_n(t) = \sum_{i=1}^{[nt]} X'_{i,n}$. For any nonnegative integer N and any function f, let $V_{N,n}(k) = \sum_{i=k-N+1}^{k-1} X'_{i,n}$ and $\Delta f(N,n,k) = f(V_{N,n}(k) + X'_{k,n}) - f(V_{N,n}(k))$ For any positive number C, let $\mathcal{F}_C$ be the class of three-times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\|h'''\|_\infty \leq C$, $\|h''\|_\infty \leq C$, $h'(0) = 0$ and $h(0) = 0$. For any function h belonging to $\mathcal{F}_C$, we have*

$$\limsup_{t\to 0} \limsup_{n\to\infty} \Big\|\frac{1}{t}\mathbb{E}\Big(h(S'_n(t)) - \sum_{k=1}^{[nt]} \Delta h(N,n,k)\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 \leq C(R_1(N,X) + R_2(N,X)).$$

Proof of Lemma 3. For any integer N, we have the decompositions

$$h(S'_n(t)) = \sum_{k=1}^{[nt]} X'_{k,n} h'(S'_{k-N,n}) + \sum_{k=1}^{[nt]} X'^2_{k,n} \int_0^1 (1-s) h''(S'_{k-1,n} + sX'_{k,n}) ds$$
$$+ \sum_{k=1}^{[nt]} X'_{k,n}(S'_{k-1,n} - S'_{k-N,n}) \int_0^1 h''(S'_{k-N,n} + s(S'_{k-1,n} - S'_{k-N,n})) ds$$

and

$$h(V_{N,n}(k) + X'_{k,n}) - h(V_{N,n}(k)) = X'^2_{k,n} \int_0^1 (1-s) h''(S'_{k-1,n} - S'_{k-N,n} + sX'_{k,n}) ds$$
$$+ X'_{k,n}(S'_{k-1,n} - S'_{k-N,n}) \int_0^1 h''(s(S'_{k-1,n} - S'_{k-N,n})) ds \,.$$

From this two decompositions, we get

$$h(S'_n(t)) - \sum_{k=1}^{[nt]} \Delta h(N,n,k) = \sum_{k=1}^{[nt]} X'_{k,n} h'(S'_{k-N,n}) + E_1 + E_2 \,, \tag{3.18}$$

where

$$E_1 = \sum_{k=1}^{[nt]} X'_{k,n}(S'_{k-1,n} - S'_{k-N,n}) \int_0^1 h''(S'_{k-N,n} + s(S'_{k-1,n} - S'_{k-N,n})) - h''(s(S'_{k-1,n} - S'_{k-N,n})) ds$$
$$E_2 = \sum_{k=1}^{[nt]} X'^2_{k,n} \int_0^1 (1-s)(h''(S'_{k-1,n} + sX'_{k,n}) - h''(S'_{k-1,n} - S'_{k-N,n} + sX'_{k,n})) ds \,.$$

Let us first study the first term on right hand in (3.18). Obviously

$$\sum_{k=1}^{[nt]} X'_{k,n} h'(S'_{k-N,n}) = \sum_{i=1}^{[nt]} \Big(\sum_{k=N+i}^{[nt]} X'_{k,n} \Big) (h'(S'_{i,n}) - h'(S'_{i-1,n})) \,.$$

Taking the conditional expectation with respect to $\mathcal{M}_{i,n}$ and using that h' is C-lipschitz,

$$\begin{aligned} \Big\| \mathbb{E}\Big(\sum_{k=1}^{[nt]} X'_{k,n} h'(S'_{k-N,n}) \Big| \mathcal{M}_{0,n} \Big) \Big\|_1 &\leq C \sum_{i=1}^{[nt]} \Big\| X'_{i,n} \sum_{k=N+i}^{[nt]} \mathbb{E}(X'_{k,n} | \mathcal{M}_{i,n}) \Big\|_1 \\ &\leq tC \max_{N \leq m \leq [nt]} n \Big\| X'_{0,n} \sum_{k=N}^{m} \mathbb{E}(X'_{k,n} | \mathcal{M}_{0,n}) \Big\|_1 \,, \end{aligned}$$

the second inequality involving the stationarity of $(X'_{i,n})$. This together with (3.18) yields

$$\limsup_{t \to 0} \limsup_{n \to \infty} \frac{1}{t} \Big\| \mathbb{E}\Big(h(S'_n(t)) - \sum_{k=1}^{[nt]} \Delta h(N,n,k) \Big| \mathcal{M}_{0,n} \Big) \Big\|_1$$
$$\leq C R_1(N,X) + \limsup_{t \to 0} \limsup_{n \to \infty} \frac{1}{t} \| E_1 + E_2 \|_1 \,.$$

Now, since h'' is C-lipschitz and bounded by C, we easily see that

$$|E_1| \le 2C \sum_{k=1}^{[nt]} |X'_{k,n}(S'_{k-1,n} - S'_{k-N,n})|(1 \wedge |S'_{k-N,n}|) \quad \text{and} \quad |E_2| \le C \sum_{k=1}^{[nt]} X'^2_{k,n}(1 \wedge |S'_{k-N,n}|)\,.$$

This completes the proof of Lemma 3.

End of the Proof of Proposition 1. Assume that $(X_{i,n})$ is **WD** and **EQ**. Let $(N_i)_{i\in\mathbb{N}^*}$ be a nondecreasing sequence converging to $N_0(X)$ such that $A(N_i)$ holds for any i in $\mathbb{N}^*$.

For any function g denote by $\bar g$ the function $y \to y^2 g(y)$. For any positive integer k, we have $\mathbb{E}(V^2_{N,n}(k)) \le N^2 \mathbb{E}(X'^2_{0,n})$. Since by assumption the sequence $n\mathbb{E}(X'^2_{0,n})$ is bounded, we infer that, for each N, the sequence $t^{-1}\sum_{k=1}^{[nt]} \mathbb{E}(V^2_{N,n}(k))$ is bounded. This fact together with $A(N_i)$ implies that, for each i and each continuous bounded function g,

$$\lim_{t\to 0} \limsup_{n\to\infty} \Big\| \mathbb{E}\Big(\frac{1}{t} \sum_{k=1}^{[nt]} \Delta \bar g(N_i, n, k) - \int g dF_{N_i} \Big| \mathcal{M}_{0,n}\Big)\Big\|_1 = 0\,. \tag{3.19}$$

If furthermore g is three times continuously differentiable with compactly supported derivatives, then $\bar g$ belongs to a class $\mathcal{F}_C$ for a certain constant C. Now Lemma 3 together with (3.19) yields

$$\limsup_{t\to 0} \limsup_{n\to\infty} \Big\| \mathbb{E}\Big(\frac{1}{t} \bar g(S'_n(t)) - \int g dF_{N_i} \Big| \mathcal{M}_{0,n}\Big)\Big\|_1 \le C(R_1(N_i, X) + R_2(N_i, X))\,. \tag{3.20}$$

From (3.20), we first derive that, for $j \ge i$,

$$\limsup_{n\to\infty} \Big\| \mathbb{E}\Big(\int g dF_{N_j} - \int g dF_{N_i} \Big| \mathcal{M}_{0,n}\Big)\Big\|_1 \le 2C(R_1(N_i, X) + R_2(N_i, X))\,, \tag{3.21}$$

so that $\limsup_{n\to\infty} \|\mathbb{E}(\int g dF_{N_j} - \int g dF_{N_i} | \cap_{k\ge n} \mathcal{M}_{0,k})\|_1 \le 2C(R_1(N_i, X) + R_2(N_i, X))$. Applying the martingale convergence theorem, and bearing in mind that each F_{N_i} is $\mathcal{M}_{0,\mathrm{inf}}$-measurable, it follows from (3.21) that

$$\text{for } j \ge i, \quad \Big\| \int g dF_{N_j} - \int g dF_{N_i} \Big\|_1 \le 2C(R_1(N_i, X) + R_2(N_i, X))\,.$$

Hence, $\int g dF_{N_i}$ converges in $\mathbb{L}^1$ to a $\mathcal{M}_{0,\mathrm{inf}}$-measurable limit $L(g)$. Now, from (3.20) again $\lim_{t\to 0} \limsup_{n\to\infty} \|\mathbb{E}(t^{-1}\bar g(S'_n(t)) - L(g) | \cap_{k\ge n} \mathcal{M}_{0,k})\|_1 = 0$, and since $L(g)$ is $\mathcal{M}_{0,\mathrm{inf}}$-measurable,

$$\lim_{t\to 0} \limsup_{n\to\infty} \Big\| \mathbb{E}\Big(\frac{1}{t} \bar g(S'_n(t)) \Big| \bigcap_{k\ge n} \mathcal{M}_{0,k}\Big) - L(g) \Big\|_1 = 0\,. \tag{3.22}$$

From (3.22), we infer that for nonnegative g, $L(g)$ is almost surely nonnegative. Furthermore, $L(f+g) = L(f) + L(g)$, $L(\alpha g) = \alpha L(g)$ and $\mathbb{E}(L(1))$ is finite. This enables us to prove the following lemma:

Lemma 4 *Let $\mathcal{C}_3$ be the space of bounded and three times continuously differentiable functions with compactly supported derivatives. There exists a $\mathcal{M}_{0,\inf}$-measurable distribution function F such that, for any function g in $\mathcal{C}_3$, $L(g) = \int g dF$ almost surely.*

Applying Lemma 4, we obtain that

$$\lim_{t\to 0}\limsup_{n\to\infty}\Big\|\mathbb{E}\Big(\frac{1}{t}\bar{g}(S'_n(t)) - \int g dF\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 = 0\,. \tag{3.23}$$

To prove that (3.23) still holds for $g(y) = \mathbb{1}_{y\leq x}$ where x is a continuity point of $x \to \mathbb{E}(F(x))$, we proceed as in Inequality (3.1), Section 3.1. Since $\mathbf{S2}(a)$ holds, we can apply Remark 1 with $Z_n(t) = [nt]\mathbb{E}(X_{0,n})$, and Proposition 1 follows.

Proof of Lemma 4. Let $g : \mathbb{R} \mapsto [0,1]$ be a function of $\mathcal{C}_3$, equal to one on $]-\infty,0]$ and to 0 on $[1,+\infty[$. Define $g_{n,x}(y) = g(n(y-x))$. For almost every ω, the sequence $L(g_{n,x})(\omega)$ is nonincreasing with n. Denote by $L(\mathbb{1}_{]-\infty,x]})(\omega)$ its limit. Since $L(g_{n,x})(\omega) \leq L(1)(\omega)\, a.s.$, the dominated convergence theorem ensures that $L(g_{n,x})$ converges to $L(\mathbb{1}_{]-\infty,x]})$ in $\mathbb{L}^1$. It is clear that if $h \geq \mathbb{1}_{]-\infty,x]}$ then $L(h) \geq L(\mathbb{1}_{]-\infty,x]})\, a.s.$, and if $x \geq y$ then $L(\mathbb{1}_{]-\infty,x]}) \geq L(\mathbb{1}_{]-\infty,y]})\, a.s.$. Therefore, on a set A of probability 1, the function from $\mathbb{Q}$ to $\mathbb{R}$: $x \to L(\mathbb{1}_{]-\infty,x]})$ is almost surely nondecreasing. Define the random function F as follows: for each ω of A, F is the unique distribution function equal to $x \to L(\mathbb{1}_{]-\infty,x]})$ on $\mathbb{Q}$. For ω in A^c, $F \equiv 0$. According to our definition, F is a $\mathcal{M}_{0,\inf}$-measurable distribution function. Now, let h be any function of $\mathcal{C}_3$ with compact support, and choose a function $h_\epsilon = \sum_{i=1}^m a_i \mathbb{1}_{]x_i,x_{i+1}]}$, with $x_i \in \mathbb{Q}$, such that $0 \leq h - h_\epsilon \leq \epsilon$. Then $|L(h) - \int h_\epsilon dF| \leq \epsilon L(1)\, a.s.$ and consequently $L(h) = \int h dF\, a.s.$. This result extends to any function of $\mathcal{C}_3$ and the proof of Lemma 4 is complete.

Proof of Corollary 3. We begin with the first part of Corollary 3. Let $(X_{i,n})$ be **WD** and **EQ** with $N_0(X) = 1$. From Proposition 1, we know that (2.4) implies **S2**. Assume that **S2** holds and let $f_x(y) = y^2\mathbb{1}_{y\leq x}$. To see that (2.4) holds, it suffices to prove that, for any continuity point of $x \to G(x) = \mathbb{E}(F(x))$,

$$\lim_{t\to 0}\limsup_{n\to\infty}\Big\|\frac{1}{t}\mathbb{E}\Big(f_x(S'_n(t)) - \sum_{k=1}^{[nt]} f_x(X'_{k,n})\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 = 0\,. \tag{3.24}$$

For any positive ϵ let $h_{x,\epsilon+}$ (resp. $h_{x,\epsilon-}$) be a positive three times continuously differentiable function, bounded by one, equal to one on the interval $]-\infty,x]$ (resp $]-\infty,x-\epsilon]$) and to zero on $[x+\epsilon,\infty[$ (resp. $[x,\infty[$). The functions $f_{x,\epsilon+}(y) = y^2 h_{x,\epsilon+}(y)$ and $f_{x,\epsilon-}(y) = y^2 h_{x,\epsilon-}(y)$ belong to a class $\mathcal{F}_C$ for a certain constant C. Using these functions, we have the inequality

$$\begin{aligned}\frac{1}{t}\Big\|\mathbb{E}\Big(f_x(S'_n(t)) - \sum_{k=1}^{[nt]} f_x(X'_{k,n})\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 &\leq \frac{1}{t}\Big\|\mathbb{E}\Big(f_{x,\epsilon+}(S'_n(t)) - \sum_{k=1}^{[nt]} f_{x,\epsilon+}(X_{k,n})\Big|\mathcal{M}_{0,n}\Big)\Big\|_1 \\ &+ \frac{1}{t}\mathbb{E}\Big(\sum_{k=1}^{[nt]}(f_{x,\epsilon+} - f_{x,\epsilon-})(X'_{k,n})\Big) + \frac{1}{t}\mathbb{E}\Big((f_{x,\epsilon+} - f_{x,\epsilon-})(S'_n(t))\Big)\,. \end{aligned} \tag{3.25}$$

The first term on right hand is well controlled via Lemma 3. From Remark 1 with $Z_n(t) = [nt]\mathbb{E}(X_{0,n}|\mathcal{B})$, we infer that $\mathbf{S2}(b)$ holds with $X'_{i,n}$ instead of $X_{i,n}$. Combining this fact with Lemma 3, we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\Big(\sum_{k=1}^{[nt]}(f_{x,\epsilon+}-f_{x,\epsilon-})(X'_{k,n})\Big)\le G(x+\epsilon)-G(x-\epsilon)\,, \tag{3.26}$$

and Property $\mathbf{S2}(b)$ with $X'_{i,n}$ instead of $X_{i,n}$ provides also

$$\limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\Big((f_{x,\epsilon+}-f_{x,\epsilon-})(S'_n(t))\Big)\le G(x+\epsilon)-G(x-\epsilon)\,. \tag{3.27}$$

Collecting (3.25), (3.26) and (3.27), we obtain (3.24). This completes the proof of the first part of Corollary 3. The second part of Corollary 3 follows from the first part by noting that, if $(X_{i,n})$ is an array with i.i.d. rows such that $n\mathbb{E}(X_{0,n})$ converges to γ and $n\mathbb{E}(X'^2_{0,n})$ is bounded (here $\mathcal{B}=\{\emptyset,\Omega\}$), then it is $\mathbf{WD}$ and $\mathbf{EQ}$ with $N_0(X)=1$.

3.5 Sufficient conditions for WD and proof of Corollary 4.

Proposition 9 *Let $(X_{i,n})$ and $\mathcal{M}_{i,n}$ be as in Theorem 1. Consider the two conditions:*

(a) $\mathbf{S2}(a)$ *holds, and* $\displaystyle\lim_{N\to\infty}\limsup_{n\to\infty}\; n\sum_{k=N}^{n}\|X'_{0,n}\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\|_1=0.$

(b) There exists an $\mathcal{M}_{0,\inf}$-measurable square integrable random variable γ such that the sequence $\|n\mathbb{E}(X_{0,n}|\mathcal{B})-\gamma\|_1$ converges to 0, and for some conjugate exponents p,q,

$$\lim_{N\to\infty}\limsup_{n\to\infty}\; n\|X'_{0,n}\|_p\sum_{k=N}^{n}\|\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\|_q=0\,.$$

We have the implications $(b)\Rightarrow(a)\Rightarrow$ $\mathbf{WD}$.

Proof of Proposition 9. The fact that $(a)\Rightarrow$ $\mathbf{WD}$ is straightforward. Applying Hölder's inequality, we easily see that (b) implies the second condition required in (a). It remains to see that (b) also implies $\mathbf{S2}(a)$, for some γ such that $\|n\mathbb{E}(X_{0,n}|\mathcal{B})-\gamma\|_1$ converges to 0. Since by assumption $\|X_{0,n}\|_1$ tends to zero as n tends to infinity, $\mathbf{S2}(a)$ follows from

$$\lim_{N\to\infty}\limsup_{n\to\infty}\Big\|\sum_{k=N}^{n}\mathbb{E}(X_{k,n}|\mathcal{M}_{0,n})-\mathbb{E}(X_{0,n}|\mathcal{B})\Big\|_1=\lim_{N\to\infty}\limsup_{n\to\infty}\Big\|\sum_{k=N}^{n}\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\Big\|_1=0\,.$$

For any conjugate exponent p,q we have the inequality

$$\Big\|\sum_{k=N}^{n}\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\Big\|_1\le\Big(n\|X'_{0,n}\|_p\sum_{k=N}^{n}\|\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\|_q\Big)^{1/2}.$$

Consequently $\mathbf{S2}(a)$ follows from (c) and the proof of Proposition 9 is complete.

Proof of Corollary 4. This corollary follows from Proposition 1 and the following proposition.

Proposition 10 *Let $X_{i,n}$, $\mathcal{M}_{i,n}$ and $S_n(t)$ be as in Theorem 1 and let $\mathcal{B} = \{\emptyset, \Omega\}$. Assume that $n\mathbb{E}(X_{0,n})$ converges to γ.*

1. *If $C_\phi(p,1)$ (resp. $C_\rho(1)$) holds then $(X_{i,n})_{i,n}$ is* **WD** *and $\lim_{M\to\infty} \limsup_{N\to\infty} R_4(M,N) = 0$.*
2. *Assume that C_0 and $C_\phi(p,1)$ (resp. $C_\rho(1)$) hold. Then $R_3(N)$ tends to zero as $N \to \infty$.*
3. *Assume that C_0 and $C_\phi(p,N)$ (resp. $C_\rho(N)$) hold. If $B(N)$ holds then $A(N)$ holds.*

Proof of Proposition 10. We do the proof under Condition C_ϕ only. In fact, the proof is the same under C_ρ by taking $p = q = 2$ everywhere and replacing $\phi_{\infty,N}^{1/p}$ by $\rho_{\infty,N}$.

Proof of 1. We first prove that if Condition $C_\phi(p,1)$ holds, then $(X_{i,n})$ is **WD**. It suffices to see that Condition (b) of Proposition 9 holds. Applying Serfling's inequality (1968), we have $\|\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\|_q \le 2\phi_{\infty,1}^{1/p}(k,n)\|X'_{0,n}\|_q$, and consequently

$$n\|X'_{0,n}\|_p \sum_{k=N}^{n} \|\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\|_q \le 2n\|X'_{0,n}\|_p\|X'_{0,n}\|_q \sum_{k=N}^{n} \phi_{\infty,1}^{1/p}(k,n)\,.$$

This inequality together with $C_\phi(p,1)$ imply that (c) holds, and the array $(X_{i,n})$ is **WD**.

It remains to see that $C_\phi(p,1)$ implies $\lim_{M\to\infty} \limsup_{N\to\infty} R_4(M,N) = 0$. From 3 of Proposition 8

$$R_4(M,N) \le \limsup_{n\to\infty} \sum_{i=M}^{n} n\|X'_{0,n}(\mathbb{E}(|X'_{i,n}|\,|\mathcal{M}_{0,n}) - \mathbb{E}(|X'_{0,n}|))\|_1\,. \tag{3.28}$$

Now $n\|X'_{0,n}(\mathbb{E}(|X'_{i,n}|\,|\mathcal{M}_{0,n}) - \mathbb{E}(|X'_{0,n}|))\|_1 \le 2n\|X'_{0,n}\|_q\|X'_{0,n}\|_p\phi_{\infty,1}^{1/p}(i,n)$ by Serfling's inequality. This inequality together with $C_\phi(p,1)$ and (3.28) gives the result.

Proof of 2. By C_0, we can choose K large enough so that $\limsup_{n\to\infty} n\mathbb{E}(X'^2_{0,n}\mathbb{1}_{|X'_{0,n}|>K}) \le \epsilon$. Hence, it follows from 2 of Proposition 8 that $R_3(N)$ tends to zero as soon as, for any $K > 0$,

$$\lim_{N\to\infty} \limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t}\Big\|(1 \wedge \overline{S'}_n(t)) \sum_{k=1}^{[nt]} \big(\mathbb{E}(X'^2_{k,n}\mathbb{1}_{|X'_{k,n}|\le K}|\mathcal{M}_{k-N,n}) - \mathbb{E}(X'^2_{0,n}\mathbb{1}_{|X'_{0,n}|\le K})\big)\Big\|_1 = 0\,.$$

Since by Lemma 2 $\limsup_{n\to\infty} \|(1 \wedge \overline{S'}_n(t))\|_2 = O(\sqrt{t})$, this equality follows from

$$\lim_{N\to\infty} \limsup_{t\to 0} \limsup_{n\to\infty} \frac{1}{t}\mathrm{Var}\Big(\sum_{k=1}^{[nt]} \mathbb{E}(X'^2_{k,n}\mathbb{1}_{|X'_{k,n}|\le K}|\mathcal{M}_{k-N,n})\Big) = 0\,. \tag{3.29}$$

Setting $Y_{k,n} = \mathbb{E}(X'^2_{k,n}\mathbb{1}_{|X'_{k,n}|\le K}|\mathcal{M}_{k-N,n})$, we have the elementary inequality

$$\frac{1}{t}\mathrm{Var}\Big(\sum_{k=1}^{[nt]} Y_{k,n}\Big) \le 2n \sum_{k=0}^{n} |\mathrm{Cov}(Y_{0,n}, Y_{k,n})|\,. \tag{3.30}$$

Now, by definition of $Y_{k,n}$ and applying Peligrad's inequality (1983), we have successively

$$
\begin{aligned}
|\mathrm{Cov}(Y_{0,n}, Y_{k,n})| &= |\mathrm{Cov}(Y_{0,n}, X'^2_{k,n} \mathbb{1}_{|X'_{k,n}| \le K})| \\
&\le 2\phi^{1/p}_{\infty,1}(k+N, n)\|X'^2_{0,n}\mathbb{1}_{|X'_{0,n}|\le K}\|_p \|X'^2_{0,n}\mathbb{1}_{|X'_{0,n}|\le K}\|_q \\
&\le 2K^2\phi^{1/p}_{\infty,1}(k+N, n)\|X'_{0,n}\|_p\|X'_{0,n}\|_q \, .
\end{aligned}
$$

This last inequality together with (3.30) yields

$$
\frac{1}{t}\mathrm{Var}\Big(\sum_{k=1}^{[nt]} \mathbb{E}(X'^2_{k,n}\mathbb{1}_{|X'_{k,n}|\le K}|\mathcal{M}_{k-N,n})\Big) \le 4K^2 n\|X'_{0,n}\|_p\|X'_{0,n}\|_q \sum_{k=0}^{n} \phi^{1/p}_{\infty,1}(k+N,n)\, . \qquad (3.31)
$$

Now $\sum_{k=0}^n \phi^{1/p}_{\infty,1}(k+N,n) \le \sum_{i=N}^n \phi^{1/p}_{\infty,1}(i,n) + \sum_{i=n-N+1}^n \phi^{1/p}_{\infty,1}(i,n)$, since $\phi_{\infty,1}(k,n)$ is nonincreasing in k. Combining this inequality with $C_\phi(p,1)$ and (3.31), we infer that (3.29) holds.

Proof of 3. With the notations of Proposition 1, set $Z_{k,n} = \Delta f_x(N,n,k)$. To prove that $B(N)$ implies $A(N)$, it suffices to see that

$$
\lim_{n\to\infty}\Big\|\sum_{k=1}^{[nt]}(\mathbb{E}(Z_{k,n}|\mathcal{M}_{0,n}) - \mathbb{E}(Z_{k,n}))\Big\|_1 = 0\, . \qquad (3.32)
$$

Since C_0 holds and $|Z_{k,n}| \le 2N(X'^2_{k-N+1} + \cdots + X'^2_{k,n})$, we can choose K large enough so that $\limsup_{n\to\infty} n\mathbb{E}(|Z_{0,n}|\mathbb{1}_{|Z_{0,n}|>K}) \le \epsilon$. Therefore, to prove (3.32) it suffices to see that for any K,

$$
\lim_{n\to\infty}\mathrm{Var}\Big(\sum_{k=1}^{[nt]}\mathbb{E}(Z_{k,n}\mathbb{1}_{|Z_{k,n}|\le K}|\mathcal{M}_{0,n})\Big) = 0\, . \qquad (3.33)
$$

Setting $W_{k,n} = \mathbb{E}(Z_{k,n}\mathbb{1}_{|Z_{k,n}|\le K}|\mathcal{M}_{0,n})$, we have the elementary inequality

$$
\mathrm{Var}\Big(\sum_{k=1}^{[nt]} W_{k,n}\Big) \le 2\sum_{i=1}^{[nt]}\sum_{j=i}^{[nt]} |\mathrm{Cov}(W_{i,n}, W_{j,n})|\, . \qquad (3.34)
$$

Now, by definition of $W_{k,n}$ and applying Peligrad's inequality (1983), we have successively

$$
\begin{aligned}
|\mathrm{Cov}(W_{i,n}, W_{j,n})| &= |\mathrm{Cov}(W_{i,n}, Z_{j,n}\mathbb{1}_{|Z_{j,n}|\le K})| \\
&\le 2\phi^{1/p}_{\infty,N}\left((j-N+1)_+, n\right)\|Z_{i,n}\mathbb{1}_{|Z_{i,n}|\le K}\|_p\|Z_{j,n}\mathbb{1}_{|Z_{j,n}|\le K}\|_q \\
&\le 8KN^2\phi^{1/p}_{\infty,N}((j-N+1)_+, n)\|X'_{0,n}\|_p\|X'_{0,n}\|_q\, .
\end{aligned}
$$

This last inequality together with (3.34) yields

$$
\mathrm{Var}\Big(\sum_{k=1}^{[nt]}\mathbb{E}(Z_{k,n}\mathbb{1}_{|Z_{k,n}|\le K}|\mathcal{M}_{0,n})\Big) \le (16KN^2 n\|X'_{0,n}\|_p\|X'_{0,n}\|_q)\frac{1}{n}\sum_{j=1}^{[nt]} j\phi^{1/p}_{\infty,N}((j-N+1)_+, n)
$$

and the right hand term tends to zero as soon as $\lim_{n\to\infty} n^{-1}\sum_{j\le n} j\phi^{1/p}_{\infty,N}(j,n) = 0$. Finally, (3.33) holds as soon as $C_\phi(p,N)$ holds, which completes the proof.

3.6 Proof of Proposition 2

The main reference here is Dedecker and Merlevède (2002), Section 4, where it is shown that property **LP** with $F = \lambda \mathbb{1}_{[0,\infty[}$ follows from both finite dimensional convergence and tightness of some sequences of signed measures. This results extends easily to any $\mathcal{M}_{0,\inf}$-measurable distribution function F. Now, finite dimensional convergence follows from **S2** as in Section 4.1 in Dedecker and Merlevède (2002). Furthermore, proceeding as in Section 4.2 of the latter and applying Theorem 15.3 in Billingsley (1968), tightness holds as soon as, setting $\delta_k = 2^{-k}$,

$$\text{for any positive } \epsilon, \quad \lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(w''(S_n, \delta_k) \geq \epsilon\Big) = 0 \tag{3.35}$$

where $w''(x,\delta) = \sup\{|x(t)-x(t_1)| \wedge |x(t_2)-x(t)|, t_1 \leq t \leq t_2, t_2 - t_1 \leq \delta\}$ is defined for x in $D([0,1])$ and $\delta > 0$. Recall that $\mathcal{B}$ is $\mathcal{I}$ or $\{\emptyset, \Omega\}$. From **S2**(a), $\|n\mathbb{E}(X_{0,n}|\mathcal{B}) - \gamma\|_1$ converges to 0. Hence $\sup_{t\in[0,1]} |[nt]\mathbb{E}(X_{0,n}|\mathcal{B}) - t\gamma|$ converges in $\mathbb{L}^1$ to 0. Let $X'_{i,n} = X_{i,n} - \mathbb{E}(X_{i,n}|\mathcal{B})$ and $S'_n = \{t \to S'_n(t) = \sum_{i=1}^{[nt]} X'_{i,n}\}$. It follows that (3.35) holds if and only if

$$\text{for any positive } \epsilon, \quad \lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(w''(S'_n, \delta_k) \geq \epsilon\Big) = 0 \tag{3.36}$$

Consequently, Proposition 2 follows straightforwadly from the two following lemmas

Lemma 5 *Let $(X'_{i,n})$ be an array with stationary rows, such that $X'_{0,n}$ converges in probability to zero. Then (3.36) holds as soon as, for any positive ϵ,*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\leq t\leq s\leq \delta} |S'_n(t)| \wedge |S'_n(\delta) - S'_n(s)| \geq \epsilon \right) = 0\,. \tag{3.37}$$

Moreover (3.37) is satisfied provided that

1. *For any positive η, there exists K such that $\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{\delta} \mathbb{P}(|S'_n(\delta)| > K) \leq \eta$.*
2. *For any real a, $\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\leq t\leq s\leq\delta} |S'_n(t)| \wedge |S'_n(s) - a| \geq 2\epsilon, |S'_n(\delta) - a| \leq \epsilon \right) = 0.$*

Lemma 6 *Let $(X_{i,n})$ be a* **WD** *and 1-***EQ** *array. Let a be a real number and g_a be a function from $\mathbb{R}$ to $[0,1]$ such that: g_a is two times continuously differentiable, $g_a(x) = 1$ if $|x-a| \leq \epsilon$ and $g_a(x) = 0$ if $|x-a| \geq 2\epsilon$. Let $f_a(x,y) = \mathbb{1}_{|x|\wedge|y-a|\geq 2\epsilon}$. Then*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{\delta} \mathbb{E}\left(\sup_{0\leq t\leq s\leq\delta} f_a\big(S'_n(t), S'_n(s)\big) g_a(S'_n(\delta)) \right) = 0\,. \tag{3.38}$$

Proof of Lemma 5. For any integer $0 \leq j \leq 2^k - 1$, let $I_{j,k} = [j2^{-k}, (j+1)2^{-k}]$. If $t_2 - t_1 \leq \delta_k$, there are two possibility: either both t_1 and t_2 belongs to the same $I_{j,k}$ or t_1 belongs to $I_{j,k}$ and t_2 to $I_{j+1,k}$. In the first case, $|x(t)-x(t_1)| \wedge |x(t_2)-x(t)| \leq A_1 + A_2 + A_3 + A_4$, with $A_1 = |x(t) - x(j2^{-k})| \wedge |x((j+1)2^{-k}) - x(t)|$, $A_2 = |x(t) - x(j2^{-k})| \wedge |x((j+1)2^{-k}) - x(t_2)|$, and $A_3 = |x(t_1) - x(j2^{-k})| \wedge |x((j+1)2^{-k}) - x(t)|$, $A_4 = |x(t_1) - x(j2^{-k})| \wedge |x((j+1)2^{-k}) - x(t_2)|$.

In the second case $|x(t) - x(t_1)| \wedge |x(t_2) - x(t)| \leq B_1 + B_2 + B_3 + B_4$, with the quantities $B_1 = |x(t) - x(j2^{-k})| \wedge |x((j+2)2^{-k}) - x(t)|$, $B_2 = |x(t) - x(j2^{-k})| \wedge |x((j+2)2^{-k}) - x(t_2)|$, $B_3 = |x(t_1) - x(j2^{-k})| \wedge |x((j+2)2^{-k}) - x(t)|$, $B_4 = |x(t_1) - x(j2^{-k})| \wedge |x((j+2)2^{-k}) - x(t_2)|$. Define the quantities $v_{k,j}(x)$, $w_{k,j}(x)$, $v_k(x)$ and $w_k(x)$ by

$$\begin{aligned}
v_{k,j}(x) &= \sup_{j2^{-k} \leq t \leq s \leq (j+1)2^{-k}} |x(t) - x(j2^{-k})| \wedge |x((j+1)2^{-k}) - x(s)| \\
w_{k,j}(x) &= \sup_{j2^{-k} \leq t \leq s \leq (j+2)2^{-k}} |x(t) - x(j2^{-k})| \wedge |x((j+2)2^{-k}) - x(s)| \\
v_k(x) &= \max_{0 \leq j \leq 2^k - 1} v_{k,j}(x) \quad \text{and} \quad w_{k,j}(x) = \max_{0 \leq j \leq 2^k - 1} w_{k,j}(x)\,.
\end{aligned}$$

From this definition $\mathbb{P}(w''(S'_n, \delta_k) \geq 4\epsilon) \leq \mathbb{P}(v_k(S'_n) \geq \epsilon) + \mathbb{P}(w_k(S'_n) \geq \epsilon)$. By subbaditivity $\mathbb{P}(v_k(S'_n) \geq \epsilon) \leq \sum_{j=0}^{2^k-1} \mathbb{P}\left(v_{k,j}(S'_n) \geq \epsilon\right)$ and $\mathbb{P}(w_k(S'_n) \geq \epsilon) \leq \sum_{j=0}^{2^k-1} \mathbb{P}\left(w_{k,j}(S'_n) \geq \epsilon\right)$. Using both the stationarity and the fact that $X'_{0,n}$ converges in probability to zero,

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}(v_{k,j}(S'_n) \geq \epsilon) &= \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{0 \leq t \leq s \leq 2^{-k}} |S'_n(t)| \wedge |S'_n(2^{-k}) - S'_n(s)| \geq \epsilon \Big) \\
\limsup_{n\to\infty} \mathbb{P}(w_{k,j}(S'_n) \geq \epsilon) &= \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{0 \leq t \leq s \leq 2^{-k+1}} |S'_n(t)| \wedge |S'_n(2^{-k+1}) - S'_n(s)| \geq \epsilon \Big).
\end{aligned}$$

From the two preceding remarks, it is clear that (3.36) follows from (3.37). Now let $(A_i)_{i \in I}$ be a finite covering of $[-K, K]$ by intervals with centers $(a_i)_{i \in I}$ and length ϵ. We have the inequality

$$\begin{aligned}
\mathbb{P}\left(\sup_{0 \leq t \leq s \leq \delta} |S'_n(t)| \wedge |S'_n(\delta) - S'_n(s)| \geq 3\epsilon \right) &\leq \mathbb{P}(|S'_n(\delta)| > K) \\
&+ \sum_{i \in I} \mathbb{P}\left(\sup_{0 \leq t \leq s \leq \delta} |S'_n(t)| \wedge |S'_n(\delta) - S'_n(s)| \geq 3\epsilon, |S'_n(\delta) - a_i| \leq \epsilon \right) \quad (3.39)
\end{aligned}$$

If $|S'_n(s) - S'_n(\delta)| \geq 3\epsilon$ and $|S'_n(\delta) - a| \leq \epsilon$ then $|S'_n(s) - a| \geq 2\epsilon$. Combining this fact together with (3.39), we infer that (3.37) follows from 1. and 2. of Lemma 5. This completes the proof.

Proof of Lemma 6. Let $f^*(k) = \max\{f_a(S'_{i,n}, S'_{j,n}), 1 \leq i \leq j \leq k\}$ if $k > 0$ and $f^*(k) = 0$ otherwise. Let $g(k) = g_a(S'_{k,n})$ if $k > 0$ and $g(k) = g_a(0)$ otherwise. Using these notations, (3.38) becomes $\lim_{\delta\to 0} \limsup_{n\to\infty} \delta^{-1}\mathbb{E}\left(f^*([n\delta])g([n\delta])\right) = 0$. We make the decomposition

$$f^*([n\delta])g([n\delta]) = \sum_{k=1}^{[n\delta]} g(k)(f^*(k) - f^*(k-1)) + \sum_{k=1}^{[n\delta]} f^*(k-1)(g(k) - g(k-1))\,. \quad (3.40)$$

To control the first term, note that for positive k,

$$f^*(k) - f^*(k-1) \leq \max_{1 \leq i \leq k} f_a(S'_{i,n}, S'_{k,n})\,. \quad (3.41)$$

Since $f_a(x, y)g_a(y) = 0$, (3.41) implies that the first term on right hand in (3.40) is 0. Hence

$$f^*([n\delta])g([n\delta]) = \sum_{k=1}^{[n\delta]} f^*(k-1)\big(g(k) - g(k-1)\big)\,. \quad (3.42)$$

Let $C = \|g_a''\|_\infty$ and $g'(k) = g_a'(S_{k,n}')$. Applying Taylor's formula, we obtain from (3.42) that

$$f^*([n\delta])g([n\delta]) \leq \sum_{k=1}^{[n\delta]} f^*(k-1)g'(k-1)X_{k,n}' + \frac{C}{2}\sum_{k=1}^{[n\delta]} f^*(k-1)X_{k,n}'^2 \,. \tag{3.43}$$

Now $|f^*(k)g'(k) - f^*(k-1)g'(k-1)| \leq f^*(k-1)|g'(k) - g'(k-1)| + |g'(k)|(f^*(k) - f^*(k-1))$. Since $g_a'(S_{k,n}')f_a(S_{i,n}', S_{k,n}') = 0$, we infer from (3.41) that the second term on right hand is zero. Since furthermore $|g'(k) - g'(k-1)| \leq C|X_{k,n}'|$, we easily obtain that for positive k,

$$|f^*(k)g'(k) - f^*(k-1)g'(k-1)| \leq Cf^*(k-1)|X_{k,n}'| \,. \tag{3.44}$$

From (3.43) and (3.44), we have, for any positive integer N,

$$f^*([n\delta])g([n\delta]) \leq \sum_{k=1}^{[n\delta]} X_{k,n}'f^*(k-N)g'(k-N) + C\sum_{k=1}^{[n\delta]}\sum_{i=(k-N)_++1}^{k} |X_{k,n}'X_{i,n}'|f^*(i-1) \,. \tag{3.45}$$

Let us first study the first term on right hand in Inequality (3.45). Setting $h(k) = f^*(k)g'(k)$, we have $\sum_{k=1}^{[n\delta]} X_{k,n}'h(k-N) = \sum_{i=1}^{[n\delta]}(\sum_{k=N+i}^{[n\delta]} X_{k,n}')(h(i) - h(i-1))$. Taking the conditional expectation with respect to $\mathcal{M}_{i,n}$ and using again (3.44), we obtain

$$\begin{aligned}\Big|\mathbb{E}\Big(\sum_{k=1}^{[n\delta]} X_{k,n}'h(k-N)\Big)\Big| &\leq C\sum_{i=1}^{[n\delta]}\Big\|X_{i,n}'\sum_{k=N+i}^{[n\delta]}\mathbb{E}(X_{k,n}'|\mathcal{M}_{i,n})\Big\|_1 \\ &\leq C\delta \max_{N\leq m\leq[n\delta]} n\Big\|X_{0,n}'\sum_{k=N}^{m}\mathbb{E}(X_{k,n}'|\mathcal{M}_{0,n})\Big\|_1 \,,\end{aligned}$$

the second inequality involving the stationarity of $(X_{i,n}')$. This together with (3.45) yields

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{\delta}\mathbb{E}\left(f^*([n\delta])g([n\delta])\right) \leq CR_1(N,X) + \limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{C}{\delta}\|Q(n,\delta)\|_1 \,, \tag{3.46}$$

where $R_1(N,X)$ is defined in (2.2) and, $Q(n,\delta) = \sum_{k=1}^{[n\delta]}\sum_{i=(k-N)_++1}^{k} |X_{k,n}'X_{i,n}'|f^*(i-1)$. Using the basic inequality $2|X_{k,n}'X_{i,n}'| \leq X_{k,n}'^2 + X_{i,n}'^2$, and the fact that f^* increases, we obtain

$$Q(n,\delta) \leq \frac{1}{2}\sum_{k=1}^{[n\delta]}\sum_{i=(k-N)_++1}^{k} X_{k,n}'^2 f^*(i-1) + \frac{N}{2}\sum_{i=1}^{[n\delta]} X_{i,n}'^2 f^*(i-1) \leq N\sum_{k=1}^{[n\delta]} X_{k,n}'^2 f^*(k-1) \,. \tag{3.47}$$

Set $\overline{S'}_{k,n} = \max\{|S_{1,n}'|, \ldots, |S_{k,n}'|\}$. Since $f^*(k-1) \leq (2\epsilon)^{-1}(\overline{S'}_{k-1,n} \wedge 1)$, Inequality (3.47) becomes, letting $C' = (2\epsilon)^{-1} \vee 1$,

$$Q(n,\delta) \leq NC'\sum_{k=1}^{[n\delta]} X_{k,n}'^2(\overline{S'}_{k-1,n} \wedge 1) \,. \tag{3.48}$$

Since $(X_{i,n})$ is 1-**EQ**, $\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{\delta}\mathbb{E}\left(f^*([n\delta])g([n\delta])\right) \leq CR_1(N,X)$ by (3.46) and (3.48). Since $(X_{i,n})$ is **WD**, the last term is as small as we wish. This completes the proof.

3.7 Proofs of Propositions 3, 4, 5, 6 and of Corollary 6

Proof of Proposition 3. Assume that $R_3(N,X)$ tends to zero as N tends to infinity and that $L_1(N_3)$ holds. Note that $(\overline{S'}_{k,n}\wedge 1)-(\overline{S'}_{k-1,n}\wedge 1)\le(|X'_{k,n}|\wedge 1)$. Hence, for $P\le N_3\vee 1$, we have that $\sum_{k=1}^{[nt]}X'^2_{k,n}(\overline{S'}_{k-1,n}\wedge 1)\le\sum_{k=1}^{[nt]}X'^2_{k,n}(\overline{S'}_{k-P,n}\wedge 1)+\sum_{k=1}^{[nt]}\sum_{i=k-P+1}^{k-1}X'^2_{k,n}(1\wedge|X'_{i,n}|)$. From $L_1(N_3)$, the expectation of the second term on right hand vanishes as n increases. Consequently, it remains to study the first term. Taking the conditional expectation with respect to $\mathcal{M}_{k-P,n}$, we obtain $\sum_{k=1}^{[nt]}\mathbb{E}(X'^2_{k,n}(\overline{S'}_{k-P,n}\wedge 1))\le\mathbb{E}((\overline{S'}_n(t)\wedge 1)\sum_{k=1}^{[nt]}\mathbb{E}(X'^2_{k,n}|\mathcal{M}_{k-P,n}))$. Consequently,

$$\limsup_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\Big(\sum_{k=1}^{[nt]}X'^2_{k,n}(\overline{S'}_{k-1,n}\wedge 1)\Big)\le R_3(P,X)\,.$$

Since $R_3(P,X)$ tends to 0 as P tends to N_3, the result follows.

Proof of Proposition 5. We have to prove that if $(X_{i,n})$ is **WD** and 0-**EQ**, then $\mathbf{S2}(b1)$ holds with $a=0$. From Remark 1 applied to $Z_n(t)=[nt]\mathbb{E}(X_{0,n}|\mathcal{B})$, it is equivalent to prove this with $X'_{i,n}=X_{i,n}-\mathbb{E}(X'_{i,n}|\mathcal{B})$ and $S'_n(t)=\sum_{i=1}^{[nt]}X'_{i,n}$. Write first

$$\mathbb{E}\Big(\frac{S'^2_n(t)}{t}(1\wedge|S'_n(t)|)\Big)\le\mathbb{E}\Big(\frac{S'^2_n(t)}{t}\mathbb{I}_{|S'_n(t)|>2\epsilon}\Big)+\frac{2\epsilon}{t}\mathbb{E}(S'^2_n(t))\le\frac{4}{t}\mathbb{E}(|S'_n(t)|-\epsilon)^2_+ +\frac{2\epsilon}{t}\mathbb{E}(S'^2_n(t)).\tag{3.49}$$

From Lemma 2, $t^{-1}\mathbb{E}(S'^2_n(t))$ is bounded, so that the second term on right hand is a small as we wish. Consequently, we infer from (3.49) that $\mathbf{S2}(b1)$ holds with $a=0$ as soon as,

$$\text{for any positive }\epsilon,\quad\lim_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\left((|S'_n(t)|-\epsilon)^2_+\right)=0\,.\tag{3.50}$$

We shall prove that (3.50) holds with $\overline{S'}_n(t)$ instead of $|S'_n(t)|$. Let $G(t,\epsilon,n)=\{S'^*_n(t)>\epsilon\}$. From Proposition 1(a) in Dedecker and Rio (2000), we have, for any positive integer N,

$$\frac{1}{t}\mathbb{E}\left((S'^*_n(t)-\epsilon)^2_+\right)\le 8\mathbb{E}\Big(\mathbb{I}_{G(t,\epsilon,n)}\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=0}^{N-1}|X'_{k,n}X'_{k+i,n}|\Big)+8n\sup_{N\le m\le[nt]}\Big\|X'_{0,n}\sum_{k=N+1}^{m}\mathbb{E}(X'_{k,n}|\mathcal{M}_{0,n})\Big\|_1.\tag{3.51}$$

Since $(X_{i,n})$ is **WD**, the second term on right hand is as small as we wish by choosing N large enough. To control the first term, note that $G(t,\epsilon,n)\le\epsilon^{-1}(1\wedge\overline{S'}_n(t))$ Since furthermore $2|X'_{k,n}X'_{l,n}|\le X'^2_{k,n}+X'^2_{l,n}$ and $(X_{i,n})$ is 0-**EQ** we infer that, for any positive ϵ, $\lim_{t\to 0}\limsup_{n\to\infty}t^{-1}\mathbb{E}(\mathbb{I}_{G(t,\epsilon,n)}\sum_{k=1}^{[nt]}\sum_{i=0}^{N-1}|X'_{k,n}X'_{k+i,n}|)=0$. Consequently, (3.51) yields

$$\text{for any positive }\epsilon,\quad\lim_{t\to 0}\limsup_{n\to\infty}\frac{1}{t}\mathbb{E}\left((S'^*_n(t)-\epsilon)^2_+\right)=0\,.\tag{3.52}$$

Of course, the same arguments apply to the array $(-X_{i,n})$, so that (3.52) holds for $\overline{S'}_n(t)$. This proves (3.50) and hence $\mathbf{S2}(b1)$. The first item of Proposition 5 follows directly from Lemma 3 by noting that the function $x\to x^2$ belongs to $\mathcal{F}_2$ and that $R_2(N,X)=0$ for any positive integer N (so that $N_0=N_1$). The second item of Proposition 5 follows directly from Proposition 1 by noting that if $k\ge N$, then $(V_{N,n}(k)+X'_{k,n})^2-(V_{N,n}(k))^2=X'^2_{k,n}+2X'_{k,n}(X'_{k-1,n}+\cdots+X'_{k-N+1,n})$.

Proof of Proposition 4. Let $(X_{i,n})$ be a **WD** array such that $n\mathbb{E}(X'^2_{0,n})$ is bounded. Assume that C_1 holds and that $L_1(N)$ is satisfied for some N in $\overline{\mathbb{N}}^*$ such that $R_1(N,X)=0$. For any positive integer P, we have

$$R_1(1,X) \leq \limsup_{n\to\infty} n \sum_{k=1}^{P-1} \|X'_{0,n}X'_{k,n}\|_1 + R_1(P,X)\,. \tag{3.53}$$

Choose a finite integer $P \leq N$ such that $R(P,X) \leq \epsilon$. We have the inequality $n\|X'_{0,n}X'_{k,n}\|_1 \leq n\|X'_{0,n}\|_2\|X'_{k,n}\mathbb{1}_{|X'_{0,n}|>\epsilon}\|_2 + n\|X'_{k,n}\|_2\|X'_{0,n}\mathbb{1}_{|X'_{0,n}|\leq\epsilon}\|_2$. Now, $L_1(N)$ means exactly that for any positive ϵ and any integer $1 \leq k < N$, the sequence $n^{1/2}\|X'_{k,n}\mathbb{1}_{|X'_{0,n}|>\epsilon}\|_2$ tends to zero. Letting first n go to infinity and next ϵ go to zero, Condition C_1 implies that $n^{1/2}\|X'_{0,n}\mathbb{1}_{|X'_{0,n}|\leq\epsilon}\|_2$ vanishes. Since furthermore $n^{1/2}\|X'_{0,n}\|_2 = n^{1/2}\|X'_{k,n}\|_2$ is bounded, we conclude that $n\|X'_{0,n}X'_{k,n}\|_1$ tends to zero. From (3.53) we infer that $R(1,X)=0$, and consequently $N_1(X)=1$.

Proofs of Proposition 6 and Corollary 6. Assume that $(X_{i,n})$ is **WD** with $N_1(X)=1$ and 1-**EQ**. From Corollary 3, $\mathbf{P1}(\gamma,a,\lambda)$ holds if and only if (2.4) holds with $F=\lambda\mathbb{1}_{[a,\infty[}$. Now (2.4) holds for F if and only if C_2 and C_3 holds. This completes the proof of Proposition 6. Since C_2 with $\mathbb{P}(a=0)=0$ implies C_1, Corollary 6 follows from Propositions 4, 6 and 3.

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