Variational method combined with Frangi vesselness for tubular object segmentation

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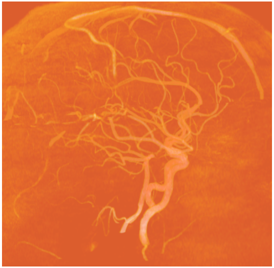
Outline

Resume

Segmentation: a double challenge



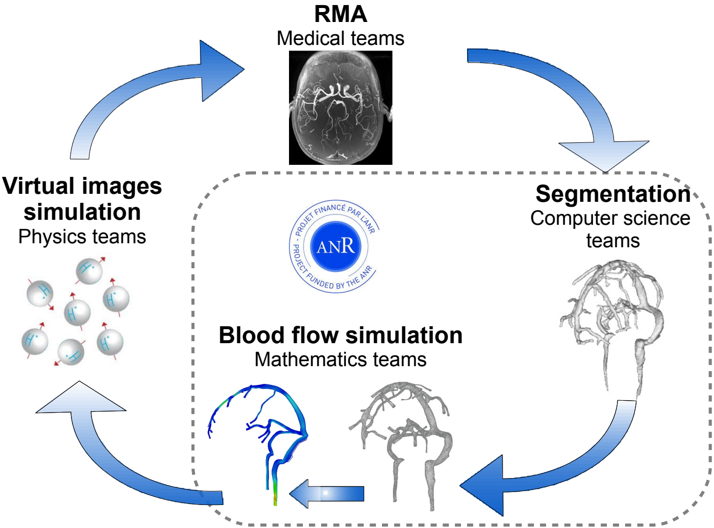
Difficulties: detect tubular structures

- very thin (generally a few pixel thick);
- corrupted by noise;
- geometrically complex.

Two induced issues

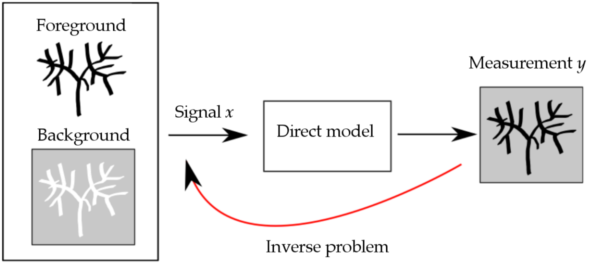
- *denoising;*
- *enhancement and segmentation.*

Context : ANR project VivaBrain



Resume

Segmentation as an inverse problem



Inverse problems are typically *ill-posed*. A problem is *well-posed* according to Hadamard if:

- the solution exists;
- it is unique;
- the solution changes smoothly if the data changes smoothly.

$\Rightarrow$ In 1943, Tikhonov proposed a method to solve ill-posed problems $Ax = y$.

Least squares method

$$f(x) = \|Ax - y\|^2 = x^T A^T Ax - 2y^T Ax + y^T y$$

A point x minimize f if and only if:

$$\nabla f(x) = 2A^T Ax - 2A^T y = 0$$
$$\Leftrightarrow A^T Ax = A^T y$$
$$\Leftrightarrow \hat{x} = (A^T A)^{-1} A^T y$$

In image processsing, A is very large and ill-conditioned
$\Rightarrow$ Tikhonov added a prior term Γ:

$$f(x) = \|Ax - y\|^2 + \lambda \|\Gamma x\|^2$$
$$\hat{x} = (A^T A + \lambda \Gamma^T \Gamma)^{-1} A^T y$$

- $\Gamma = I$ encourages solutions to have low norm;
- $\Gamma = \nabla$ encourages solutions to have low variation;
- $\Gamma = \Delta$ encourages solutions to have low curvature.

Tikhonov model

The Tikhonov model can be used for image denoising or restoration. We observe an image $f : \Omega \subset \mathbb{R}^N \mapsto \mathbb{R}$ as:

$$f = u + n$$

$u =$original image
$n =$additive noise

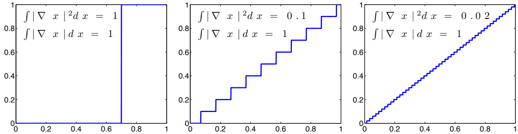
The solution of the problem is given by the minimization of an energy-functional composed of two terms:

$$\min_u \underbrace{\frac{1}{2}\int_\Omega |\nabla u|^2}_{\text{regularization term}} + \underbrace{\frac{\lambda}{2}\int_\Omega \|u-f\|^2 dx}_{\text{data fidelity term}}$$

Varying models

ROF model (1992) : the quadratic regularization is replaced with an L1 norm, the Total Variation (TV) term

$$\min_u \underbrace{\int_\Omega |\nabla u|}_{\text{TV term}} + \underbrace{\frac{\lambda}{2}\int_\Omega \|u - f\|^2 dx}_{\text{data fidelity term}}$$

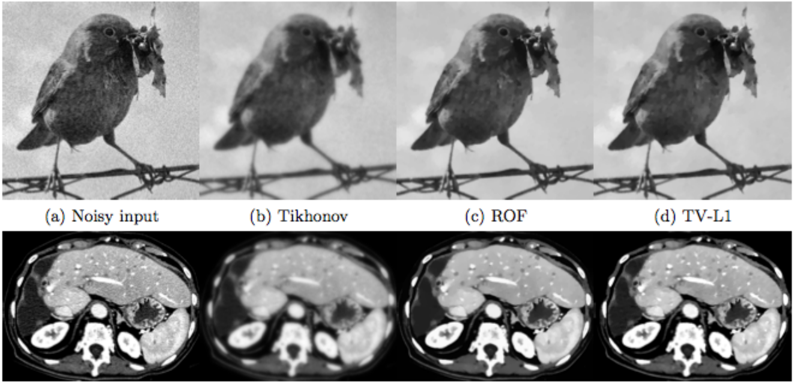


(a) (b) (c)

TV-L1 model (1992) : the L1 norm is used for the data term

$$\min_u \underbrace{\int_\Omega |\nabla u|}_{\text{TV term}} + \underbrace{\lambda \int_\Omega |u - f| dx}_{\text{data fidelity term}}$$

Comparison of different variational denoising models I



(a) Noisy input (b) Tikhonov (c) ROF (d) TV-L1

Figure: Top: example with artificial Gaussian noise; bottom: CT image with natural noise. [Unger, 2012]

Comparison of different variational denoising models II

Data fidelity or regularization ?

- *A priori about the perturbation type* $\Rightarrow$ *choice of a data fidelity*
- *A priori about the desired smoothing* $\Rightarrow$ *choice of a regularization*

Data fidelity

- $|u - f|$ *(median): impulse noise - outliers*
- $\|u - f\|^2$ *(average): gaussian noise*

Regularization

- $|\nabla u|$*: denoise + preserve smooth transitions and edges*
- $\|\nabla u\|^2$*: denoise + preserve smooth transitions*

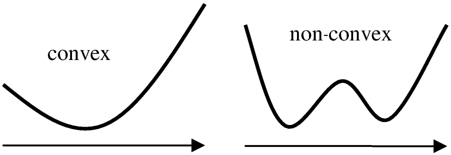
Convex minimization problem

Problem

$$\min_{x \in \mathbb{R}^N} F(x)$$

where $F : \mathbb{R}^N \rightarrow]-\infty, +\infty]$ *is a convex energy.*

Importance of the convexity ?



If the function is convex then local minimum = global minimum

If the function is non-convex and the initial condition is not well placed $\Rightarrow$ the algorithm is stuck in a local minimum

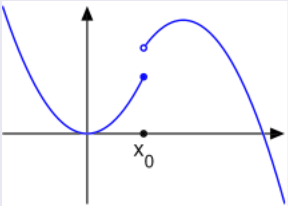
Proximal point methods I

Proper function

A function is proper if and only if it is not identically equal to $+\infty$ *and its domain* $\text{dom} f = \{x \in \mathbb{R}^N : f(x) < +\infty\}$ *is non-empty.*

Lower semicontinuous convex function

A function is lower semicontinuous (l.s.c.) if and only if $\forall x_0 \in \mathbb{R}^N,\ f(x_0) \leq \lim_{x \to x_0} \inf f(x)$



Proximal point methods II

Proximity operator

Let f(x) be a proper l.s.c. convex function, the proximity operator x associated to a function f is the operator defined by:

$$prox_{\tau f(x)} = \arg\min_{y} \left\{ f(y) + \frac{\|y - x\|^2}{2\tau} \right\}$$

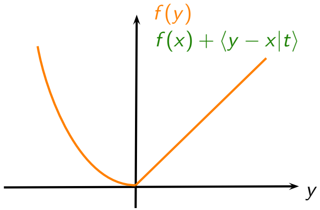
$$x_p = \text{prox}_{\tau f(x)} \Leftrightarrow \exists g_p \in \partial f(x_p),\ x_p = x - \tau g_p$$

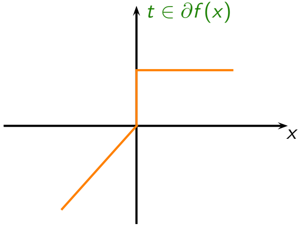
Proximal point algorithm

$$x_{k+1} = x_k - \tau_k g_{k+1}, \quad avec\ g_{k+1} \in \partial f(x_k + 1)$$

Implicit subgradient descent

Examples of subgradient





- If f is differentiable $x \in \mathbb{R}^N$, then $\partial f(x) = \{\nabla f(x)\}$
- If $f = |.|$, then

$$\forall x \in \mathbb{R}^N, \partial f(x) = \begin{cases} \{\text{sign}(x)\} & \text{if } x \neq 0 \\ \text{[-1,+1]} & \text{if } x = 0 \end{cases}$$

Several classes of algorithms

Name	Problem	Algorithm
FB	$f(x) = g(x) + h(x)$ g differentiable	$x_{k+1} = \text{prox}_{\tau_k h}(x_k - \tau_k \nabla g(x_{k+1}))$
ISTA (FISTA, Twist Nesterov)	$g(x) = \|Ax - b\|^2$ $h(x) = \lvert \cdot \rvert$	$x_{k+1} = \text{prox}_{\tau_k g}(x_k - \tau_k A^T(Ax_k - b))$
DR	$f(x) = g(x) + h(x)$ $(\text{ri dom } g) \cap (\text{ri dom } h) \neq \emptyset$	$x_k = \text{prox}_{\tau h}\, y_k$ $y_{k+1} = y_k + \lambda_k(\text{prox}_{\tau g}(2x_k - y_k) - x_k)$
PPXA	$f(x) = f_1(x) + \ldots + f_m(x)$ $(\text{ri dom } f_1) \cap (\text{ri dom } f_2)$ $\ldots \cap (\text{ri dom } f_m) \neq \emptyset$	For $i = 1, \ldots, m$ $p_{i,k} = \text{prox}_{\tau_k f_i}\, y_k \; (p_k = \sum p_{i,k})$ $y_{i,k+1} = y_{i,k} + \lambda_k(2p_k - x_k - pi, k_k)$ End $x_{k+1} = x_k + \lambda_k(p_k - x_k)$
Primal-dual	$f(x) = g(Kx) + h(x)$	$y_{k+1} = \text{prox}_{\sigma g^*}(y_k + \sigma K\bar{x}_k)$ $x_{k+1} = \text{prox}_{\tau h}(x_k - \tau K^T y_{k+1})$ $\bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k)$

Resume

Primal-dual algorithm (Chambolle et Pock, 2011)

Let F and G be two proper, l.s.c., convex functions and K a linear operator, the general problem is defined by:

$$\min_{x} F(Kx) + G(x) \tag{1}$$

By applying the Fenchel-Moreau theorem, we obtain the following saddle point problem:

$$\min_{x} \max_{y} \langle Kx, y \rangle + G(x) - F^*(y) \tag{2}$$

where $F^*(y) = \sup_{x \in X} \langle y, x \rangle - F(x)$ is the conjugate function. The idea consists of perfoming simultaneously an approximate gradient ascent in the dual variable y and gradient descent in the primal variable x

$$y = \text{prox}_{\sigma F^*} \tilde{y} = \arg\min_{y} \left\{ \frac{\|y - \tilde{y}\|^2}{2\sigma} + F^*(y) \right\}$$

$$x = \text{prox}_{\tau G} \tilde{x} = \arg\min_{x} \left\{ \frac{\|x - \tilde{x}\|^2}{2\tau} + G(x) \right\}$$

ROF model I

The proximity operator for the primal variable can be computed as:

$$x = \text{prox}_{\tau G_{TV-L2}} \tilde{x} = \arg\min_x \left\{ \frac{\|x - \tilde{x}\|^2}{2\tau} + \frac{\lambda}{2}\|x - f\|^2 \right\}$$

To solve this minimization problem we look to the corresponding Euler-Lagrange equation:

$$\frac{1}{\tau}(x - \tilde{x}) + \lambda(x - f) = 0$$

Thus, the solution is given by the following:

$$x = \frac{\tilde{x} + \lambda\tau f}{1 + \lambda\tau}$$

ROF model II

We finally have to compute the proximity operator for the dual update as:

$$y = \text{prox}_{\sigma F^*} \tilde{y} = \arg\min_{y} \left\{ \frac{\|y - \tilde{y}\|^2}{2\sigma} + F^*(y) \right\}$$

Now, we have to determine the conjugate F^* of F by duality :

$$\begin{aligned} F(Kx) = \|\nabla x\|_{2,1} &= \sup\{\langle \xi, \nabla x\rangle_{X^*} \ : |\xi_{i,j}| \leq 1 \, \forall i,j\} \\ &= \sup\{-\langle \text{div}\, \xi, x\rangle_{X} \ : |\xi_{i,j}| \leq 1 \, \forall i,j\} \\ &= \sup_{p} \langle p, x\rangle_X - \delta_P(p) \end{aligned}$$

where $P = \{p = -\text{div}\, \xi \in X \ : |\xi_{i,j}| \leq 1 \, \forall i,j\}$ and $\delta_P(p)$ defined by:

$$\delta_P(p) = \begin{cases} 0 & \text{si } p \in P \\ +\infty & \text{si } p \notin P \end{cases}$$

So $F^*(y) = \delta_P(y)$ and $y = \frac{\tilde{y}}{\max(1, |\tilde{y}|)}$

TV-L1 model

The proximity operator for the dual variable $\text{prox}_{\sigma F^*}$ is the same. The proximity operator for the primal can be computed as:

$$x = \text{prox}_{\tau G_{TV-L1}}(\tilde{x}) = \arg\min_x \left\{ \frac{\|x - \tilde{x}\|^2}{2\tau} + \lambda \|x - f\| \right\}$$

with the corresponding Euler-Lagrange equation:

$$\frac{1}{\tau}(x - \tilde{x}) + \lambda \frac{x - f}{|x - f|} = 0$$

As a result, we arrive at the following soft thresholding schema:

$$x = \begin{cases} \tilde{x} - \tau\lambda & \text{if } \tilde{x} - f > \tau\lambda \\ \tilde{x} + \tau\lambda & \text{if } \tilde{x} - f < -\tau\lambda \\ f & \text{if } |\tilde{x} - f| \leq \tau\lambda \end{cases}$$

Results (grey level and color)

Original

Noisy

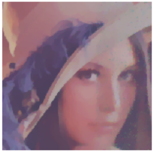
Original

Noisy

ROF lambda=4

TVL1 lambda=1

ROF lambda=8



TVL1 lambda=2

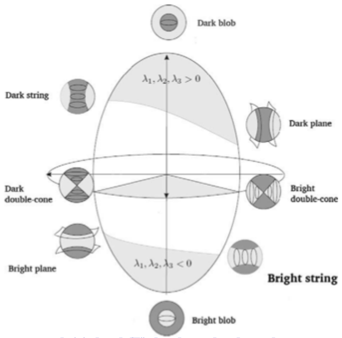
Resume

How to detect tubular structures ?

The segmentation can use the differential properties of the image
Gradient = geometric information of objects (edges, texture,...)
Hessian = shape characteristics of objects (tube, plane, blob,...)

By eigenvalue analysis, the Hessian matrix can be decomposed into three eigenvalues λ_1, λ_2, λ_3 ($\lambda_1 \leq \lambda_2 \leq \lambda_3$). For an ideal tubular structure in a 3D image, we have:

$$|\lambda_1| \approx 0$$
$$|\lambda_1| \ll |\lambda_2|$$
$$|\lambda_2| \approx |\lambda_3|$$



Frangi's vesselness function (1998)

$$\mathcal{V}(x) = \begin{cases} 0 & \text{if } \lambda_2 > 0 \text{ or } \lambda_3 > 0 \\ (1 - e^{-\frac{-R_A^2}{2\alpha^2}}) \cdot e^{-\frac{-R_B^2}{2\beta^2}} \cdot (1 - e^{-\frac{-S^2}{2\gamma^2}}) & \text{otherwise} \end{cases}$$

with

$$R_A = \left|\frac{\lambda_2}{\lambda_3}\right|$$
$$R_B = \frac{|\lambda_1|}{\sqrt{|\lambda_2 \lambda_3|}}$$
$$S = \|H_\sigma\| = \sqrt{\Sigma_j \lambda_j^2}$$

R_A differentiates between plane- and line-like objects, R_B differentiates blob-like ones, and S accounts for the intensity difference between objects and background.

Inclusion of vesselness to the model

ROF model:

$$\min_u \int_\Omega |\nabla u| \quad + \quad \underbrace{\lambda}_{\text{vesselness}} \int_\Omega \|u - f\|^2 dx$$

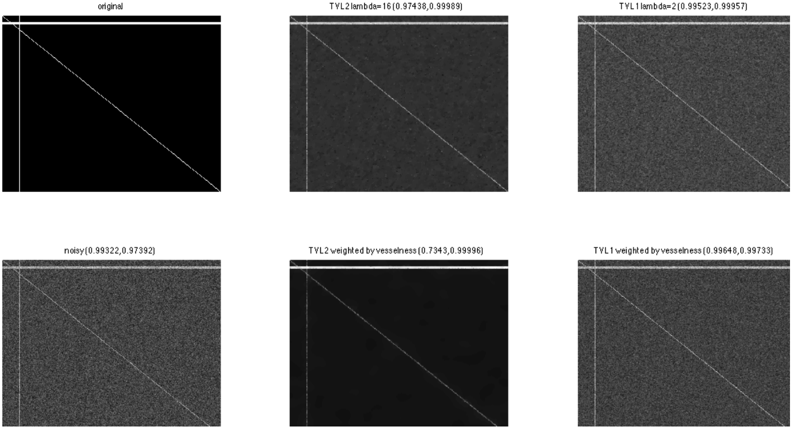
TV-L1 model:

$$\min_u \int_\Omega |\nabla u| \quad + \quad \underbrace{\lambda}_{\text{vesselness}} \int_\Omega |u - f| dx$$

where $\lambda = \lambda_{Reg} \left[\alpha + (1-\alpha)\mathcal{V}(x)\right]$ and $\alpha \in [0, 1]$

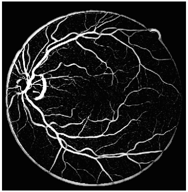
$\mathcal{V}(x)$ = Frangi's vesselness = matrix !

Synthetic 2D image results

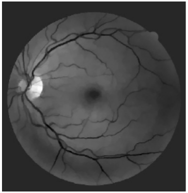


Retinal image results (DRIVE database)

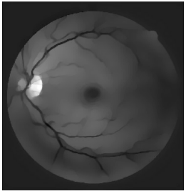
Original image



Vesselness



TV-L1



ROF

To segment, we have to modify the data fidelity
$\Rightarrow$ Chan-Vese model

$$\text{Data fidelity} = \int_\Omega \underbrace{\|u\|^2 \|c_1 - f\|^2}_{\text{region of interest}} + \underbrace{\|1 - u\|^2 \|c_2 - f\|^2}_{\text{background}} \, dx$$

Resume

Conclusion and outlook

2D Results:

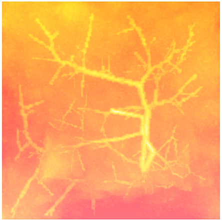
Include vesselness ⇒ detect tubular structures

✓ Enhancement

✗ Segmentation

Outlook:

- Include the Chan-Vese model to the fidelity ⇒ segment vessels
- Test on 3D synthetic images



Useful links

- olivia.miraucourt@orange.fr
- http://olivia.miraucourt.pagesperso-orange.fr
- http://numtourcfd.univ-reims.fr