Rigid and Articulated Motion

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September 25, 2000

Preliminary DRAFT

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2.1 Motion and coordinate frames

In order to study the position of one rigid body with respect to another rigid body we attach coordinate frames to each body. Equivalently, the motion of a rigid body may be studied by attaching a coordinate frame to the initial and final positions of the body. Let M be a point of a rigid body or an object and we attach a frame F to this object. The object moves and let's denote by F_0 the same coordinate frame F in its initial (static) position. Therefore point M has two distinct physical positions, one before the motion, M_0 and one after the motion which is also called its current position. Let $\boldsymbol{x}_0$ be the coordinates of M_0 with respect to F_0 or the coordinates of M with respect to F. The coordinates of M with respect to the fixed frame F_0 are denoted by $\boldsymbol{x}$ and they are given by:

$$\boldsymbol{x} = \mathbf{R}\boldsymbol{x}_0 + \boldsymbol{t} \tag{1}$$

This is illustrated on Figure 1. In three dimensions $\mathbf{R}$ is a 3×3 matrix and $\boldsymbol{t}$ is a 3-vector. The rigidity constraint implies that $\mathbf{R}$ is an orthogonal matrix, $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ and from this equality we immediately find that $\det(\mathbf{R}) = \pm 1$.

Orthogonal matrices with $\det(\mathbf{R}) = 1$ are rotations while those with $\det(\mathbf{R}) = -1$ are reflections. Only rotations are used to define the motion of a rigid body because rotations leave the coordinate frames unchanged.

2.2 The components of motions: rotation and translation

Therefore, the components of rigid motion are a rotation and a translation and they act on the coordinates of a 3-D point as described by the equation above. Motions or displacements of rigid bodies form a group, the SE(3) group and there are various parameterizations of the elements of this group. Representing elements of SE(3) by a rotation matrix and a translation vector is not the most convenient representation and a homogeneous representation is usually being used. Meanwhile it is possible to recall that the unity element is formed by the identity matrix $\mathbf{I}$ and by zero vector $\mathbf{0}$ and the inverse of $\mathbf{R}$ and $\boldsymbol{t}$ is $\mathbf{R}^\top$ and $-\mathbf{R}^\top\boldsymbol{t}$.

Another straightforward formula is the composition of two motions. Indeed, if we combine eq. 1 with another motion of the same frame, say from position F to position F' we obtain:

$$\boldsymbol{x}' = \mathbf{R}'(\mathbf{R}\boldsymbol{x}_0 + \boldsymbol{t}) + \boldsymbol{t}' = \mathbf{R}'\mathbf{R}\boldsymbol{x}_0 + \mathbf{R}'\boldsymbol{t} + \boldsymbol{t}'$$

2.3 Changing coordinate frames

An important aspect of rigid motion representation with coordinate frames is the relationship that exists between various choices of coordinate frames of *the same* rigid

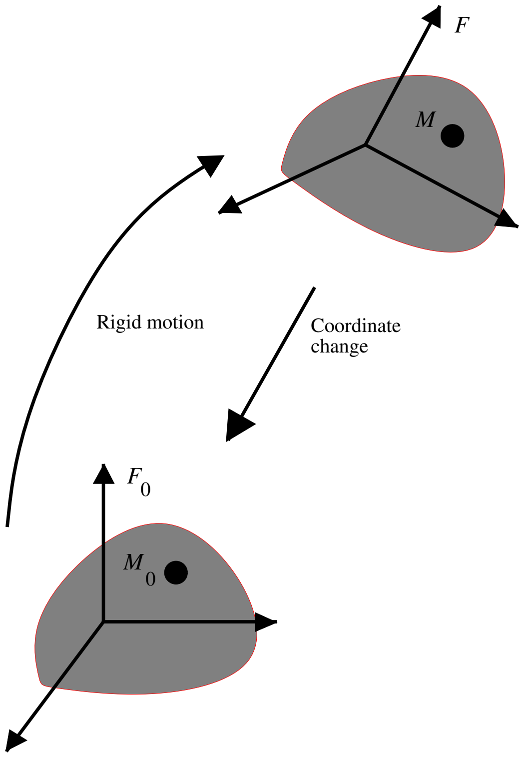

Figure 1: The motion of a rigid object is represented by a rotation and a translation of a coordinate frame.

motion. Let two frames be associated with the same object and let $\mathbf{R}_c$ and $\boldsymbol{t}_c$ be the rotation and translation of one frame with respect to another. Hence, the parameters associated with these matrix and vector are the same before and after a rigid motion. The displacement of the first frame is described by a rotation and a translation indexed by 1 and the displacement of the second frame is described by a rotation and a translation indexed by 2. We have by the formula of motion composition:

$$\mathbf{R}_c(\mathbf{R}_1\boldsymbol{x}_0 + \boldsymbol{t}_1) + \boldsymbol{t}_c = \mathbf{R}_2(\mathbf{R}_c\boldsymbol{x}_0 + \boldsymbol{t}_c) + \boldsymbol{t}_2$$

By separating rotations from translations we obtain:

$$\mathbf{R}_c\mathbf{R}_1 = \mathbf{R}_2\mathbf{R}_c \tag{2}$$

$$\boldsymbol{t}_c + \mathbf{R}_c\boldsymbol{t}_1 = \mathbf{R}_2\boldsymbol{t}_c + \boldsymbol{t}_2 \tag{3}$$

The two displacements, $\mathbf{R}_1$, $\boldsymbol{t}_1$ and $\mathbf{R}_2$, $\boldsymbol{t}_2$ associated with the same motion but with different choices of coordinate frames, Figure 2, are similar or conjugated and this similarity will play a crucial role in the study of motion in non-metric spaces.

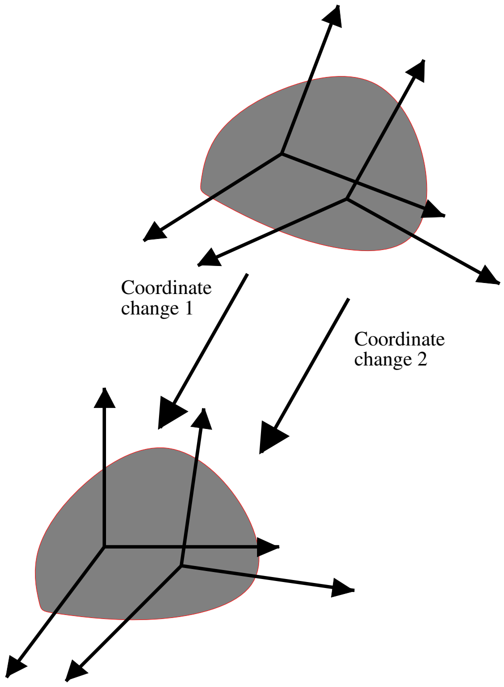

Figure 2: The motion of a rigid object may be represented the displacement of any frame attached to the object.

3 Rotations

3.1 Orthogonal matrices and rotations

Rotations are orthogonal matrices with determinant equal to 1. They form the matrix group denoted $SO(3)$, the special orthogonal group. Several parameterizations of this group were proposed including the roll-pitch-yaw (three rotations around three orthogonal axes) representations, the quaternion representation and so forth. Here we find convenient to perform a change of coordinate frame such that the new frame is aligned with the axis of rotation.

We briefly recall the eigenvalues and eigenvectors of a rotation together with their geometric meaning. The real Jordan form and real Jordan decomposition of a rotation are immediately derived. We also establish the equivalence between the real Jordan decomposition, the Rodrigues formula and the exponential representation.

3.2 Eigenvalues and eigenvectors

A 3×3 rotation matrix has one eigenvalue equal to 1 and two eigenvalues which are complex cojugate:

$$\begin{aligned} \lambda &= 1 \\ \mu &= \cos\theta + i\sin\theta \\ \mu^\star &= \cos\theta - i\sin\theta \end{aligned}$$

From the three eigenvalues we find the trace of a rotation matrix:

$$\mathrm{trace}\,(\mathbf{R}) = 1 + 2\cos\theta \tag{4}$$

The eigenvector associated with λ is the direction in space which remains fixed during rotation and is therefore called the axis of rotation:

$$\mathbf{R}\boldsymbol{n} = \boldsymbol{n} \tag{5}$$

The other two eigenvectors are complex conjugate and let us denote them by $\boldsymbol{u}$ and $\boldsymbol{u}^\star$:

$$\begin{aligned} \mathbf{R}\boldsymbol{u} &= \mu\boldsymbol{u} && (6) \\ \mathbf{R}\boldsymbol{u}^\star &= \mu^\star\boldsymbol{u}^\star && (7) \end{aligned}$$

The eigenvalues and eigenvectors of the transpose (inverse) of a rotation are:

$$\begin{aligned} \mathbf{R}^\top\boldsymbol{n} &= \boldsymbol{n} \\ \mathbf{R}^\top\boldsymbol{u} &= \mu^\star\boldsymbol{u} \\ \mathbf{R}^\top\boldsymbol{u}^\star &= \mu\boldsymbol{u}^\star \end{aligned}$$

It is possible to combine the two complex conjugate eigenvectors into two real vectors:

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{u} + \boldsymbol{u}^\star \\ \boldsymbol{v}^\star &= i(\boldsymbol{u} - \boldsymbol{u}^\star) \end{aligned}$$

3.3 Real Jordan factorization

From the formulae above it is possible to write the action of the rotation matrix $\mathbf{R}$ onto the real vectors $\boldsymbol{v}$ and $\boldsymbol{v}^\star$:

$$\begin{aligned} \mathbf{R}\boldsymbol{v} &= \cos\theta\boldsymbol{v} + \sin\theta\boldsymbol{v}^\star \\ \mathbf{R}\boldsymbol{v}^\star &= -\sin\theta\boldsymbol{v} + \cos\theta\boldsymbol{v}^\star \end{aligned}$$

The rotation matrix can now be written as:

$$\mathbf{R} = \boldsymbol{\Sigma} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\Sigma}^\top = \boldsymbol{\Sigma}\mathbf{J}_\theta\boldsymbol{\Sigma}^\top \tag{8}$$

The 3×3 matrix $\boldsymbol{\Sigma} = (\boldsymbol{v}\ \boldsymbol{v}^\star\ \boldsymbol{n})$ is a rotation. This decomposition is known as the real Jordan factorisation (RJF) and matrix $\mathbf{J}_\theta$ is the real Jordan form of a rotation. There is a simple geometric interpretation of this decomposition which is illustrated on Figure 3. The object undergoes a pure rotation but for convenience we introduce an arbitrary translation along the axis of rotation. The rotation decomposes as a rotation $\boldsymbol{\Sigma}^\top$ followed by a rotation around an axis of angle θ and then again a rotation $\boldsymbol{\Sigma}$. Notice that the only parameter needed to describe this motion is the angle of rotation, the other parameters remain fixed.

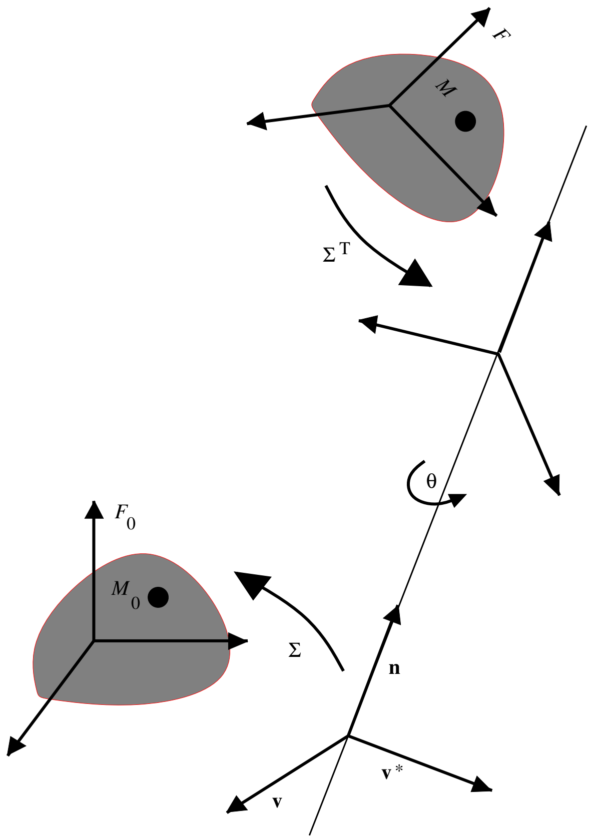

Figure 3: The real Jordan factorization of a rotation where an arbitrary translation along the axis of rotation was introduced artificially for the clarity of the drawing.

3.4 Ambiguity of RJF

The decomposition of a rotation just described is not unique. Indeed, consider all the matrices that commute with the real Joradan form $\mathbf{J}_\theta$:

$$\mathbf{C}\mathbf{J}_\theta = \mathbf{J}_\theta\mathbf{C}$$

The most general form of this commutator is:

$$\mathbf{C} = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

All the real Jordan decompositions of a rotation are:

$$\mathbf{R} = \mathbf{\Sigma}\mathbf{J}_\theta\mathbf{\Sigma}^\top = \mathbf{\Sigma}\mathbf{J}_\theta\mathbf{C}\mathbf{C}^{-1}\mathbf{\Sigma}^\top = (\mathbf{\Sigma}\mathbf{C})\mathbf{J}_\theta(\mathbf{\Sigma}\mathbf{C})^{-1}$$

Matrix $\mathbf{C}$ is orthogonal and its determinant is equal to $c(a^2+b^2)$. One may further constrain this matrix to be a rotation which gives $c(a^2+b^2)=1$.

3.5 Rodrigues formula for 3×3 rotations

In this section we show that the real Jordan decomposition of a rotation matrix as given by eq. (8) is nothing else but the well known Rodrigues formula. We write the real Jordan decomposition as follows:

$$\mathbf{R} = \mathbf{\Sigma}(\mathbf{J}_\theta - \mathbf{I})\mathbf{\Sigma}^\top + \mathbf{I}$$

By substituting $\mathbf{J}_\theta$ with its expression and with $\mathbf{\Sigma} = (\boldsymbol{v}\ \boldsymbol{v}^\star\ \boldsymbol{n})$ we obtain:

$$\mathbf{R} = (\cos\theta - 1)(\boldsymbol{v}\boldsymbol{v}^\top + \boldsymbol{v}^\star\boldsymbol{v}^{\star\top}) + \sin\theta(\boldsymbol{v}^\star\boldsymbol{v}^\top - \boldsymbol{v}\boldsymbol{v}^{\star\top}) + \mathbf{I}$$

Matrix $\mathbf{\Sigma}$ being orthogonal we have $\boldsymbol{v}\times\boldsymbol{v}^\star = \boldsymbol{n}$ and we obtain:

$$\boldsymbol{v}^\star\boldsymbol{v}^\top - \boldsymbol{v}\boldsymbol{v}^{\star\top} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = \mathbf{S}(\boldsymbol{n})$$

and:

$$\boldsymbol{v}\boldsymbol{v}^\top + \boldsymbol{v}^\star\boldsymbol{v}^{\star\top} = -\mathbf{S}(\boldsymbol{n})^2$$

We obtain the Rodriques formula for rotations:

$$\mathbf{R} = \mathbf{I} + \sin\theta\mathbf{S}(\boldsymbol{n}) + (1-\cos\theta)\mathbf{S}(\boldsymbol{n})^2 \tag{9}$$

The skew-symmetric matrix $\mathbf{S}(\boldsymbol{n})$ is also called the *tangent operator* associated with SO(3), i.e., so(3) and will be denoted by:

$$\widehat{\mathbf{R}} = \mathbf{S}(\boldsymbol{n}) \tag{10}$$

With this notation, the Rodrigues formula will be written as:

$$\mathbf{R} = \mathbf{I} + \sin\theta \widehat{\mathbf{R}} + (1 - \cos\theta)\widehat{\mathbf{R}}^2$$

The advantage of the Rodrigues formula over the real Jordan factorization is that the former is unique while the latter is not. To be convinced, notice that we have:

$$\mathbf{S}(\boldsymbol{n}) = \boldsymbol{\Sigma} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\Sigma}^\top = \widehat{\mathbf{R}}$$

Matrix $\boldsymbol{\Sigma}$ may be replaced by $\boldsymbol{\Sigma}\mathbf{C}$ without modifying the expression of $\mathbf{S}(\boldsymbol{n})$.

3.6 Exponential representation

The Rodrigues formula is simply the matrix exponential:

$$\exp(\widehat{\mathbf{R}}\theta) = \mathbf{I} + \widehat{\mathbf{R}}\theta + \frac{(\widehat{\mathbf{R}}\theta)^2}{2!} + \frac{(\widehat{\mathbf{R}}\theta)^3}{3!} + \dots$$

Indeed, it is straightforward to verify that we have:

$$\begin{array}{rcl} \widehat{\mathbf{R}}^{2n+1} & = & (-1)^n \widehat{\mathbf{R}} \\ \widehat{\mathbf{R}}^{2n+2} & = & (-1)^n \widehat{\mathbf{R}}^2 \end{array}$$

The matrix exponential becomes:

$$\exp(\widehat{\mathbf{R}}\theta) = \mathbf{I} + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin\theta} \widehat{\mathbf{R}} + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right)}_{1-\cos\theta} \widehat{\mathbf{R}}^2$$

and hence we obtain:

$$\mathbf{R} = \exp(\widehat{\mathbf{R}}\theta) = \mathbf{I} + \sin\theta \widehat{\mathbf{R}} + (1 - \cos\theta)\widehat{\mathbf{R}}^2 \tag{11}$$

Therefore, a rotation matrix can be described by its angle of rotation and its axis of rotation $\boldsymbol{n}$ or, equivalently, its *tangent operator* $\widehat{\mathbf{R}}$. The formulae for extracting these elements from a rotation matrix are very simple:

$$\theta = \arccos\left(\frac{\text{trace}\,(\mathbf{R}) - 1}{2}\right) \tag{12}$$

and:

$$\widehat{\mathbf{R}} = \frac{1}{2\sin\theta}\left(\mathbf{R} - \mathbf{R}^\top\right) \tag{13}$$

4 Homogeneous (projective) coordinates

So far we represented points with three-dimensional vectors. We introduced displacements as an action onto these vectors. Displacements were rerpresented by a rotation matrix and a translation vector. Therefore the action of displacement consists of left multiplication of a vector by a matrix followed by the summation of two vectors. This representation is not well suited if one wants to study motion in a more general framework and, in particular, when motion is observed with one or several cameras.

Homogeneous or projective coordinates allow the representation of the 3-D vector space as a subspace of the 3-D projective space, $\mathcal{P}^3$. Consequently actions onto elements of $\mathcal{P}^3$ will belong to the group of projective transformations. Elements of $\mathcal{P}^3$ are 4-vectors defined up to a scale factor. Therefore projective transformations are represented by 4×4 regular matrices. Displacements form a subgroup of the group of projective transformations and they are represented by 4×4 matrices with a special structure.

The homogeneous (projective) representation makes possible the algebraic manipulation of points at infinity just as any other points. It also makes easy the use of point-hyperplane duality which in the case of $\mathcal{P}^3$ is the point-plane duality.

Intuitively, consider a sphere of radius r whose center is the origin $(0,0,0)$. A point onto this sphere has coordinates (a,b,c) with the constraint $a^2+b^2+c^2=r^2$. Let the radius grow to infinity. At the limit the sphere is infinitely large and its curvature becomes null. Therefore, there is no difference between an infinitely large sphere and a plane. This plane is called the plane at infinity π_∞ and one issue is the algebraic representation of points belonging to this plane.

The point $(\lambda a, \lambda b, \lambda c)$ lies along a line passing through the origin and through the point (a,b,c). At the limit, we have $\lambda=\infty$ and we obtain the point (∞,∞,∞). This representation is not very useful because all the points lying on π_∞ will have the same coordinates.

In order to be able to represent points at infinity let us introduce the four-vector (a,b,c,d) and let us define the homogeneous or projective equality in the following manner. $(a,b,c,d) \simeq (a',b',c',d')$ if and only if there exists a non null real number s such that $(a,b,c,d) = s(a',b',c',d')$. Therefore (a,b,c,d) is projectively equal to $(\frac{a}{d},\frac{b}{d},\frac{c}{d},1)$. A point at infinity may therefore be characterized by $d\rightarrow 0$ or:

$$\lim_{d\rightarrow 0}\left(\frac{a}{d},\frac{b}{d},\frac{c}{d},1\right) \simeq \lim_{d\rightarrow 0}(a,b,c,d) = (a,b,c,0)$$

Therefore the point $(a,b,c,0)$ lies on a sphere of infinite radius or, equivalently, the plane at infinity. Notice that two diametrically opposed points onto this sphere have the same projective coordinates. The point (a,b,c) simply becomes $(a,b,c,1)$ or more generally $(\lambda a,\lambda b,\lambda c,\lambda)$.

More formal definitions of projective space and projective coordinates are to be found in various textbooks such as [3, 1].

We consider again eq. (1) which describes a rigid displacement. If we add a fourth component to 3-vectors $\boldsymbol{x}$ and $\boldsymbol{x}_0$ we obtain:

$$\begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \simeq \begin{bmatrix} \mathbf{R} & \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_0 \\ 1 \end{pmatrix} \tag{14}$$

This formula is strictly equivalent with eq. (1). Notice that we have the following equality:

$$\begin{pmatrix} \boldsymbol{x}' \\ 0 \end{pmatrix} \simeq \begin{bmatrix} \mathbf{R} & \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_0 \\ 0 \end{pmatrix}$$

which means that points at infinity remain at infinity after a displacement. This transformation is illustrated on Figure 4. In fact the equation above is equivalent to $\boldsymbol{x}' = \mathbf{R}\boldsymbol{x}_0$ and it illustrates the fact that at each direction $\boldsymbol{x}'$ corresponds a point at infinity $(\boldsymbol{x}'\ 0)^\top$. The fact that displacements keep the infinity at infinity is equivalent with the fact that parallelism is invariant under rigid motion. This property is linked with the structure of the fourth row of a displacement matrix $\mathbf{T}$:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix} \tag{15}$$

We consider now the general form of a 4×4 full rank matrix and we call it $\mathbf{H}$. The difference between $\mathbf{T}$ and $\mathbf{H}$ is that the last row of $\mathbf{H}$ has no particular structure:

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \tag{16}$$

Clearly, left multiplication of a point at infinity with this matrix gives, in the general case, a point which is not at infinity any more:

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \simeq \mathbf{H} \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

and $t' \neq 0$ in the general case. This is illustrated on Figure 5. General projective transformations do not preserve parallelism and therefore they do not preserve metric structure. They define a non-metric geometry. Since projective transformations are linear, coplanar points remain coplanar. Therefore, the plane at infinity is transformed into another plane. Under metric transformations this plane is easy to identify. Under projective transformations this plane has no special algebraic structure and one difficulty associated with projective space is to identify this plane. Methods for indentifying the plane at infinity have been thorougly studied in the computer vision literature.

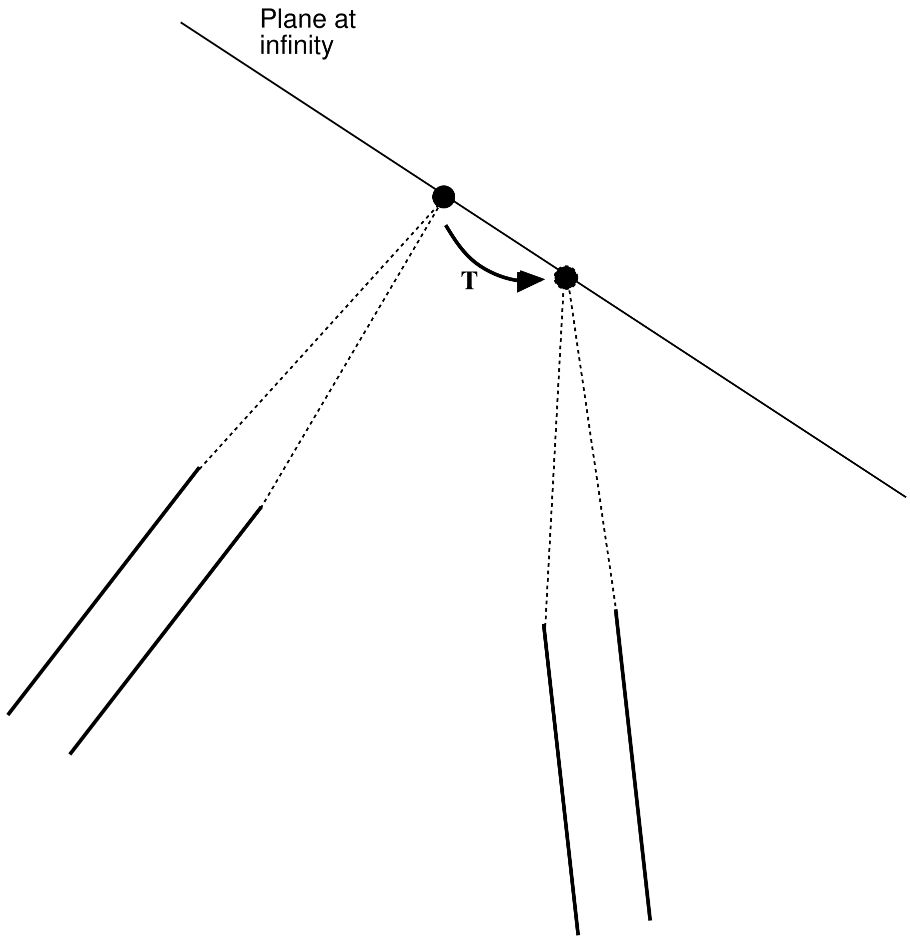

Figure 4: A point at infinity may well be viewed as the point of intersection of two parallel lines. Therefore, a rigid displacement moves a point at infinity onto another point at infinity thus preserving parallelism.

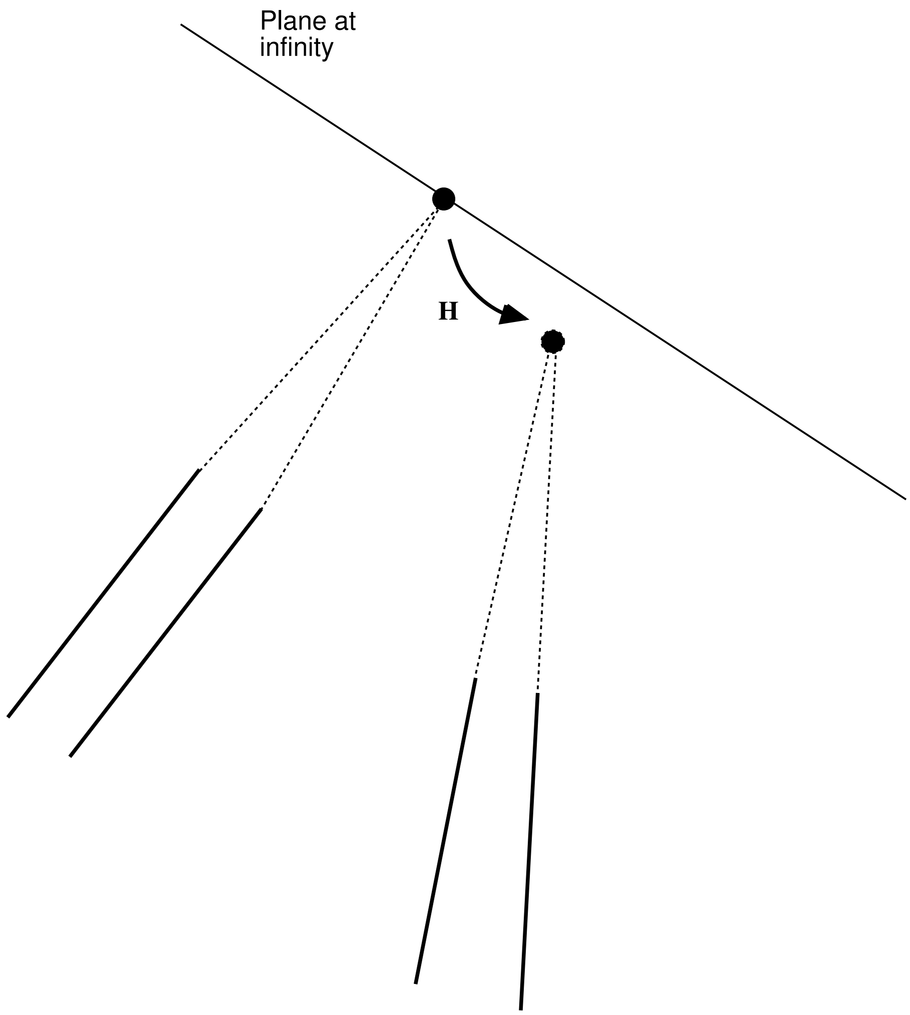

Figure 5: A projective transformation takes a point at infinity into a point which is not at infinity anymore. Parallelism is not preserved anymore.

5 Displacements

The 4×4 matrices $\mathbf{T}$ of the form of eq. (15) describe rigid motion, or displacements, or Euclidean transformations of the 3-D space. They have a group structure and this group is called $SE(3)$, the special Eucldidean group. $\mathbf{T}$ acts on 4-vectors which will be designated by upper slanted bold characters $\boldsymbol{X}, \boldsymbol{X}_0$, and so forth with $\boldsymbol{X}^\top = (X, Y, Z, W)$. If we have $W \neq 0$ we obtain the Euclidean coordinates of the corresponding point $(X/W, Y/W, Z/W, 1)$. If $T = 0$ the point is at infinity and it corresponds to the direction vector $(X, Y, Z, 0)$.

The inverse of a displacement matrix is:

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix}$$

Using this homogeneous representation of displacements, the composition of two motions and the change of coordinate frames for a motion are straightforward. The composition of two motions $\mathbf{T}$ followed by $\mathbf{T}'$ is simply $\mathbf{T}'\mathbf{T}$.

A change in coordinate frames of a motion, i.e., equations (2) and (3) become:

$$\mathbf{T}_2 = \mathbf{T}_c \mathbf{T}_1 \mathbf{T}_c^{-1} \tag{17}$$

In projective space there is no formal difference between points and hyperplanes – this is the duality principle. Let point $\boldsymbol{X}$ belong to plane $\boldsymbol{p}$. This writes as $\boldsymbol{p}^\top \boldsymbol{X} = 0$. We wish that this property remains valid after a transformation $\mathbf{T}$: $\boldsymbol{p}'^\top \boldsymbol{X}' = 0$ or $\boldsymbol{p}'^\top \mathbf{T} \boldsymbol{X} = 0$. Therefore $\boldsymbol{p}^\top = \boldsymbol{p}'^\top \mathbf{T}$ and:

$$\boldsymbol{p}' = \mathbf{T}^{-\top} \boldsymbol{p}$$

with:

$$\mathbf{T}^{-\top} = \begin{bmatrix} & 0 \\ \mathbf{R} & 0 \\ & 0 \\ -\boldsymbol{t}^\top \mathbf{R} & 1 \end{bmatrix} \tag{18}$$

5.1 Eigenvalues and eigenvectors

Using the structure of a displacement matrix and from the eigenvalues of a rotation matrix it is straightforward to compute the eigenvalues of $\mathbf{T}$ which are the **double** eigenvalue $\lambda = 1$, μ, and $\mu^\star$ already defined in §3.2. We have as well:

$$\text{trace}\,(\mathbf{T}) = 2(1 + \cos\theta)$$

$$\det(\mathbf{T}) = 1$$

We determine first the eigenvector associated with $\lambda = 1$, i.e., $\mathbf{T}\boldsymbol{X} = \boldsymbol{X}$ and we obtain the following equation:

$$(\mathbf{R} - \mathbf{I})\boldsymbol{x} + W\boldsymbol{t} = 0$$

We use the same notations as in §3.2. Matrix $\mathbf{R} - \mathbf{I}$ is of rank 2 and therefore the 3-vector $\boldsymbol{x}$ must belong to the null space of this matrix. Consequently there are two solutions for $\boldsymbol{x}$: the null vector $(0,0,0)$ or the eigenvector of $\mathbf{R}$, $\boldsymbol{n}$, which is the axis of rotations. We must consider three cases:

1. *Pure rotations.* In this case we have $\boldsymbol{t} = 0$ and there are two possible eigenvectors:

$$\boldsymbol{N} = \begin{pmatrix} \boldsymbol{n} \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   The eigen space associated with $\lambda = 1$ is therefore spanned by these two points. The former is the intersection of the rotation axis with the plane at infinity and the latter is the origin. The point:

$$\boldsymbol{X} = \boldsymbol{N} + W\boldsymbol{O}$$

   belongs to a projective line going through the origin and which can be metrically interpreted as the axis of rotation. In other words, under pure rotation every point along the axis of rotation is a fixed point.

2. *General motions.* These motions are also called screws and they are characterized by a non-null rotation and non-null translation. The same analysis as above leads to two solutions: $\boldsymbol{N}$ and the point $(0,0,0,0)$. However the latter is not an element of projective space and therefore in this case there is only one eigenvector, $\boldsymbol{N}$.

3. *Pure translations.* In this case eigenvalue $\lambda = 1$ has algebraic multiplicity equal to 4. The equation to be solved in this case is $T\boldsymbol{t} = 0$ which yields $T = 0$. The eigenspace is:

$$\boldsymbol{X} = \begin{pmatrix} X \\ Y \\ Z \\ 0 \end{pmatrix} = X \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + Y \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Z \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

   These points form the plane at infinity and they are the fixed points of a pure translational motion.

The following table summarizes the eigvenvector analysis of displacements. The two other eigenvalues are complex conjuguate, i.e., §3.2. The associated eigenvectors are:

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{u} \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{U}^\star = \begin{pmatrix} \boldsymbol{u}^\star \\ 0 \end{pmatrix}$$

$\lambda = 1$	General motion	Rotation	Translation
Algebraic multiplicity	2	2	4
Geometric multiplicity	1	2	3

Table 1: The algebraic and geometric multiplicities of the eigenvalue 1 depends on the motion type.

They belong to the absolute conic which is a curve lying onto the plane at infinity. The absolute conic has as equation $X^2 + Y^2 + Z^2 = 0$ and hence it is an imaginary curve. This curve remains fixed under rigid motion and each motion has two fixed points onto this curve. From this pair of complex conjugate vectors it is possible to obtain the following real vectors which will be useful to obtain the real Jordan factorization of $\mathbf{T}$:

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{u} + \boldsymbol{u}^\star \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{V}^\star = \begin{pmatrix} i(\boldsymbol{u} - \boldsymbol{u}^\star) \\ 0 \end{pmatrix}$$

Figure 6: The fixed points under general motion, under rotation (the axis of rotation) and under translation (the plane at infinity).

6 Real Jordan factorization of displacements

It is now possible to compute the real Jordan decomposition of a displacement. Generally speaking, this decomposition corresponds to a coordinate transformation and we will use the following notation:

$$\mathbf{T} = \mathbf{U}\mathbf{J}_T\mathbf{U}^{-1} \tag{19}$$

with [2]:

$$\mathbf{J}_T = \begin{bmatrix} & & & 0 \\ & \mathbf{J}_\theta & & 0 \\ & & & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix $\mathbf{U}$ represents a displacement as well. In practice we have to distinguish between motions with rotation and motions without rotation (pure translations). The former farther decomposes into general motions and planar motions.

The real Jordan forms will be denoted $\mathbf{J}_T$ in the general case, $\mathbf{J}_R$ for pure rotations and planar motions, and $\mathbf{J}_t$ for pure translations. Displacement $\mathbf{U}$ is not unique. Indeed consider all the matrices $\mathbf{C}$ commuting with the real Jordan form. If $\mathbf{U}\mathbf{J}_T\mathbf{U}^{-1}$ is a Jordan factorization, then $(\mathbf{U}\mathbf{C})\mathbf{J}_T(\mathbf{U}\mathbf{C})^{-1}$ is an admissible factorization as well. The ambiguities of the Jordan factorization will be made explicit.

6.1 Screw, rotational, and planar motions

Using the real Jordan decomposition of a rotation, eq. (8) we are left with the translational part of the displacement:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\sigma} \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & \mathbf{J}_\theta & & 0 \\ & & & s \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\sigma} \\ 0\,0\,0 & 1 \end{bmatrix}^{-1}$$

Or:

$$\mathbf{R} = \boldsymbol{\Sigma}\mathbf{J}_\theta\boldsymbol{\Sigma}^\top$$

and

$$\mathbf{R}\boldsymbol{\sigma} + \boldsymbol{t} = s\boldsymbol{n} + \boldsymbol{\sigma} \tag{20}$$

where $\boldsymbol{n}$ is the third column of $\boldsymbol{\Sigma}$, i.e., the axis of rotation. s is the amount of slide along the axis of rotation and is easily obtained from the previous equation (left multiply by $\boldsymbol{n}^\top$ and notice that $\boldsymbol{n}^\top\mathbf{R} = \boldsymbol{n}^\top$:

$$s = \boldsymbol{n}^\top\boldsymbol{t}$$

Therefore, there will be two different real Jordan decomposition, $s = 0$ for pure rotations or planar motions and $s \neq 0$ for the general case.

We establish the decomposition for each special case and for the general case:

6.1.1 Pure rotations

In this case we have $s = 0$ and $\boldsymbol{t} = 0$. eq. (20) above becomes:

$$(\mathbf{R} - \mathbf{I})\,\boldsymbol{\sigma} = \mathbf{0}$$

The solution is $\boldsymbol{\sigma} = \alpha\boldsymbol{n}$. Moreover, the translation must be choosen such that the origin of the new frame is somewhere along the axis of rotation. Obviously, one may take $\alpha = 0$ and we obtain the following decomposition:

$$\mathbf{T}_R = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} & 0 \\ \mathbf{J}_\theta & 0 \\ & 0 \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}^\top & \mathbf{0} \\ 0\,0\,0 & 1 \end{bmatrix} \tag{21}$$

6.1.2 Planar motions

In this case we have $s = 0$ and $\boldsymbol{n} \perp \boldsymbol{t}$. The eq. (20) above becomes:

$$(\mathbf{R} - \mathbf{I})\,\boldsymbol{\sigma} = -\boldsymbol{t}$$

Let's write vector $\boldsymbol{\sigma}$ as a linear combination of $\boldsymbol{n}$ and of a vector perpendicular to $\boldsymbol{n}$:

$$\boldsymbol{\sigma} = \alpha\boldsymbol{n} + \boldsymbol{\sigma}_\perp$$

We obtain:

$$(\mathbf{R} - \mathbf{I})\,\boldsymbol{\sigma} = (\mathbf{R} - \mathbf{I})\,\boldsymbol{\sigma}_\perp = -\boldsymbol{t}$$

One may choose $\alpha = 0$ and hence the translation vector $\boldsymbol{\sigma}_\perp$ absorbs the effect of $\boldsymbol{t}$. In practice, one has to solve the following set of linear equations:

$$\left\{ \begin{array}{rcr} (\mathbf{R} - \mathbf{I})\,\boldsymbol{\sigma}_\perp & = & -\boldsymbol{t} \\ \boldsymbol{n}^\top\boldsymbol{\sigma}_\perp & = & 0 \end{array} \right.$$

The Jordan decomposition becomes:

$$\mathbf{T}_\perp = \begin{bmatrix} \mathbf{R} & \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\sigma}_\perp \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} & 0 \\ \mathbf{J}_\theta & 0 \\ & 0 \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}^\top & -\boldsymbol{\Sigma}^\top\boldsymbol{\sigma}_\perp \\ 0\,0\,0 & 1 \end{bmatrix} \tag{22}$$

6.1.3 Screw motions

. The translation vector $\boldsymbol{\sigma}$ is given by:

$$(\mathbf{R} - \mathbf{I})\,\boldsymbol{\sigma} = (\boldsymbol{n} \cdot \boldsymbol{t})\boldsymbol{n} - \boldsymbol{t}$$

We notice that vector $(\boldsymbol{n}\cdot\boldsymbol{t})\boldsymbol{n}-\boldsymbol{t}$ lies in a plane perpendicular to $\boldsymbol{n}$. Therefore we obtain a solution by solving a set of linear equations similar to the planar motion case. the RJF becomes:

$$\mathbf{T}=\begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\sigma}_\perp \\ 0\,0\,0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{J}_\theta & \begin{matrix}0\\0\\s\end{matrix} \\ 0\,0\,0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}^\top & -\boldsymbol{\Sigma}^\top\boldsymbol{\sigma}_\perp \\ 0\,0\,0 & 1 \end{bmatrix} \tag{23}$$

This is illustrated on Figure 7.

6.1.4 Pure translations

The real Jordan decomposition of a pure translation has to be treated differently. Indeed, the decomposition writes:

$$\mathbf{T}_t=\begin{bmatrix} \mathbf{I} & \boldsymbol{t} \\ 0\,0\,0 & 1 \end{bmatrix}=\begin{bmatrix} \mathbf{Q} & \boldsymbol{\sigma} \\ 0\,0\,0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I} & \begin{matrix}0\\0\\s\end{matrix} \\ 0\,0\,0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{Q} & \boldsymbol{\sigma} \\ 0\,0\,0 & 1 \end{bmatrix}^{-1}$$

with the notation $\mathbf{Q}=(\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3)$ we obtain the following constraint:

$$\boldsymbol{\sigma}+\boldsymbol{t}=s\boldsymbol{q}_3+\boldsymbol{\sigma}$$

which yield the following solution:

$$s=||\boldsymbol{t}||\ ,\ \boldsymbol{q}_3=\frac{\boldsymbol{t}}{||\boldsymbol{t}||}\ ,\ \boldsymbol{q}_1\perp\boldsymbol{t}\ ,\ \boldsymbol{q}_2\perp\boldsymbol{t}$$

The decompositon of a pure translation is:

$$\mathbf{T}_t=\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \frac{\boldsymbol{t}}{||\boldsymbol{t}||} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I} & \begin{matrix}0\\0\\||\boldsymbol{t}||\end{matrix} \\ 0\,0\,0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \frac{\boldsymbol{t}}{||\boldsymbol{t}||} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \tag{24}$$

Finally, all possible real Jordan forms of a displacement are summarized on Table 6.1.4

6.2 Ambiguity of real Jordan forms

From the above computations it is clear that the real Jordan decomposition of displacements is not unique. Indeed, eq. (19) can be written as:

$$\mathbf{T}=\mathbf{U}\mathbf{J}\mathbf{C}_{(.)}\mathbf{C}_{(.)}^{-1}\mathbf{U}^{-1}=(\mathbf{U}\mathbf{C}_{(.)})\ \mathbf{J}\ (\mathbf{U}\mathbf{C}_{(.)})^{-1}$$

if and only if $\mathbf{C}_{(.)}$ commutes with $\mathbf{J}$. In §3.4 we characterized the ambiguity of the real Jordan decomposition of a rotation. Using the same notations we obtain:

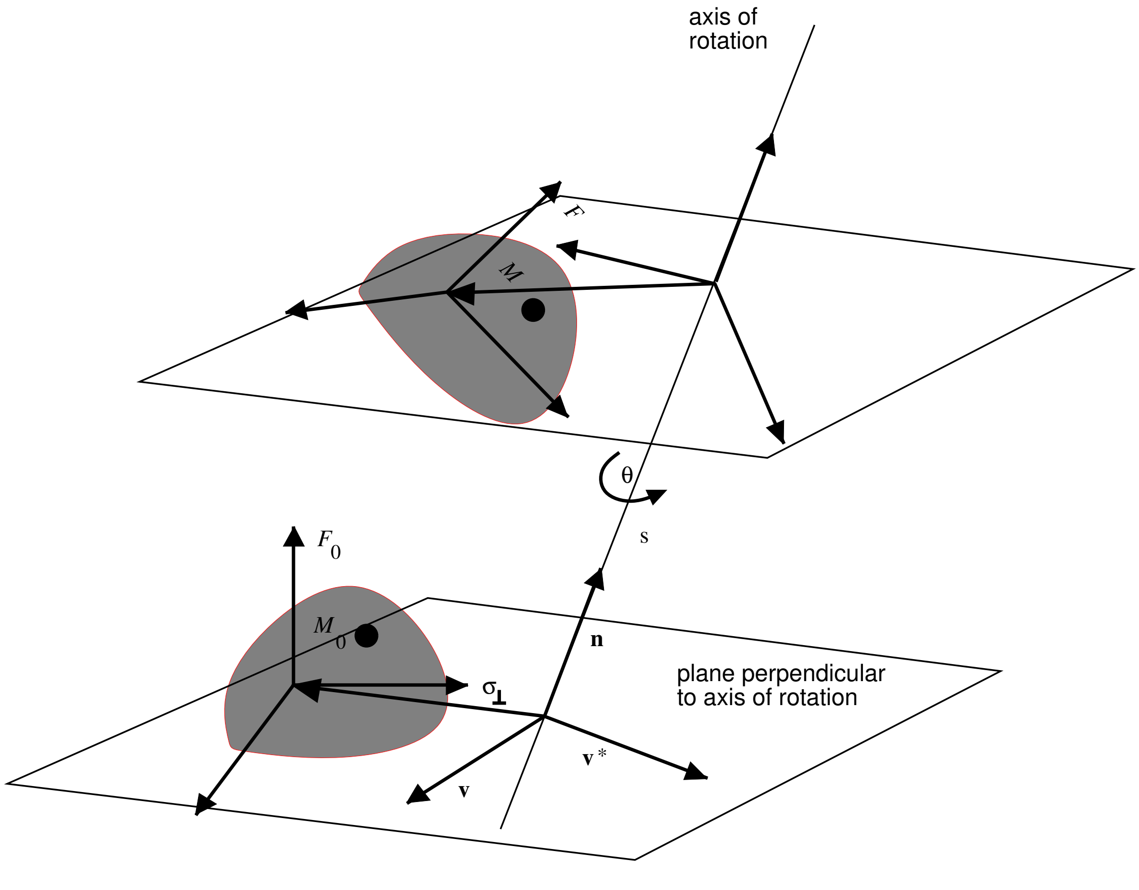

Figure 7: The real Jordan factorization of a displacement.

Screw motion	Planar/rotation	Translation
$\mathbf{J}_T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\mathbf{J}_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\mathbf{J}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Table 2: This table sumarizes all possible real Jordan forms of a displacement. Notice that the RJF of a planar motion cannot be distiguished from the RJF of a pure rotation.

1. For a *general motion* the commuting matrix is:

$$\mathbf{C}_T = \begin{bmatrix} & & & 0 \\ & \mathbf{C} & & 0 \\ & & & d \\ 0 & 0 & 0 & f \end{bmatrix}$$

2. For a *planar motion* or a *pure rotation* the commuting matrix is:

$$\mathbf{C}_R = \begin{bmatrix} & & & 0 \\ & \mathbf{C} & & 0 \\ & & & d \\ 0 & 0 & e & f \end{bmatrix}$$

3. Finally, for a *pure translation* the commuting matrix has a more general form:

$$\mathbf{C}_t = \begin{bmatrix} c_{11} & c_{12} & 0 & c_{14} \\ c_{21} & c_{22} & 0 & c_{24} \\ c_{31} & c_{32} & a & c_{34} \\ 0 & 0 & 0 & a \end{bmatrix}$$

6.3 Rodrigues formula for planar motions

The real Jordan factorization of planar motions is given by eq. (22) which can be written compactly as:

$$\mathbf{T}_\perp = \mathbf{U}\mathbf{J}_R\mathbf{U}^{-1}$$

where the sub-script $\perp$ indicates that the translation vector is in a plane perpendicular to the rotation axis. **Notice that this representation includes pure rotations as a special case of planar motion**. As for 3×3 matrices we have:

$$\mathbf{T}_\perp = \mathbf{U}(\mathbf{J}_R - \mathbf{I})\mathbf{U}^{-1} + \mathbf{I}$$

After some obvious algebra we obtain the following formula:

$$\mathbf{T}_\perp = \mathbf{I} + \sin\theta\widehat{\mathbf{T}}_\perp + (1 - \cos\theta)\widehat{\mathbf{T}}_\perp^2 \tag{25}$$

with:

$$\hat{\mathbf{T}}_\perp = \begin{bmatrix} \widehat{\mathbf{R}} & -\widehat{\mathbf{R}}\boldsymbol{\sigma}_\perp \\ \mathbf{0}^\top & 0 \end{bmatrix} \tag{26}$$

This matrix plays the same role for planar motion as $\widehat{\mathbf{R}}$ for rotations. As it will be explained below it describes the "tangent operator" associated with a displacement.

The formula above allow us to obtain a simple relationship between the tangent operator and the matrix representing the planar motion. Indeed we have:

$$\mathbf{T}_\perp^{-1} = (\mathbf{U}\mathbf{J}_R\mathbf{U}^{-1})^{-1} = \mathbf{U}\mathbf{J}_R^{-1}\mathbf{U}^{-1}$$

and

$$\mathbf{J}_R^{-1}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\top = \mathbf{J}_R(-\theta)$$

We obtain:

$$\mathbf{T}_\perp^{-1}(\theta) = \mathbf{T}_\perp(-\theta)$$

Using the Rodrigues formula we obtain:

$$\hat{\mathbf{T}}_\perp = \frac{1}{2\sin\theta}\left(\mathbf{T}_\perp - \mathbf{T}_\perp^{-1}\right) \tag{27}$$

7 The tangent operator and the kinematic screw

7.1 Definition and coordinate change

We consider again the motion of a rigid body. As already outlined in §2.1 the motion is characterized by a moving frame with respect to a fixed frame and point M has homogeneous coordinates $\boldsymbol{X}$ after the motion and $\boldsymbol{X}_0$ before the motion, e.g., Figure 1:

$$\boldsymbol{X}(t) = \mathbf{T}(t)\boldsymbol{X}_0$$

The velocity of point M is therefore given by:

$$\dot{\boldsymbol{X}} = \dot{\mathbf{T}}\boldsymbol{X}_0$$

where a dot overlapping a variable denotes the time derivative. By combining these two relationships we obtain an equation relating the velocity of a point to its current coordinates:

$$\dot{\boldsymbol{X}} = \dot{\mathbf{T}}\mathbf{T}^{-1}\boldsymbol{X} \tag{28}$$

Matrix $\dot{\mathbf{T}}\mathbf{T}$ is called "the tangent operator" and it will be denoted by the matrix overlapped by a tilde ($\tilde{}$):

$$\tilde{\mathbf{T}} = \dot{\mathbf{T}}\mathbf{T}$$

By combining eq. (28) with eq. (17) we obtain:

$$\tilde{\mathbf{T}}_2 = \mathbf{U}\tilde{\mathbf{T}}_1\mathbf{U}^{-1} \tag{29}$$

7.2 The case of a rotation

The tangent operator of a rotation matrix $\mathbf{R}$ is obviously $\dot{\mathbf{R}}\mathbf{R}^\top$. Moreover, by taking the time derivative of $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ we obtain:

$$\dot{\mathbf{R}}\mathbf{R}^\top + \mathbf{R}\dot{\mathbf{R}}^\top = 0$$

Therefore the tangent operator is a skew-symmetric matrix and it will be denoted by $\widetilde{\mathbf{R}}$:

$$\widetilde{\mathbf{R}} = \dot{\mathbf{R}}\mathbf{R}^\top = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = S(\boldsymbol{\omega})$$

The point-velocity under pure rotation is:

$$\dot{\boldsymbol{x}} = \widetilde{\mathbf{R}}\boldsymbol{x} = S(\boldsymbol{\omega})\boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}$$

In order to get some intuition about the meaning of this tangent operator, we apply it to the Jordan form of a rotation and then use the rule for coordinate change. We have:

$$\tilde{\mathbf{J}}_\theta = \dot{\mathbf{J}}_\theta\mathbf{J}_\theta^\top = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\theta}S\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

By applying the rule of coordinate change of the tangent operator, i.e., eq. (29) to eq. (8) we obtain:

$$\widetilde{\mathbf{R}} = \mathbf{\Sigma}\tilde{\mathbf{J}}_\theta\mathbf{\Sigma}^\top = \dot{\theta}\mathbf{S}(\boldsymbol{n}) = \dot{\theta}\widehat{\mathbf{R}} \tag{30}$$

where $\boldsymbol{n}$ is the axis of rotation and the third column of matrix $\mathbf{\Sigma}$. By identification we obtain the following formula for the angular velocity:

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{n} \tag{31}$$

Therefore, the angular velocity is a 3-vector aligned with the axis of rotation and whose module is equal to the time derivative of the angle of rotation.

However, a pure rotational motion may be described by a 4×4 homogeneous matrix as well. Using the real Jordan decomposition of such a matrix, i.e. eq. (21), we obtain the following expression for the tangent operator:

$$\widetilde{\mathbf{T}}_R = \begin{bmatrix} \dot{\theta}\widehat{\mathbf{R}} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} \tag{32}$$

7.3 The case of a translation

The tangent operator of the RJF for a pure translation is:

$$\tilde{\mathbf{J}}_t = \dot{\mathbf{J}}_t\mathbf{J}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{s} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \dot{s}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \dot{s}\widehat{\mathbf{J}}_t$$

Using the notations introduced in §6.1.4 we have:

$$\boldsymbol{t} = ||\boldsymbol{t}||\frac{\boldsymbol{t}}{||\boldsymbol{t}||} = s\boldsymbol{q}_3$$

Since $\boldsymbol{q}_3$ is associated with the coordinate change it remains fixed over time and hence the translational velocity is:

$$\boldsymbol{v} = \dot{\boldsymbol{t}} = \dot{s}\boldsymbol{q}_3$$

The tangent operator becomes in this case:

$$\widetilde{\mathbf{T}}_t = \mathbf{T}_c\tilde{\mathbf{J}}_t\mathbf{T}_c^{-1} = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \dot{s}\boldsymbol{q}_3 \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} = \dot{s}\widehat{\mathbf{T}}_t \tag{33}$$

The point velocity is in this case:

$$\dot{\boldsymbol{x}} = \widetilde{\mathbf{T}}_t\boldsymbol{x} = \boldsymbol{v}_0 \tag{34}$$

7.4 The case of a planar motion

The real Jordan form of a planar motion is identical to the real Jordan form of a pure rotation. The corresponding tangent operator is:

$$\tilde{\mathbf{J}}_R = \begin{bmatrix} \tilde{\mathbf{J}}_\theta & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

From the real Jordan decomposition of a planar motion and the definition of the tangent operator, its straightforward to find that:

$$\tilde{\mathbf{T}}_\perp = \begin{bmatrix} \widetilde{\mathbf{R}} & -\widetilde{\mathbf{R}}\boldsymbol{\sigma}_\perp \\ \mathbf{0}^\top & 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}\widehat{\mathbf{R}} & -\dot{\theta}\widehat{\mathbf{R}}\boldsymbol{\sigma}_\perp \\ \mathbf{0}^\top & 0 \end{bmatrix} = \dot{\theta}\widehat{\mathbf{T}}_\perp \tag{35}$$

with:

$$\widehat{\mathbf{T}}_\perp = \begin{bmatrix} \widehat{\mathbf{R}} & -\widehat{\mathbf{R}}\boldsymbol{\sigma}_\perp \\ \mathbf{0}^\top & 0 \end{bmatrix} \tag{36}$$

With the notation:

$$\boldsymbol{v}_\perp = \boldsymbol{\sigma}_\perp \times \boldsymbol{\omega} = -\widetilde{\mathbf{R}}\boldsymbol{\sigma}_\perp$$

We obtain the following expression for the velocity of a point:

$$\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x} + \boldsymbol{v}_\perp$$

7.5 The case of a general motion

The tangent operator of the real Jordan form of a general motion is:

$$\tilde{\mathbf{J}}_T = \begin{bmatrix} & 0 \\ \tilde{\mathbf{J}}_\theta & 0 \\ & \dot{s} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

In this case the tangent operator has the following structure:

$$\tilde{\mathbf{T}} = \begin{bmatrix} \widetilde{\mathbf{R}} & -\widetilde{\mathbf{R}}\boldsymbol{\sigma}_\perp + \dot{s}\boldsymbol{n} \\ \mathbf{0}^\top & 0 \end{bmatrix} \tag{37}$$

Moreover we have:

$$\dot{s} = \frac{d(\boldsymbol{n} \cdot \boldsymbol{t})}{dt} = \boldsymbol{n} \cdot \dot{\boldsymbol{t}}$$

The translational component of the tangent operator becomes:

$$\boldsymbol{v} = \boldsymbol{\sigma}_\perp \times \boldsymbol{\omega} + (\boldsymbol{n} \cdot \dot{\boldsymbol{t}})\boldsymbol{n} = \boldsymbol{v}_\perp + \boldsymbol{v}_\parallel$$

The velocity of a point in the general case is the well known formula:

$$\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x} + \boldsymbol{v} \tag{38}$$

Screw	Planar	Rotation	Translation
$\begin{bmatrix} \dot{\theta}\widehat{\mathbf{R}} & -\dot{\theta}\widehat{\mathbf{R}}\boldsymbol{\sigma}_\perp + \dot{s}\boldsymbol{n} \\ \mathbf{0}^\top & 0 \end{bmatrix}$	$\begin{bmatrix} \dot{\theta}\widehat{\mathbf{R}} & -\dot{\theta}\widehat{\mathbf{R}}\boldsymbol{\sigma}_\perp \\ \mathbf{0}^\top & 0 \end{bmatrix}$	$\begin{bmatrix} \dot{\theta}\widehat{\mathbf{R}} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$	$\begin{bmatrix} \mathbf{0}_{3\times 3} & \dot{s}\boldsymbol{q}_3 \\ \mathbf{0}^\top & 0 \end{bmatrix}$

Table 3: This table sumarizes all possible parameterizations of the tangent operators.

7.6 Chain rule for tangent operators

When two motions are combined we obtain the following transformation:

$$\mathbf{T} = \mathbf{T}_2\mathbf{T}_1$$

A point in the moving frame may, again, be expressed with respect to its coordinates in the fixed frame:

$$\boldsymbol{X} = \mathbf{T}_2\mathbf{T}_1\boldsymbol{X}_0$$

The time-derivative of this expression is:

$$\begin{aligned} \dot{\boldsymbol{X}} &= \dot{\mathbf{T}}_2\mathbf{T}_1\boldsymbol{X}_0 + \mathbf{T}_2\dot{\mathbf{T}}_1\boldsymbol{X}_0 \\ &= \dot{\mathbf{T}}_2\mathbf{T}_2^{-1}\underbrace{\mathbf{T}_2\mathbf{T}_1\boldsymbol{X}_0}_{\boldsymbol{X}} + \mathbf{T}_2\dot{\mathbf{T}}_1\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}\underbrace{\mathbf{T}_2\mathbf{T}_1\boldsymbol{X}_0}_{\boldsymbol{X}} \\ &= \widetilde{\mathbf{T}}_2\boldsymbol{X} + \mathbf{T}_2\widetilde{\mathbf{T}}_1\mathbf{T}_2^{-1}\boldsymbol{X} \end{aligned}$$

Therefore, the chain rule for tangent operator $\mathbf{T} = \mathbf{T}_2\mathbf{T}_1$ is, Figure 8:

$$\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}_2 + \mathbf{T}_2\widetilde{\mathbf{T}}_1\mathbf{T}_2^{-1} \tag{39}$$

The chain rule for tangent operators will allow us to write the Jacobian matrix associated with a articulated mechanism such as a robot manipulator. Notice that, in order to be summed up, the two tangent operators $\widetilde{\mathbf{T}}_1$ and $\widetilde{\mathbf{T}}_2$ described in the same coordinate frame.

7.7 The kinematic screw

The kinematic screw is formally defined by eq. (38). It is a six dimensional vector composed of two vectors, $\boldsymbol{\omega}$ and $\boldsymbol{v}$, Figure 9. The first vector describes the angular velocity $(d\theta/dt)$ around the instantaneous axis of rotation and the second vector describes the linear velocity associated with the origin of the coordinate frame.

7.7.1 Coordinate change

Both these vectors depend on the choice of the coordinate frame. Therefore there is a formula for coordinate change which can directly be derived from eq. (29). By

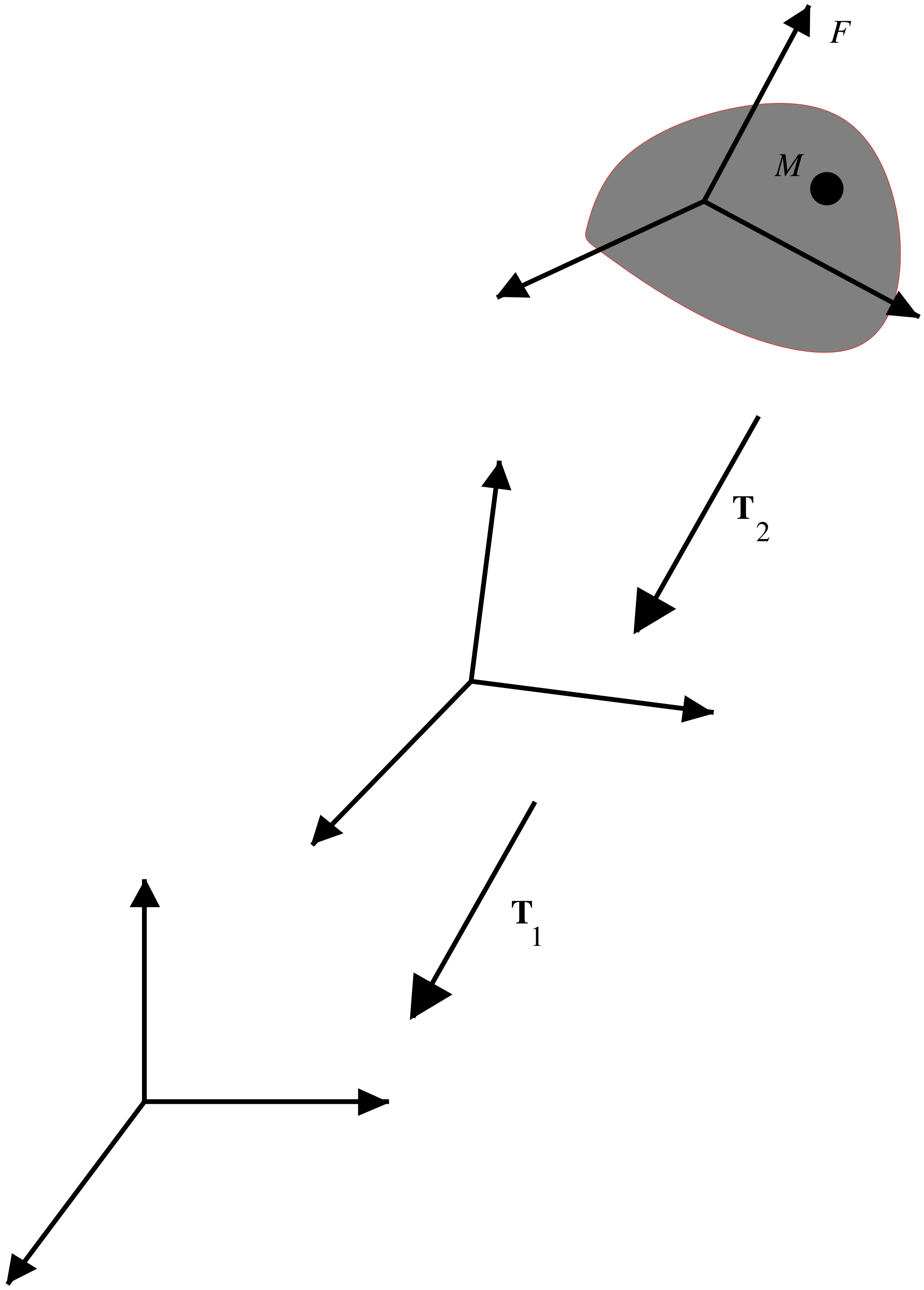

Figure 8: Two (or several) motions may be combined using a simple chain rule. This will be the basis for characterizing the links of a articulated mechanism or the virtual link between a sensor and an observed motion.

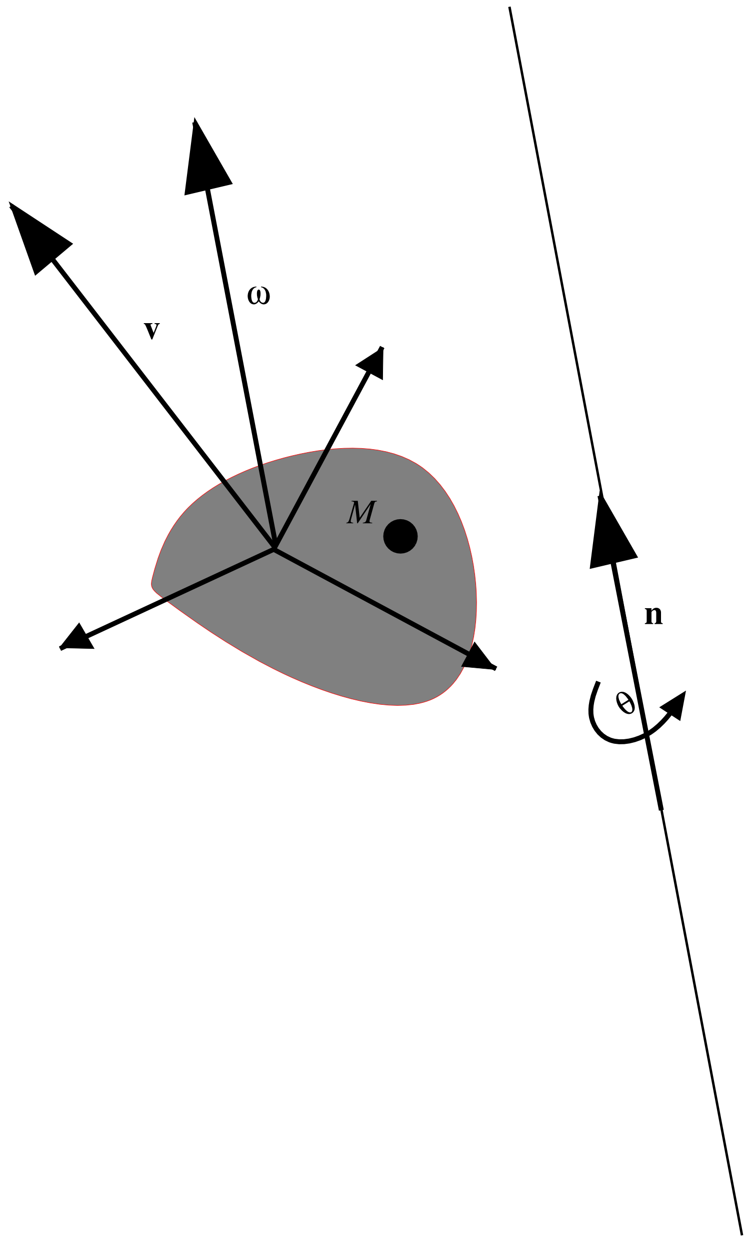

Figure 9: The kinematic screw is composed of two 3-vectors, one representing the rotational velocity and one representing the translational velocity of the origin of the coordinate frame.

developping the right side of this equation we obtain:

$$\begin{bmatrix} \mathbf{R}_c S(\boldsymbol{\omega}_1)\mathbf{R}_c^\top & -\mathbf{R}_c S(\boldsymbol{\omega}_1)\mathbf{R}_c^\top \boldsymbol{t}_c + \mathbf{R}_c \boldsymbol{v}_1 \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

By identification we therefore have $S(\boldsymbol{\omega}_2) = \mathbf{R}_c S(\boldsymbol{\omega}_1)\mathbf{R}_c^\top$ which can be written as $\boldsymbol{\omega}_2 = \mathbf{R}_c\boldsymbol{\omega}_1$. We also have $\boldsymbol{v}_2 = S(\boldsymbol{\omega}_2)\boldsymbol{t}_c + \mathbf{R}_c\boldsymbol{v}_1$ which can be written as $\boldsymbol{v}_2 = S(\boldsymbol{t}_c)\mathbf{R}_c\boldsymbol{\omega}_1 + +\mathbf{R}_c\boldsymbol{v}_1$. We obtain:

$$\begin{pmatrix} \boldsymbol{\omega}_2 \\ \boldsymbol{v}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_c & \mathbf{0} \\ S(\boldsymbol{t}_c)\mathbf{R}_c & \mathbf{R}_c \end{bmatrix} \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{v}_1 \end{pmatrix} \tag{40}$$

7.7.2 Chain rule

The chain rule for tangent operators can be written using the kinematic screw notation, i.e., by combining equations (39) and (40):

$$\begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega}_2 \\ \boldsymbol{v}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{R}_2\boldsymbol{\omega}_1 \\ \boldsymbol{t}_2 \times \mathbf{R}_2\boldsymbol{\omega}_1 + \mathbf{R}_2\boldsymbol{v}_1 \end{pmatrix} \tag{41}$$

8 Exponential representation of displacements

9 Kinematic chains

9.1 Kinematic chain of an articulated mechanism

We consider the kinematic model of an articulated mechanism. Without loss of generality we consider a mechanism formed by an open loop of rigid bodies. Let n be the number of rigid bodies. Each such body has one degree of freedom associated with it: a rotation or a translation. These one degree of freedom motions can be represented either by matrix $\mathbf{J}_R$ or by matrix $\mathbf{J}_t$, i.e., Table 6.1.4. Therefore, each rigid body motion is described by the following matrix:

$$\mathbf{T}_i = \mathbf{L}_i \mathbf{J}_i$$

where the first matrix, $\mathbf{L}_i$ is constant over time while the second matrix describes a one-dimensional motion parameterized either by a rotation angle θ or by a slide s, Figure 10. We denote by $\boldsymbol{q}$ the vector of dimension n whose components are parameterized by each individual motion: $\boldsymbol{q} = (q_1 \dots q_n)^\top$. Therefore, the motion associated with the chain of rigid bodies can be written as:

$$\mathbf{T}_n(q_n) \dots \mathbf{T}_1(q_1) = \mathbf{L}_n \mathbf{J}_n(q_n) \dots \mathbf{L}_1 \mathbf{J}_1(q_1)$$

Let a physical point M be associated with the hand-frame – this frame may in general be different than the n-th body's frame introduced above. Similarly, there is a base frame which is different than the first body's frame. Hence we have:

$$\begin{aligned} \boldsymbol{X}(\boldsymbol{q}) &= \mathbf{T}_h \mathbf{T}_n(q_n) \dots \mathbf{T}_1(q_1) \mathbf{T}_b \boldsymbol{X}_0 \\ &= \mathbf{T}(\boldsymbol{q}) \boldsymbol{X}_0 \end{aligned}$$

which expresses the change of coordinates of M from the moving frame $\mathbf{T}_h$ to the fixed frame $\mathbf{T}_b$.

9.2 Zero-reference kinematic model

We consider now the mechanism above in some fixed position. Such a position is characterized by the vector $\boldsymbol{q}_0$. Without loss of generality one may take $\boldsymbol{q}_0 = (0 \dots 0)^\top$. For this configuration we have:

$$\boldsymbol{X}(\boldsymbol{q}_0) = \mathbf{T}_h \mathbf{T}_n(0) \dots \mathbf{T}_1(0) \mathbf{T}_b \boldsymbol{X}_0 = \mathbf{T}(\boldsymbol{q}_0) \boldsymbol{X}_0$$

By eliminating $\boldsymbol{X}_0$ we obtain a relationship between $\boldsymbol{X}(\boldsymbol{q})$ and $\boldsymbol{X}(\boldsymbol{q}_0)$:

$$\boldsymbol{X}(\boldsymbol{q}) = \mathbf{T}(\boldsymbol{q}) \mathbf{T}(\boldsymbol{q}_0)^{-1} \boldsymbol{X}(\boldsymbol{q}_0)$$

We have:

$$\begin{aligned} \mathbf{T}_z(\boldsymbol{q}, \boldsymbol{q}_0) &= \mathbf{T}(\boldsymbol{q}) \mathbf{T}(\boldsymbol{q}_0)^{-1} \\ &= \mathbf{T}_h \mathbf{T}_n(q_n) \dots \mathbf{T}_1(q_1) \mathbf{T}_1(0)^{-1} \dots \mathbf{T}_n(0)^{-1} \mathbf{T}_h^{-1} \end{aligned}$$

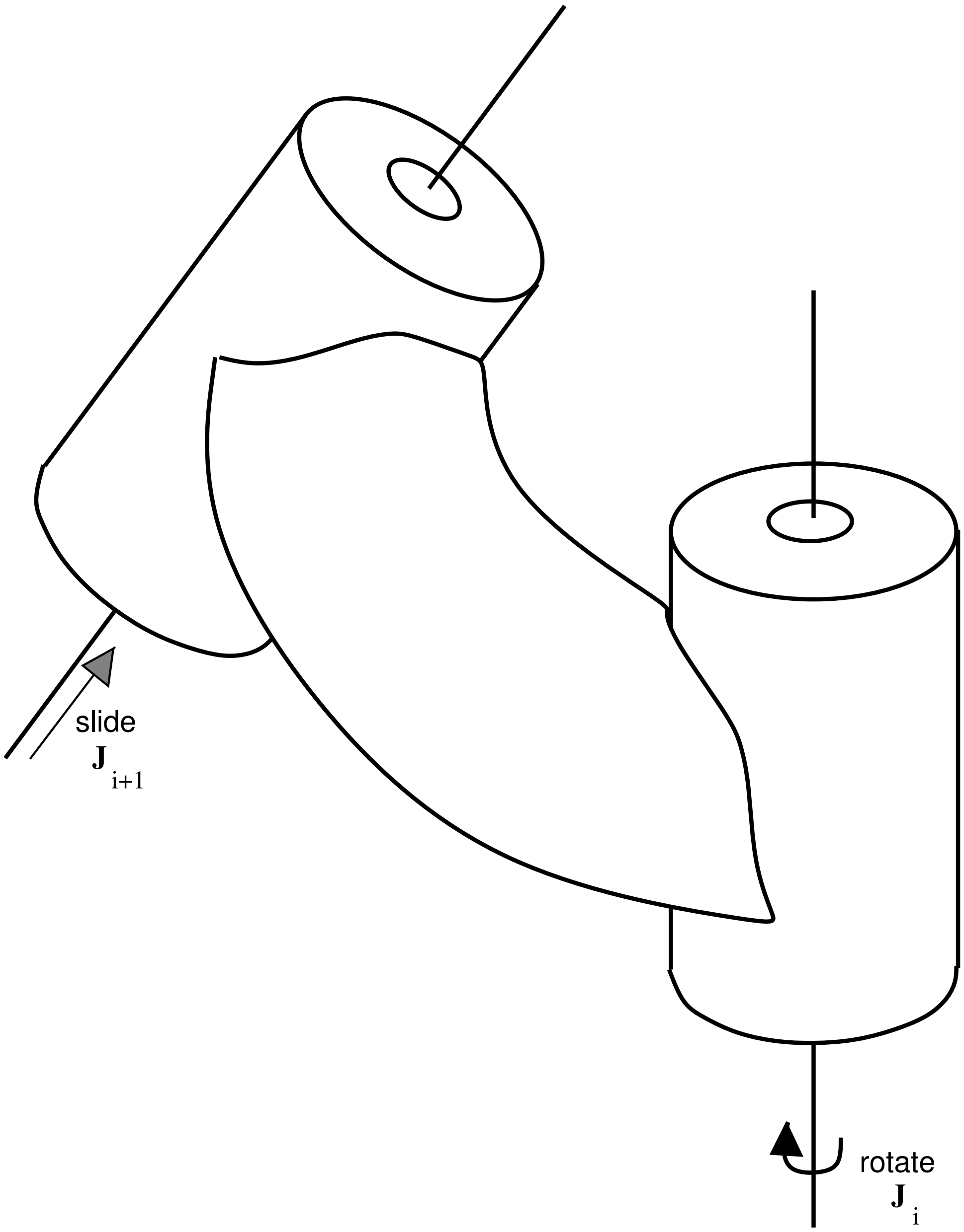

Figure 10: An articulated mechanism is formed of rigid bodies such as this one. The rigid body i separates two motions by a fixed matrix $\mathbf{L}_i$.

After some obvious matrix manipulations we obtain the following formula for the zero-reference kinematic model:

$$\mathbf{T}_z(\boldsymbol{q}, \boldsymbol{q}_0) = \mathbf{Q}_n(q_n) \dots \mathbf{Q}_1(q_1) \tag{42}$$

with:

$$\mathbf{Q}_i(q_i) = \mathbf{T}_h \mathbf{T}_n(0) \dots \mathbf{T}_{i+1}(0) \mathbf{T}_i(q_i) \mathbf{T}_i(0)^{-1} \mathbf{T}_{i+1}(0)^{-1} \dots \mathbf{T}_n(0)^{-1} \mathbf{T}_h^{-1} \tag{43}$$

Each rigid body transformation $\mathbf{T}_i(q_i)\mathbf{T}_i(0)^{-1}$ is:

$$\mathbf{T}_i(q_i)\mathbf{T}_i(0)^{-1} = \mathbf{L}_i \mathbf{J}_i(q_i) \mathbf{J}_i(0)^{-1} \mathbf{L}_i^{-1}$$

Matrices $\mathbf{J}_i(q_i)$ and $\mathbf{J}_i(0)^{-1}$ belong to the same one dimensional motion group. Therefore, each rigid body transformation is of the form:

$$\mathbf{Q}_i(q_i) = \mathbf{D}\mathbf{J}_R(\theta_i)\mathbf{D}^{-1}$$

or

$$\mathbf{Q}_i(q_i) = \mathbf{D}\mathbf{J}_t(s_i)\mathbf{D}^{-1}$$

where $\mathbf{D}$ is a fixed transformation mapping the motion i from its local "natural" frame to the hand frame. Therefore, the zero-reference kinematic model of an articulated mechanism, eq. (42), is a combination of n one-parameter motions which are all represented in the same reference frame.

The important feature to keep in mind is that the similarity transformations which map $\mathbf{J}_i$ onto $\mathbf{Q}_i$ do not change the nature of the motions – they remain one-parameter motions.

9.3 Kinematic model and sensor frame

The zero-reference kinematic model describes the motion of a point M attached to the n-th body. More precisely, the equation:

$$\boldsymbol{X}(\boldsymbol{q}) = \mathbf{T}_z(\boldsymbol{q}, \boldsymbol{q}_0)\boldsymbol{X}(\boldsymbol{q}_0)$$

expresses the coordinates of the moving point $M(\boldsymbol{q})$ in the fixed zero-reference frame associated with the hand. Let $\mathbf{D}_0$ the matrix mapping these coordinates from the hand frame to an arbitrary sensor frame. We obtain:

$$\underbrace{\mathbf{D}_0\boldsymbol{X}(\boldsymbol{q})}_{\boldsymbol{X}^s(\boldsymbol{q})} = \mathbf{D}_0\mathbf{T}_z(\boldsymbol{q}, \boldsymbol{q}_0)\boldsymbol{X}(\boldsymbol{q}_0) = \mathbf{D}_0\mathbf{T}_z(\boldsymbol{q}, \boldsymbol{q}_0)\mathbf{D}_0^{-1}\underbrace{\mathbf{D}_0\boldsymbol{X}(\boldsymbol{q}_0)}_{\boldsymbol{X}^s(\boldsymbol{q})}$$

Therefore the zero-reference kinematic model may be represented in any sensor frame, whether this sensor is fixed onto the hand (eye-in-hand) or not (independent-eye). The sensor-centered zero-reference kinematic model writes:

$$\begin{aligned}
\mathbf{T}_s(\boldsymbol{q}, \boldsymbol{q}_0) &= \mathbf{D}_0\mathbf{T}_z(\boldsymbol{q}, \boldsymbol{q}_0)\mathbf{D}_0^{-1} \\
&= \mathbf{D}_0\mathbf{Q}_n(q_n)\mathbf{D}_0^{-1} \dots \mathbf{D}_0\mathbf{Q}_1(q_1)\mathbf{D}_0^{-1} \\
&= \mathbf{D}_n(q_n) \dots \mathbf{D}_1(q_1)
\end{aligned} \tag{44}$$

Therefore, each rigid-body elementary motion of the articulated mechanism is represented in a sensor frame – the frame where the motion is to be measured. This concept is illustrated on Figure 11.

9.4 The Jacobian of the zero-reference kinematic model

First we derive an expression for the tangent operator associated with the zero-reference kinematic model in sensor frame. Eq. (44) maps the coordinates of the moving point M in the fixed sensor frame, when the articulated mechanism is in it's zero configuration:

$$\boldsymbol{X}^s(\boldsymbol{q}) = \mathbf{T}_s(\boldsymbol{q}, \boldsymbol{q}_0)\boldsymbol{X}^s(\boldsymbol{q}_0)$$

The tangent operator allows one to compute the velocity of a point with respect to its own coordinates:

$$\dot{\boldsymbol{X}}^s(\boldsymbol{q}) = \widetilde{\mathbf{T}}_s(\boldsymbol{q}, \boldsymbol{q}_0)\boldsymbol{X}^s(\boldsymbol{q})$$

Notice that each elementray motion $\mathbf{D}(q)$ is a one dimensional Lie group and hence its tangent operator is of the form:

$$\widetilde{\mathbf{D}}(q) = \dot{q}\widehat{\mathbf{D}}(q)$$

where $\dot{q}$ is either an angular velocity $\dot{\theta}$ or a translational velocity $\dot{s}$.

We apply the chain rule and obtain the following expression:

$$\begin{aligned}
\widetilde{\mathbf{T}}_s(\boldsymbol{q}, \boldsymbol{q}_0) &= \dot{\mathbf{T}}_s(\boldsymbol{q}, \boldsymbol{q}_0)\mathbf{T}_s(\boldsymbol{q}, \boldsymbol{q}_0)^{-1} \\
&= \widetilde{\mathbf{D}}_n + \mathbf{D}_n\widetilde{\mathbf{D}}_{n-1}\mathbf{D}_n^{-1} + \ldots + \mathbf{D}_n \ldots \mathbf{D}_2\widetilde{\mathbf{D}}_1\mathbf{D}_2^{-1} \ldots \mathbf{D}_n^{-1} \\
&= \dot{q}_n\widehat{\mathbf{D}}_n + \dot{q}_{n-1}\mathbf{D}_n\widehat{\mathbf{D}}_{n-1}\mathbf{D}_n^{-1} + \ldots + \dot{q}_1\mathbf{D}_n \ldots \mathbf{D}_2\widehat{\mathbf{D}}_1\mathbf{D}_2^{-1} \ldots \mathbf{D}_n^{-1} \qquad (45)
\end{aligned}$$

The elements of each tangent operator $\widetilde{\mathbf{D}}_i$ define the angular and linear velocities of rigid body i with respect to rigid body $i-1$. Pre and post multiplication by matrices $\mathbf{D}_{i+1}$ and $\mathbf{D}_{i+1}^{-1}$ simply transforms these velocities from rigid body i to the sensor frame. Denoting the transformed tangent operators with an overscript s we have:

$$\widetilde{\mathbf{T}}_s(\boldsymbol{q}, \boldsymbol{q}_0) = \dot{q}_n\widehat{\mathbf{D}}_n^s + \dot{q}_{n-1}\widehat{\mathbf{D}}_{n-1}^s + \ldots + \dot{q}_1\widehat{\mathbf{D}}_1^s$$

Equivalently, this relationship may be written using the kinematic screw representation of angular and linear velocities:

$$\begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix} = \dot{q}_n \begin{pmatrix} \boldsymbol{\omega}_n \\ \boldsymbol{v}_n \end{pmatrix} + \dot{q}_{n-1} \begin{pmatrix} \boldsymbol{\omega}_{n-1} \\ \boldsymbol{v}_{n-1} \end{pmatrix} + \ldots + \dot{q}_1 \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{v}_1 \end{pmatrix}$$

It is important to notice that this parameterization is possible only when each individual rigid-body motion is a one-parameter motion: pure rotation around an axis or translation along a direction.

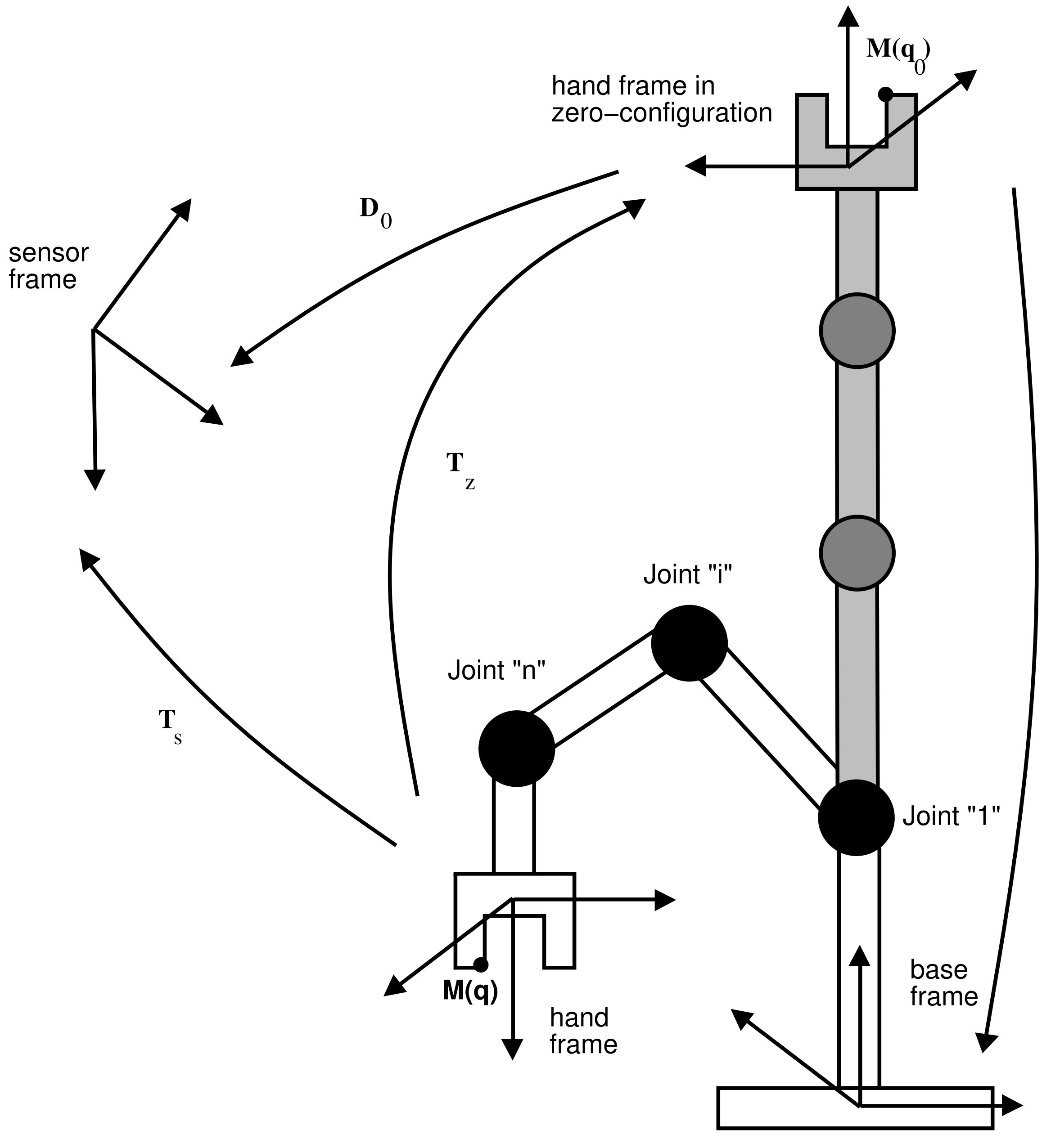

Figure 11: The zero-reference model is defined using a fixed position of the mechanism. The latter may well be described in some external sensor frame which remains fixed with respect to the zero-reference.

We obtain an expression for the Jacobian matrix which computes the hand velocities measured relative to the fixed sensor frame as a function of the joint velocities:

$$\begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix} = \begin{bmatrix} \boldsymbol{\omega}_n & \boldsymbol{\omega}_{n-1} & \dots & \boldsymbol{\omega}_1 \\ \boldsymbol{v}_n & \boldsymbol{v}_{n-1} & & \boldsymbol{v}_1 \end{bmatrix} \begin{pmatrix} \dot{q}_n \\ \dot{q}_{n-1} \\ \vdots \\ \dot{q}_1 \end{pmatrix} \tag{46}$$

9.5 Examples of robots and their kinematics

10 Camera geometry

10.1 One camera

10.2 Two cameras

10.3 Motion observation with one or two cameras

11 Linking a camera with a robot

11.1 Possible configurations

11.2 The image Jacobian

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