Distributing Knowledge into Simple Bases

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Abstract

Understanding the behavior of belief change operators for fragments of classical logic has received increasing interest over the last years. Results in this direction are mainly concerned with adapting representation theorems. However, fragment-driven belief change also leads to novel research questions. In this paper we propose the concept of belief distribution, which can be understood as the reverse task of merging. More specifically, we are interested in the following question: given an arbitrary knowledge base K and some merging operator Δ, can we find a profile E and a constraint μ, both from a given fragment of classical logic, such that $\Delta_\mu(E)$ yields a result equivalent to K? In other words, we are interested in seeing if K can be distributed into knowledge bases of simpler structure, such that the task of merging allows for a reconstruction of the original knowledge. Our initial results show that merging based on drastic distance allows for an easy distribution of knowledge, while the power of distribution for operators based on Hamming distance relies heavily on the fragment of choice.

1 Introduction

Belief change and belief merging have been topics of interest in Artificial Intelligence for three decades [Alchourrón *et al.*, 1985; Katsuno and Mendelzon, 1991; Konieczny and Pino Pérez, 2002]. However, the restriction of such operators to specific fragments of propositional logic has received increasing attention only in the last years [Delgrande *et al.*, 2013; Creignou *et al.*, 2014a; 2014b; Zhuang and Pagnucco, 2012; Zhuang *et al.*, 2013; Zhuang and Pagnucco, 2014; Delgrande and Peppas, 2015; Haret *et al.*, 2015]. Mostly, the question tackled in these works is "How should rationality postulates and change operators be adapted to ensure that the result of belief change belongs to a given fragment?". Surprisingly, the question concerning the extent to which the result of a belief change operation can deviate from the fragment under consideration has been neglected so far. In order to tackle this question, we focus here on a certain form of reverse merging. The question is, given an *arbitrary* knowledge base K and some IC-merging (i.e. merging with integrity constraint, see [Konieczny and Pino Pérez, 2002]) operator Δ, can we find a profile E, *i.e.* a tuple of knowledge bases, and a constraint μ, both from a given *fragment* of classical logic, such that $\Delta_\mu(E)$ yields a result equivalent to K? In other words, we are interested in seeing if K can be distributed into knowledge bases of simpler structure, such that the task of merging allows for a reconstruction of the original knowledge. We call this operation *knowledge distribution*.

Studying the concept of knowledge distribution can be motivated from different points of view. First, consider a scenario where the storage devices have limited expressibility, for instance, databases or logic programs. Our analysis will show which merging operators are required to reconstruct arbitrary knowledge stored in such a set of limited devices. Second, distribution can also be understood as a tool to hide information; only users who know the used merging operator (which thus acts as an encryption key) are able to faithfully retrieve the distributed knowledge. Given the high complexity of belief change (even for revision in "simple" fragments like $Horn$ and $2CNF$ [Eiter and Gottlob, 1992; Liberatore and Schaerf, 2001; Creignou *et al.*, 2013]), brute-force attack to guess the merging operator is unthinkable. Finally, from the theoretical perspective our results shed light on the power of different merging operators when applied to profiles from certain fragments. In particular, our results show that merging $1CNF$ formulas via the Hamming-distance based operator $\Delta^{H,\Sigma}$ does not need additional care, since the result is guaranteed to stay in the fragment.

Related Work. Previous work on merging in fragments of propositional logic proposed an adaptation of existing belief merging operators to ensure that the result of merging belongs to a given fragment [Creignou *et al.*, 2014b], or modified the rationality postulates in order to function in the $Horn$ fragment [Haret *et al.*, 2015]. Our approach is different, since we do not require that the result of merging stays in a given fragment. On the contrary, we want to decompose arbitrary bases into a fragment-profile. Recent work by Liberatore has also addressed a form of meta-reasoning over belief change operators. In [Liberatore, 2015a], the input is a profile of knowledge bases with the expected result of merging R, and the aim is to determine the reliability of the bases (for instance, represented by weights) which allow the obtaining of R. In another paper, Liberatore [2015b] identifies, given a sequence of belief revisions and their results, the initial pre-order which characterizes the revision operator. Finally, even if our approach may seem related to *Knowledge Compilation* (KC) [Darwiche and Marquis, 2002; Fargier and Marquis, 2014; Marquis, 2015], both methods are in fact conceptually different. KC aims at modifying a knowledge base K into a knowledge base K' such that the most important queries for a given application (consistency checking, clausal entailment, model counting, ...) are simpler to solve with K'. Here, we are interested in the extent to which it is possible to *equivalently represent* an arbitrary knowledge base by simpler fragments when using merging as a recovery operation.

Main Contributions. We formally introduce the concept of knowledge distributability, as well as a restricted version of it where the profile is limited to a single knowledge base (simplifiability). We show that for drastic distance arbitrary knowledge can be distributed into bases restricted to mostly any kind of fragment, while simplifiability is limited to trivial cases. On the other hand, for Hamming-distance based merging the picture is more opaque. We show that for *1CNF*, distributability w.r.t. $\Delta^{H,\Sigma}$ is limited to trivial cases, while slightly more can be done with $\Delta^{H,GMin}$ and $\Delta^{H,GMax}$. For *2CNF* we show that arbitrary knowledge can be distributed and even be simplified. Finally, we discuss the *Horn* fragment for which the results for $\Delta^{H,\Sigma}$, $\Delta^{H,GMin}$ and $\Delta^{H,GMax}$ are situated in between the two former fragments.

2 Background

Fragments of Propositional Logic. We consider $\mathcal{L}$ as the language of propositional logic over some fixed alphabet $\mathcal{U}$ of propositional atoms. We use standard connectives $\vee$, $\wedge$, $\neg$, and constants $\top$, $\bot$. A clause is a disjunction of literals. A clause is called *Horn* if at most one of its literals is positive. An interpretation is a set of atoms (those set to true). The set of all interpretations is $2^{\mathcal{U}}$. Models of a formula φ are denoted by $Mod(\varphi)$. A knowledge base (KB) is a finite set of formulas and we identify models of a KB K via $Mod(K) = \bigcap_{\varphi \in K} Mod(\varphi)$. A profile is a finite non-empty tuple of KBs. Two formulae φ_1, φ_2 (resp. KBs K_1, K_2) are equivalent, denoted $\varphi_1 \equiv \varphi_2$ (resp. $K_1 \equiv K_2$), when they have the same set of models.

We use a rather general and abstract notion of fragments.

Definition 1. A mapping $Cl : 2^{2^{\mathcal{U}}} \longrightarrow 2^{2^{\mathcal{U}}}$ is called *closure-operator* if it satisfies the following for any $\mathcal{M}, \mathcal{N} \subseteq 2^{\mathcal{U}}$:

- If $\mathcal{M} \subseteq \mathcal{N}$, then $Cl(\mathcal{M}) \subseteq Cl(\mathcal{N})$
- If $|\mathcal{M}| = 1$, then $Cl(\mathcal{M}) = \mathcal{M}$
- $Cl(\emptyset) = \emptyset$.

Definition 2. $\mathcal{L}' \subseteq \mathcal{L}$ is called a *fragment* if it is closed under conjunction (i.e., $\varphi \wedge \psi \in \mathcal{L}'$ for any $\varphi, \psi \in \mathcal{L}'$), and there exists an associated closure-operator Cl such that (1) for all $\psi \in \mathcal{L}'$, $Mod(\psi) = Cl(Mod(\psi))$ and (2) for all $\mathcal{M} \subseteq 2^{\mathcal{U}}$ there is a $\psi \in \mathcal{L}'$ with $Mod(\psi) = Cl(\mathcal{M})$. We often denote the closure-operator Cl associated to a fragment $\mathcal{L}'$ as $Cl_{\mathcal{L}'}$.

Definition 3. For a fragment $\mathcal{L}'$, we call a finite set $K \subseteq \mathcal{L}'$ an $\mathcal{L}'$-knowledge base. An $\mathcal{L}'$-profile is a profile over $\mathcal{L}'$-knowledge bases. A KB $K \subseteq \mathcal{L}$ is called $\mathcal{L}'$*-expressible* if there exists an $\mathcal{L}'$-KB K', such that $K \equiv K'$.

Many well known fragments of propositional logic are indeed captured by our notion. For the Horn-fragment $\mathcal{L}_{Horn}$, i.e. the set of all conjunctions of Horn clauses over $\mathcal{U}$, take the operator $Cl_{\mathcal{L}_{Horn}}$ defined as the fixed point of the function

$$Cl^1_{\mathcal{L}_{Horn}}(\mathcal{M}) = \{\omega_1 \cap \omega_2 \mid \omega_1, \omega_2 \in \mathcal{M}\}.$$

The fragment $\mathcal{L}_{2CNF}$ which is restricted to formulas over clauses of length at most 2 is linked to the operator $Cl_{\mathcal{L}_{2CNF}}$ defined as the fixed point of the function $Cl^1_{\mathcal{L}_{2CNF}}$ given by

$$Cl^1_{\mathcal{L}_{2CNF}}(\mathcal{M}) = \{\mathrm{maj}_3(\omega_1, \omega_2, \omega_3) \mid \omega_1, \omega_2, \omega_3 \in \mathcal{M}\}.$$

Here, we use the ternary majority function $\mathrm{maj}_3(\omega_1, \omega_2, \omega_3)$ which yields an interpretation containing those atoms which are true in at least two out of $\omega_1, \omega_2, \omega_3$. Finally, we are also interested in the $\mathcal{L}_{1CNF}$ fragment which is just composed of conjunctions of literals; its associated operator $Cl_{\mathcal{L}_{1CNF}}$ is defined as the fixed point of the function

$$\begin{aligned} Cl^1_{\mathcal{L}_{1CNF}}(\mathcal{M}) \quad = \quad & \{\omega_1 \cap \omega_2, \omega_1 \cup \omega_2 \mid \omega_1, \omega_2 \in \mathcal{M}\} \cup \\ & \{\omega_3 \mid \omega_1 \subseteq \omega_3 \subseteq \omega_2; \omega_1, \omega_2 \in \mathcal{M}\}. \end{aligned}$$

Note that full classical logic is given via the identity closure operator $Cl_{\mathcal{L}}(\mathcal{M}) = \mathcal{M}$.

Merging Operators. We focus on *IC-merging*, where a profile is mapped into a KB, such that the result satisfies some integrity constraint. Postulates for IC-merging have been stated in [Konieczny and Pino Pérez, 2002]. We recall a specific family of IC-merging operators, based on distances between interpretations, see also [Konieczny *et al.*, 2004].

Definition 4. A distance between interpretations is a mapping d from two interpretations to a non-negative real number, such that for all $\omega_1, \omega_2, \omega_3 \subseteq \mathcal{U}$, (1) $d(\omega_1, \omega_2) = 0$ iff $\omega_1 = \omega_2$; (2) $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$; and (3) $d(\omega_1, \omega_2) + d(\omega_2, \omega_3) \geq d(\omega_1, \omega_3)$. We will use two specific distances:

drastic distance $D(\omega_1, \omega_2) = 1$ if $\omega_1 = \omega_2$, 0 otherwise;

Hamming distance $H(\omega_1, \omega_2) = |(\omega_1 \setminus \omega_2) \cup (\omega_2 \setminus \omega_1)|$.

We overload the previous notations to define the distance between an interpretation ω and a KB K: if d is a distance between interpretations, then $d(\omega, K) = \min_{\omega' \in Mod(K)} d(\omega, \omega')$. Next, an aggregation function must be used to evaluate the distance between an interpretation and a profile.

Definition 5. An aggregation function $\otimes$ associates a non-negative number to every finite tuple of non-negative numbers, such that (1) if $x \leq y$, then $\otimes(x_1, \ldots, x, \ldots, x_n) \leq \otimes(x_1, \ldots, y, \ldots, x_n)$; (2) $\otimes(x_1, \ldots, x_n) = 0$ iff $x_1 = \cdots = x_n = 0$; (3) for every non-negative number x, $\otimes(x) = x$. As aggregation functions, we will consider the sum Σ, $GMax$ and $GMin$[1], defined as follows. Given a profile $(K_1, \dots, K_n)$, let $V_\omega = (d_1^\omega, \dots, d_n^\omega)$ be the vector of distances such that $d_i^\omega = d(\omega, K_i)$. $GMax(d_1^\omega, \dots, d_n^\omega)$ (resp. $GMin(d_1^\omega, \dots, d_n^\omega)$) is defined by ordering V_ω in decreasing (resp. increasing) order. Given two interpretations ω_1, ω_2, $GMax(d_1^{\omega_1}, \dots, d_n^{\omega_1}) \leq GMax(d_1^{\omega_2}, \dots, d_n^{\omega_2})$ (resp. $GMin(d_1^{\omega_1}, \dots, d_n^{\omega_1}) \leq GMin(d_1^{\omega_2}, \dots, d_n^{\omega_2})$) is defined by comparing them w.r.t. the lexicographic ordering.

Finally, let d be a distance, ω an interpretation and $E = (K_1, \dots, K_n)$ a profile. Then, $d^\otimes(\omega, E) = \otimes(d(\omega, K_1), \dots, d(\omega, K_n))$. If there is no ambiguity about the function $\otimes$, we write $d(\omega, E)$ for $d^\otimes(\omega, E)$.

Definition 6. For any distance d between interpretations, and any aggregation function $\otimes$, the merging operator $\Delta^{d,\otimes}$ is a mapping from a profile E and a formula μ to a KB, such that $Mod(\Delta_\mu^{d,\otimes}(E)) = \min(Mod(\mu), \leq_E^{d,\otimes})$, with $\omega_1 \leq_E^{d,\otimes} \omega_2$ iff $d^\otimes(\omega_1, E) \leq d^\otimes(\omega_2, E)$.

When we consider a profile containing a single knowledge base K, all aggregation functions are equivalent; we write $\Delta_\mu^d(K)$ instead of $\Delta_\mu^{d,\otimes}((K))$ for readability. For drastic distance, $GMin$, $GMax$, and Σ are equivalent for arbitrary profiles. Thus, whenever we show results for $\Delta^{D,\Sigma}$, these carry over to $\Delta^{D,GMin}$ and $\Delta^{D,GMax}$.

3 Main Concepts and General Results

We now give the central definition for a knowledge base being distributable into a profile from a certain fragment with respect to a given merging operator.

Definition 7. Let Δ be a merging operator, $K \subseteq \mathcal{L}$ be an arbitrary KB, and $\mathcal{L}'$ be a fragment. K is called *$\mathcal{L}'$-distributable* w.r.t. Δ if there exists an $\mathcal{L}'$-profile E and a formula $\mu \in \mathcal{L}'$, such that $\Delta_\mu(E) \equiv K$.

Example 1. Let $\mathcal{U} = \{a, b\}$ and consider $K = \{a \vee b\}$ which we want to check for $\mathcal{L}_{Horn}$-distributability w.r.t. operator $\Delta^{H,\Sigma}$. We have $Mod(K) = \{\{a\}, \{b\}, \{a, b\}\}$, thus K is not $\mathcal{L}_{Horn}$-expressible (note that $Cl_{\mathcal{L}_{Horn}}(Mod(K)) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \neq Mod(K)$), otherwise K would be distributable in a simple way (see Proposition 1 below).

Take the $\mathcal{L}_{Horn}$-profile $E = (K_1, K_2)$ with $K_1 = \{a \wedge b\}$, $K_2 = \{\neg a \vee \neg b\}$, together with the empty constraint $\mu = \top$. We have $Mod(K_1) = \{\{a, b\}\}$, $Mod(K_2) = \{\{a\}, \{b\}, \emptyset\}$. In the following matrix, each line corresponds to the distance between a model of μ and a KB from the profile E, or between a model of μ and the profile using the Σ-aggregation over the distances to the single KBs.

	K_1	K_2	Σ
$\{a, b\}$	0	1	1
$\{a\}$	1	0	1
$\{b\}$	1	0	1
$\emptyset$	2	0	2

We observe that $Mod(\Delta_\mu^{H,\Sigma}(E)) = \{\{a\}, \{b\}, \{a, b\}\}$, thus $\Delta_\mu^{H,\Sigma}(E) \equiv K$ as desired. It is easily checked that also other aggregations work: $\Delta_\mu^{H,GMax}(E) \equiv \Delta_\mu^{H,GMin}(E) \equiv K$. $\diamond$

Next, we recall that IC-merging of a single KB yields revision. Thus, the concept we introduce next is also of interest, as it represents a certain form of reverse revision.

Definition 8. Let Δ be a merging operator, $K \subseteq \mathcal{L}$ an arbitrary KB, and $\mathcal{L}'$ a fragment. K is called *$\mathcal{L}'$-simplifiable* w.r.t. Δ if there exists an $\mathcal{L}'$-KB K' and $\mu \in \mathcal{L}'$, such that $\Delta_\mu(K') \equiv K$.

As we will see later, the KB K from Example 1 cannot be $\mathcal{L}_{Horn}$-simplified w.r.t. Δ^H; in other words, we need here at least two KBs to "express" K. However, it is rather straightforward that any $\mathcal{L}'$-expressible KB can be $\mathcal{L}'$-simplified.

Proposition 1. *For every fragment $\mathcal{L}'$ and every KB K, it holds that K is $\mathcal{L}'$-simplifiable (and thus also $\mathcal{L}'$-distributable) w.r.t. Δ, whenever K is $\mathcal{L}'$-expressible.*

Proof. Let K' be an $\mathcal{L}'$-KB equivalent to K, and let $\mu = (\bigwedge_{\varphi \in K'} \varphi)$. Thus, $\mu \in \mathcal{L}'$ by definition of fragments and it is easily verified that $\Delta_\mu(K') \equiv K$. $\square$

Next, we show that in order to determine whether a KB K is $\mathcal{L}'$-distributable, it is sufficient to consider constraints μ such that $Mod(\mu) = Cl_{\mathcal{L}'}(Mod(K))$.

Proposition 2. *Let $K \in \mathcal{L}$ be a KB, $\mathcal{L}'$ be a fragment, E an $\mathcal{L}'$-profile and $\mu \in \mathcal{L}'$. Then $\Delta_\mu(E) \equiv K$ implies $\Delta_{\mu'}(E) \equiv K$ for any μ' such that $Mod(\mu') = Cl_{\mathcal{L}'}(Mod(K))$.*

Proof. Let $\Delta = \Delta^{d,\otimes}$. By Definition 6, $Mod(K) = \min(Mod(\mu), \leq_E^{d,\otimes})$, hence $Mod(K) \subseteq Mod(\mu)$. Moreover, μ is $\mathcal{L}'$-closed, so $Cl_{\mathcal{L}'}(Mod(K)) = Mod(\mu') \subseteq Mod(\mu)$. We get $Mod(K) \subseteq Mod(\mu') \subseteq Mod(\mu)$. Thus, $Mod(K) = \min(Mod(\mu'), \leq_E^{d,\otimes})$, i.e. $\Delta_{\mu'}(E) \equiv K$. $\square$

Next, we give two positive results for distributing knowledge in any fragment. The key idea is to use KBs in the profile which have exactly one model (our notion of fragment guarantees existence of such KBs). The first result is independent of the distance notion but requires $GMin$ as the aggregation function. The second result is for drastic distance and thus works for any of the aggregation functions we consider.

Theorem 3. *Let d be a distance and $\mathcal{L}'$ be a fragment. Then for every KB K, such that for all distinct $\omega_1, \omega_2 \in Mod(K)$, $d(\omega_1, \omega_2) = e$ for some $e > 0$, it holds that K is $\mathcal{L}'$-distributable w.r.t. $\Delta^{d,GMin}$.*

Proof. Build the $\mathcal{L}'$-profile E such that for each $\omega \in Mod(K)$, there is a KB with ω as its only model. Thus all models of K get a $GMin$-vector $(0, e, e, e, e, \dots)$. All interpretations from $Cl_{\mathcal{L}'}(Mod(K)) \setminus Mod(K)$ get a vector $(f, g, \dots)$ with $f > 0$. Hence, we have $\min(Mod(\mu), \leq_E^{d,GMin}) = Mod(K)$ using $\mu \in \mathcal{L}'$ with $Mod(\mu) = Cl_{\mathcal{L}'}(Mod(K))$. $\square$

Theorem 4. *For every fragment $\mathcal{L}'$ and every knowledge base K, it holds that K is $\mathcal{L}'$-distributable w.r.t. $\Delta^{D,\oplus}$, for $\oplus \in \{\Sigma, GMin, GMax\}$.*

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[1] $GMax$ and $GMin$ are known as *leximax* and *leximin* respectively. These functions return a vector of numbers, not a single number. However, $GMax$ (resp. $GMin$) can be associated with an aggregation function as defined in Def. 5 which yields the same vector ordering than $GMax$ (resp. $GMin$). We do a slight abuse by using $GMax$ and $GMin$ as the names of aggregation functions. See [Konieczny *et al.*, 2002].

The proof is similar to the one of Theorem 3 and thus omitted. For simplifiability w.r.t. drastic distance based operators, Proposition 1 cannot be improved, as we show next.

Theorem 5. *For every fragment $\mathcal{L}'$ and every KB K, K is $\mathcal{L}'$-*simplifiable *w.r.t. Δ^D iff K is $\mathcal{L}'$-expressible.*

Proof. The if-direction is by Proposition 1. For the other direction, suppose K is not $\mathcal{L}'$-expressible. We show that for any $\mathcal{L}'$-KB K', $\Delta^D_\mu(K') \not\equiv K$ with $\mu = Cl_{\mathcal{L}'}(K)$. By Proposition 2 the result then follows. Now suppose there exists an $\mathcal{L}'$-KB K' such that $\Delta^D_\mu(K') \equiv K$. First observe that since K is not $\mathcal{L}'$-expressible, $Mod(\mu) \supset Mod(K)$. Since we are working with drastic distance, in order to promote models of K, we also need them in K', hence $Mod(K') \supseteq Mod(K)$ and since K' is from $\mathcal{L}'$ we have $Mod(K') \supseteq Cl_{\mathcal{L}'}(K) = Mod(\mu)$. Thus there exists $\omega \in Cl_{\mathcal{L}'}(Mod(K)) \setminus Mod(K)$ having distance 0 to K', and thus $\omega \in \Delta^D_\mu(K')$. Since $\omega \notin Mod(K)$, this yields a contradiction to $\Delta^D_\mu(K') \equiv K$. □

4 Hamming Distance and Specific Fragments

We first consider the simplest fragment under consideration, namely conjunction of literals. As it turns out, (non-trivial) distributability for this fragment w.r.t. $\Delta^{H,\Sigma}$ is not achievable. We then see that more general fragments allow for non-trivial distributions. In particular, we show that every KB is distributable (and even simplifiable) in the *2CNF* case, and we finally give a few observations for $\mathcal{L}_{Horn}$.

4.1 The 1CNF Fragment

The following technical result is important to prove the main result in this section.

Lemma 6. *For any $\mathcal{L}_{1CNF}$-profile $E = (K_1, \dots, K_n)$ and interpretations ω_1, ω_2, it holds that:*

$$H(\omega_1, E) + H(\omega_2, E) = H(\omega_1 \cap \omega_2, E) + H(\omega_1 \cup \omega_2, E).$$

Proof. It suffices to show that for each K_i in profile E, $H(\omega_1, K_i)+H(\omega_2, K_i) = H(\omega_1\cap\omega_2, K_i)+H(\omega_1\cup\omega_2, K_i)$. Indeed, summing up these equalities over all $K_i \in E$, we get

$$\Sigma_{K_i\in E}H(\omega_1, K_i) + \Sigma_{K_i\in E}H(\omega_2, K_i) = \Sigma_{K_i\in E}H(\omega_1 \cap \omega_2, K_i) + \Sigma_{K_i\in E}H(\omega_1 \cup \omega_2, K_i).$$

Since $H(\omega, E) = \Sigma_{K_i\in E}H(\omega, K_i)$, for any interpretation ω, our conclusion then follows immediately.

Thus, take ω'_1, ω'_2 to be two interpretations that are closest to ω_1 and ω_2, respectively, among the models of $Mod(K_i)$. In other words, $H(\omega_1, \omega'_1) = \min_{\omega\in Mod(K_i)} H(\omega_1, \omega)$ and $H(\omega_2, \omega'_2) = \min_{\omega\in Mod(K_i)} H(\omega_2, \omega)$. By induction on the number of propositional atoms in $\mathcal{L}$, we can show that $\omega'_1\cap\omega'_2$ and $\omega'_1 \cup \omega'_2$ are closest in $Mod(K_i)$ to $\omega_1 \cap \omega_2$ and $\omega_1 \cup \omega_2$, respectively. Thus, we have that $H(\omega_1, K_i) = H(\omega_1, \omega'_1)$, $H(\omega_2, K_i) = H(\omega_2, \omega'_2)$, $H(\omega_1\cap\omega_2, K_i) = H(\omega_1\cap\omega_2, \omega'_1\cap \omega'_2)$, $H(\omega_1 \cup \omega_2, K_i) = H(\omega_1 \cup \omega_2, \omega'_1 \cup \omega'_2)$, and our problem reduces to showing that $H(\omega_1, \omega'_1) + H(\omega_2, \omega'_2) = H(\omega_1\cap\omega_2, \omega'_1\cap\omega'_2)+H(\omega_1\cup\omega_2, \omega'_1\cup\omega'_2)$. By using induction on the number of propositional atoms in $\mathcal{L}$ again, we can show that this equality holds. The argument runs as follows: in the base case, when the alphabet consists of just one propositional atom, the equality is shown to be true by checking all the cases. For the inductive step we assume the claim holds for an alphabet of size n and show that it also holds for an alphabet of size $n+1$. Concretely, we analyze the way in which the Hamming distances between interpretations change when we add a propositional atom to the alphabet. An analysis of all the possible cases shows that the equality holds. □

Next we observe certain patterns of interpretations that indicate whether a KB is $\mathcal{L}_{1CNF}$-expressible or not.

Definition 9. If K is a knowledge base, then a pair of interpretations ω_1 and ω_2 are called *critical with respect to K* if $\omega_1 \not\subseteq \omega_2$ and $\omega_2 \not\subseteq \omega_1$, and one of the following cases holds:

1. $\omega_1, \omega_2 \in Mod(K)$ and $\omega_1 \cap \omega_2, \omega_1 \cup \omega_2 \notin Mod(K)$,
2. $\omega_1, \omega_2, \omega_1 \cap \omega_2 \in Mod(K)$ and $\omega_1 \cup \omega_2 \notin Mod(K)$,
3. $\omega_1, \omega_2, \omega_1 \cup \omega_2 \in Mod(K)$ and $\omega_1 \cap \omega_2 \notin Mod(K)$,
4. $\omega_1 \cap \omega_2, \omega_1 \cup \omega_2 \in Mod(K)$ and $\omega_1, \omega_2 \notin Mod(K)$, or
5. $\omega_1, \omega_1 \cap \omega_2, \omega_1 \cup \omega_2 \in Mod(K)$ and $\omega_2 \notin Mod(K)$.

Lemma 7. *If a KB K is not $\mathcal{L}_{1CNF}$-expressible, then there exist $\omega_1, \omega_2 \in Cl_{\mathcal{L}_{1CNF}}(K)$ being critical with respect to K.*

Proof. The fact that K is not $\mathcal{L}_{1CNF}$-expressible implies that either: (i) K is not closed under intersection or union, or (ii) there are $w_1, w_2, w_3 \in Cl_{\mathcal{L}_{1CNF}}(K)$ such that $w_1 \subseteq w_3 \subseteq w_2$, and $w_1, w_2 \in Mod(K), w_3 \notin Mod(K)$. Case (i) implies that there exist $w_1, w_2 \in Mod(K)$ such that one of Cases 1-3 from Definition 9 holds. If we are in Case (ii), then consider the interpretation $w_4 = (w_2\setminus w_3) \cup w_1$. Clearly, $w_1 \subseteq w_4 \subseteq w_2$, hence $w_4 \in Cl_{\mathcal{L}_{1CNF}}(K)$. Also, $w_3 \cap w_4 = w_1$ and $w_3 \cup w_4 = w_2$. There are two sub-cases to consider here. If $w_4 \notin Mod(K)$, then we are in Case 4 of Definition 9. If $w_4 \in Mod(K)$, then we are in Case 5 of Definition 9. □

We can now state the central result of this section.

Theorem 8. *A KB K is $\mathcal{L}_{1CNF}$-distributable with respect to $\Delta^{H,\Sigma}$ if and only if K is $\mathcal{L}_{1CNF}$-expressible.*

Proof. If part is by Proposition 1. *Only if part:* let K be a KB that is not $\mathcal{L}_{1CNF}$-expressible. We will show that it is not $\mathcal{L}_{1CNF}$-distributable w.r.t. $\Delta^{H,\Sigma}$. Suppose, on the contrary, that K is $\mathcal{L}_{1CNF}$-distributable. Then there exists an $\mathcal{L}_{1CNF}$ profile $E = (K_1, \dots, K_n)$ such that $\Delta^{H,\Sigma}_\mu(E) \equiv K$, where $Mod(\mu) = Cl_{\mathcal{L}_{1CNF}}(Mod(K))$ (cf. Proposition 2).

By Lemma 7, there exist interpretations $\omega_1, \omega_2 \in Mod(\mu)$ that are critical with respect to K. By Lemma 6, we have

$$H(\omega_1, E)+H(\omega_2, E)=H(\omega_1\cap\omega_2, E)+H(\omega_1\cup\omega_2, E). \quad (1)$$

Let us now do a case analysis depending on the type of critical pair we are dealing with. If we are in Case 1 of Definition 9, then it needs to be the case that $H(\omega_1, E) = H(\omega_2, E) = m$, $H(\omega_1 \cap \omega_2, E) = m + k_1$ and $H(\omega_1 \cup \omega_2, E) = m + k_2$, for some integers $m \geq 0$ and $k_1, k_2 > 0$. Plugging these numbers into Equality (1), we get that $2m = 2m + k_1 + k_2$ and $k_1 + k_2 = 0$. Since $k_1, k_2 > 0$, we have arrived at a contradiction. If we are in Case 2, then it needs to be the case that $H(\omega_1 \cap \omega_2, E) = H(\omega_1 \cup \omega_2, E) = m$, $H(\omega_1, E) =$

$m+k_1$ and $H(\omega_2, E) = m+k_2$, for some integers $m \geq 0$ and $k_1, k_2 > 0$. Plugging these numbers into Equality (1) again, we get a contradiction along the same lines as in Case 1. If we are in Case 3, then it needs to hold that $H(\omega_1, E) = H(\omega_1 \cap \omega_2, E) = H(\omega_1 \cup \omega_2, E) = m$, $H(\omega_2, E) = m + k$, for some integers $m \geq 0$ and $k > 0$. Plugging these numbers into Equality (1) gives us $2m + k = 2m$ and hence $k = 0$. Since $k > 0$, we have arrived at a contradiction. Cases 4 and 5 are entirely similar. $\square$

In other words, for any $\mathcal{L}_{1CNF}$-profile and $\mu \in 1CNF$, $\Delta_\mu^{H,\Sigma}$ is guaranteed to be $\mathcal{L}_{1CNF}$-expressible as well. As we have already shown in Theorem 3, this is not necessarily the case if we replace Σ by *GMin*. The following example shows how to obtain a similar behavior for *GMax*; we then generalize this idea below.

Example 2. Let $\mathcal{U} = \{a, b\}$ and $K = \{a \vee b, \neg a \vee \neg b\}$. We have $Mod(K) = \{\{a\}, \{b\}\}$. K is not $\mathcal{L}_{1CNF}$-expressible, since $Cl_{\mathcal{L}_{1CNF}}(Mod(K)) = 2^{\mathcal{U}}$. Let K_S be the $\mathcal{L}_{1CNF}$-KB with a single model S for any $S \subseteq \mathcal{U}$ and let us have a look at the following distance matrix for μ with $Mod(\mu) = Cl_{\mathcal{L}_{1CNF}}(Mod(K))$, $E = (K_{\{a\}}, K_{\{b\}})$, and $E' = (K_\emptyset, K_{\{a,b\}})$.

	$K_\emptyset$	$K_{\{a\}}$	$K_{\{b\}}$	$K_{\{a,b\}}$	$H^{GMin}(E)$	$H^{GMax}(E')$
$\emptyset$	0	1	1	2	$(1,1)$	$(2,0)$
$\{a\}$	1	0	2	1	$(0,2)$	$(1,1)$
$\{b\}$	1	2	0	1	$(0,2)$	$(1,1)$
$\{a,b\}$	2	1	1	0	$(1,1)$	$(2,0)$

The lexicographic order of the involved vectors is $(0,2) < (1,1) < (2,0)$. We thus get that $\Delta_\mu^{H,GMin}(E) \equiv K$ (see also Theorem 3), and on the other hand, $\Delta_\mu^{H,GMax}(E') \equiv K$. $\diamond$

Theorem 9. *Any KB K such that $Mod(K) = \{\omega, \omega'\}$ is $\mathcal{L}_{1CNF}$-distributable with respect to $\Delta^{H,GMax}$.*

Proof. If K is $\mathcal{L}_{1CNF}$-expressible, then the conclusion follows from Proposition 1. If K is not $\mathcal{L}_{1CNF}$-expressible, then consider the set $Cl_{\mathcal{L}_{1CNF}}(Mod(K)) \setminus Mod(K) = \{\omega_1, \ldots, \omega_n\}$. We define the profile $E = (K_1, \ldots, K_n)$, where $Mod(K_i) = \{\mathcal{U} \setminus \omega_i\}$, for $i \in \{1, \ldots, n\}$. We show that $\Delta_\mu^{H,GMax}(E) \equiv K$, where $Mod(\mu) = Cl_{\mathcal{L}_{1CNF}}(Mod(K))$. First, we have that $H(\omega_i, \mathcal{U} \setminus \omega_i) = |\mathcal{U}|$, which implies that $H^{GMax}(\omega_i, E) = GMax(|\mathcal{U}|, \ldots)$, for any $i \in \{1, \ldots, n\}$. Furthermore, since $H(\omega, \mathcal{U} \setminus \omega_i) < |\mathcal{U}|$ and $H(\omega', \mathcal{U} \setminus \omega_i) < |\mathcal{U}|$, for any $i \in \{1, \ldots, n\}$, it follows that $\omega <_E^{H,GMax} \omega_i$ and $\omega' <_E^{H,GMax} \omega_i$. Next, we show that $H^{GMax}(\omega, E) = H^{GMax}(\omega', E)$. Consider the vectors $V = (H(\omega, \omega_1), \ldots, H(\omega, \omega_n))$ and $V' = (H(\omega', \omega_1), \ldots, H(\omega', \omega_n))$. Our claim is that $GMax(V) = GMax(V')$. To see why, notice that the elements in $Cl_{\mathcal{L}_{1CNF}}(Mod(K))$ form a complete subset lattice with $\omega \cup \omega'$ and $\omega \cap \omega'$ as the top and bottom elements, respectively. Let us write $H(\omega, \omega') = m$. This lattice has 2^m elements, and the maximum distance of two elements in it is m. Thus, the vector V is the vector of distances between ω and every other element in this lattice, except itself and ω'. A similar consideration holds for V'. Hence V and V' are vectors of length 2^{m-2} whose elements are $m-1, m-2, \ldots, 1$. We can actually count how many times each number appears in V and V'. The number of interpretations in the lattice that are at distance of 1 from ω (and ω') is $\binom{1}{m}$: thus, $m-1$ appears $\binom{1}{m}$ times in V (and V'). The number of interpretations that are at distance 2 from ω (and ω') is $\binom{2}{m}$, thus $m-2$ appears $\binom{2}{m}$ times in V and V'. We iterate this argument for every distance, up to 1. It is then easy to see that, based on these considerations, V and V' are equal when sorted in descending order. Our conclusion follows from this. $\square$

4.2 The 2CNF Fragment

We show that every knowledge base K can be distributed in the fragment $\mathcal{L}_{2CNF}$. Even a single $\mathcal{L}_{2CNF}$ knowledge base is enough to represent K. Before giving the general result, we sketch the idea via an example.

Example 3. Let K be a KB with $Mod(K) = \{\{a, b\}, \{b, c, e\}, \{a, c, d\}\}$. We observe that K is not $\mathcal{L}_{2CNF}$-expressible since $Cl_{\mathcal{L}_{2CNF}}(Mod(K)) = Mod(K) \cup \{a, b, c\}$. However, we can give an $\mathcal{L}_{2CNF}$-KB K' using three new atoms x, y, z to penalize the undesired interpretation $\{a, b, c\}$ such that $\Delta_\mu^H(K') \equiv K$, with $\mu \in \mathcal{L}_{2CNF}$ of the form $Mod(\mu) = Cl_{\mathcal{L}_{2CNF}}(Mod(K))$. To this end, assume K' with $Mod(K') = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ of the form

$$\begin{aligned} \omega_1 &= \{a, b, x, y\}, & \omega_2 &= \{b, c, e, x, z\}, \\ \omega_3 &= \{a, c, d, y, z\}, & \omega_4 &= \{a, b, c, x, y, z\}. \end{aligned}$$

One can verify that $Cl_{\mathcal{L}_{2CNF}}(Mod(K')) = Mod(K')$. Thus, K' can be picked from $\mathcal{L}_{2CNF}$. We use μ such that $Mod(\mu) = Cl_{\mathcal{L}_{2CNF}}(Mod(K))$ and get distances

	ω_1	ω_2	ω_3	ω_4	min
$\{a,b\}$	2	5	5	4	2
$\{b,c,e\}$	4	2	6	4	2
$\{a,c,d\}$	4	6	2	4	2
$\{a,b,c\}$	3	4	4	3	3

Here, each line gives the distance between a model of μ and a model of K' (ω_i columns), or between a model of μ and K' (min column). The key observation is that pairs from x, y, z as used in ω_1, ω_2, ω_3 give minimal distances 2 while the remaining interpretation ω_4, which corresponds to the closure of K, contains all three new atoms (since $maj_3(\{x, y\}, \{x, z\}, \{y, z\}) = \{x, y, z\}$). $\diamond$

Theorem 10. *Any KB K is $\mathcal{L}_{2CNF}$-simplifiable w.r.t. Δ_μ^H.*

Proof. We have to show that for any KB K, there exists an $\mathcal{L}_{2CNF}$-KB K' and a formula $\mu \in \mathcal{L}_{2CNF}$ such that $\Delta_\mu^H(K') \equiv K$. If K is $\mathcal{L}_{2CNF}$-expressible, the result is due to Proposition 1. So suppose that K is not $\mathcal{L}_{2CNF}$-expressible and let $Mod(K) = \{\omega_1, \ldots, \omega_n\}$. Consider a set of new atoms $A = \{a_1, \ldots, a_n\}$, and for each $\omega_i \in Mod(K)$, let $\omega_i' = \omega_i \cup A \setminus \{a_i\}$. We define the $\mathcal{L}_{2CNF}$-KB K' and $\mu \in \mathcal{L}_{2CNF}$ such that $Mod(K') = Cl_{\mathcal{L}_{2CNF}}(\{\omega_i' \mid \omega_i \in Mod(K)\})$ and $Mod(\mu) = Cl_{\mathcal{L}_{2CNF}}(Mod(K))$. Let $\Omega' = \{\omega_i' \mid \omega_i \in Mod(K)\}$. We first show that for each $\omega \in Mod(K') \setminus \Omega'$, $A \subseteq \omega$. Indeed, for any triple $\omega_j, \omega_k, \omega_l \in Mod(K)$, such that $\omega_{jkl} = \mathrm{maj}_3(\omega_j, \omega_k, \omega_l) \notin Mod(K)$, we observe that $\mathrm{maj}_3(\omega'_j, \omega'_k, \omega'_l) = \omega_{jkl} \cup \mathrm{maj}_3(A \setminus \{a_j\}, A \setminus \{a_k\}, A \setminus \{a_l\}) = \omega_{jkl} \cup A$. Thus, for each $\omega \in Cl^1_{\mathcal{L}_{2CNF}}(\Omega') \setminus \Omega'$, $A \subseteq \omega$. Recall that $Mod(K') = Cl_{\mathcal{L}_{2CNF}}(\Omega')$. It follows quite easily that each further interpretation $\omega \in Cl_{\mathcal{L}_{2CNF}}(\Omega') \setminus (Cl^1_{\mathcal{L}_{2CNF}}(\Omega') \cup \Omega')$, also satisfies $A \subseteq \omega$.

This shows that each model of K' contains at least $n-1$ atoms from A. Thus, for every model $\omega_i \in K$, $H(\omega_i, K') = H(\omega_i, \omega'_i) = n-1$. It remains to show that for each $\omega \in Mod(\mu) \setminus Mod(K)$, $H(\omega, K') \geq n$. First, let $\omega' \in \Omega'$. Since $\omega \notin Mod(K)$, $\omega' \setminus A \neq \omega$ and since ω' contains $n-1$ elements from A, we have $H(\omega, \omega') \geq n$. As shown above all other interpretations $\omega'' \in Mod(K') \setminus \Omega'$ contain all n atoms from A, thus $H(\omega, \omega'') \geq n$, too. □

As an immediate consequence, we obtain that any KB K is $\mathcal{L}_{2CNF}$-distributable w.r.t. $\Delta^{H,\otimes}$ for *any* aggregation function $\otimes$. Note that this result is in strong contrast to the $\mathcal{L}_{1CNF}$ fragment, where only $\mathcal{L}_{1CNF}$-expressible KBs are $\mathcal{L}_{1CNF}$-distributable w.r.t. $\Delta^{H,\Sigma}$.

4.3 The Horn-Fragment

We now turn our attention to the $\mathcal{L}_{Horn}$ fragment. Recall Example 1 where we have shown how to distribute some non $\mathcal{L}_{Horn}$-expressible KB using a profile over two $\mathcal{L}_{Horn}$-KBs. Our first result shows that in this example case we cannot reduce to profiles of a single KB, i.e. that there are KBs which are $\mathcal{L}_{Horn}$-distributable but not $\mathcal{L}_{Horn}$-simplifiable.

Proposition 11. *A KB K with $Mod(K) = \{\omega_1, \omega_2, \omega_3\}$, where $\omega_3 = \omega_1 \cup \omega_2$, $H(\omega_1, \omega_2) = 2$ and ω_1, ω_2 are incomparable, is not $\mathcal{L}_{Horn}$-simplifiable w.r.t. Δ^H.*

Proof. The situation described in the proposition corresponds to a KB $K = \{\omega \cup \{a\}, \omega \cup \{b\}, \omega \cup \{a, b\}\}$ with ω some interpretation which does not contain a or b. We need $Mod(\mu) = \{\omega, \omega \cup \{a\}, \omega \cup \{b\}, \omega \cup \{a, b\}\}$, as required by Proposition 2. We want to identify a $\mathcal{L}_{Horn}$-KB K' such that $\Delta^H_\mu(K') \equiv K$. This means that ω is the single model of μ which is not minimal w.r.t. the Hamming distance. Let ω'_1 be the model in K' closest to $\omega_1 = \omega \cup \{a\}$ and ω'_2 the one closest to $\omega_2 = \omega \cup \{b\}$. We need $a \in \omega'_1$ and $b \in \omega'_2$; otherwise $H(\omega, \omega'_1) < H(\omega_1, \omega'_1)$ or $H(\omega, \omega'_2) < H(\omega_2, \omega'_2)$; further we need $b \notin \omega'_1$ and $a \notin \omega'_2$; otherwise $H(\omega_3, \omega'_1) < H(\omega_1, \omega'_1)$ or $H(\omega_3, \omega'_2) < H(\omega_2, \omega'_2)$. Hence ω'_1 and ω'_2 are incomparable thus also $\omega'_1 \cap \omega'_2 \in Mod(K')$, since K' is a Horn KB. But then $H(\omega, \omega'_1 \cap \omega'_2) \leq H(\omega_1, \omega'_1)$. □

Our last result shows that Δ^H nonetheless increases the range of $\mathcal{L}_{Horn}$-simplifiable KBs compared to Δ^D.

Proposition 12. *Any knowledge base K with $Mod(K) = \{\omega_1, \omega_2\}$ is $\mathcal{L}_{Horn}$-simplifiable w.r.t. Δ^H.*

Proof. If ω_1, ω_2 are comparable, we can apply Proposition 1. Thus, assume ω_1, ω_2 are incomparable and let $d_1 = |\omega_1 \setminus \omega_2|$ and $d_2 = |\omega_2 \setminus \omega_1|$. W.l.o.g. assume $d_1 \leq d_2$. Also note that $d_1 > 0$. We use K' with $Mod(K') = \{\omega_1^+, \omega_1 \cup \omega_2\}$ where ω_1^+ adds d_1 elements from $\omega_2 \setminus \omega_1$ to ω_1. Thus, $\omega_1^+ \subseteq \omega_1 \cup \omega_2$ and we can choose K' from $\mathcal{L}_{Horn}$. Moreover, let $\mu \in \mathcal{L}_{Horn}$ such that $Mod(\mu) = \{\omega_1, \omega_2, \omega_1 \cap \omega_2\}$. We have the following distances (note that $d(\omega_2, \omega_1^+) = d_1 + (d_2 - d_1)$).

	ω_1^+	$\omega_1 \cup \omega_2$	K'
ω_1	d_1	d_2	d_1
ω_2	d_2	d_1	d_1
$\omega_1 \cap \omega_2$	$2d_1$	$d_1 + d_2$	$> d_1$

Hence, $\Delta^H_\mu(K') \equiv K$ as desired. □

	1CNF	*2CNF*	*Horn*
simplifiable w.r.t. Δ^D	×	×	×
simplifiable w.r.t. Δ^H	×	✓	○
distributable w.r.t. $\Delta^{D,\Sigma}$	✓	✓	✓
distributable w.r.t. $\Delta^{H,\Sigma}$	×	✓	–
distributable w.r.t. $\Delta^{H,GMax}$	–	✓	–
distributable w.r.t. $\Delta^{H,GMin}$	–	✓	–

Table 1: Summary of Results

5 Conclusion

In this paper we have proposed the notion of distributability and we have studied the properties of several merging operators with respect to different fragments of propositional logic. Our results are summarized in Table 1.

Symbol × means that only "trivial" KBs (belonging to the considered fragment) can be distributed with the corresponding operator. Alternately, ✓ means that any KB can be distributed. Symbol – means we know that some non-trivial KBs can be distributed, and finally ○ means that some, but not all, non-trivial KBs can be simplified. Interestingly, the picture emerging from Table 1 is that merging operators behave quite differently depending on the distance and aggregation function employed, in a way that does not lend itself to simple categorization. For instance, our results on simplifiability imply that using Dalal revision to $\mathcal{L}_{1CNF}$ KBs never takes us outside the *1CNF* fragment; applying the same revision operator to $\mathcal{L}_{2CNF}$ KBs can produce any KB in $\mathcal{L}$; and applying it to $\mathcal{L}_{Horn}$ KBs can produce some, though not all possible KBs.

Several questions are still open for future work. We plan to study the exact characterization of what can (and cannot) be distributed, in order to replace the symbols – and ○ in the Table 1. Other merging operators can also be integrated to our study. Some of our results on distributability require the addition of new atoms to the interpretations. We want to determine whether similar results can be obtained without modifying the set of propositional variables. We are also interested in the number of KBs needed to distribute knowledge: given an integer n, a KB K and a merging operator Δ, is it possible to distribute K w.r.t. Δ such that the resulting profile contains at most n KBs? This paper was a first step to understand the limits of distributability; the actual construction of the profile and complexity of this process are important questions that will be tackled in future research. Finally, we also consider applying the concept of distributability to non-classical formalisms, in particular in connection with merging operators proposed for logic programs [Delgrande *et al.*, 2013].

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