Anisotropic Mumford-Shah Model

Workshop Imagerie Angiographie

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University of Orléans

June 2015

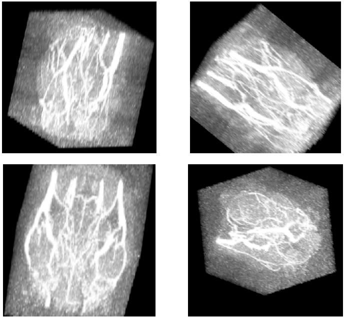

Figure : Mouse Brain Angiography

We want to perform the segmentation of the 3-D blood network. Segmentation of the biggest vessels are easily done by classical methods. The problem comes from the detection of thin tubular structures with section and intensity comparable to noise.

At the small scale of thin structures, the only criterium to discriminate vessels from noise is the tubular geometry. So, we want to define an energy which contains a *geometric prior* on the image we want to get.

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A first approach may consist in taking an energy defined as

$$\mathcal{E} = \mathcal{E}_{\text{classical}} + \mathcal{E}_{\text{geometric}}$$

where we choose the first term as a well known energy (Rudin-Osher-Fatemi, Modica-Mortola, Mumford-Shah,...) and the second term is a distance from the given image to an geometric model.

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has several drawbacks:

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has several drawbacks:

1) $\nearrow$ complexity $\Rightarrow$ $\searrow$ robustness,
2) $\nearrow$ complexity $\Rightarrow$ $\searrow$ interpretation,
3) the *Classical* Energy contains itself a geometric prior.

We rather incorporate the geometric prior *inside* the classical energy.

For binary images, we proposed an energy based the classical Modica-Mortola model in such a way that the geometric prior is incorporated inside the energy.

Anisotropic Bimodal Energy for Segmentation of thin tubes and its approximation with Γ-convergence, Advances in Calculus of Variation, V., 2015

This work may also be done for a large class of energies. In this presentation we will present it in the context of the Mumford-Shah model.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $g : \Omega \to \mathbb{R}$ an image, Mumford-Shah energy is defined as

$$\mathcal{E}(u,K) = \int_{\Omega \setminus K} (u-g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \mathcal{H}^{n-1}(K).$$

where $K \subset \Omega$ is compact, and $u \in W^{1,2}(\Omega \setminus K)$.

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where $K \subset \Omega$ is compact, and $u \in W^{1,2}(\Omega \setminus K)$.

We consider $g = \alpha \mathbf{1}_E$ and $u = g$, then

$$\mathcal{E}(\alpha \mathbf{1}_E, \partial E) \leq \mathcal{E}(0, \emptyset) \Leftrightarrow \mathcal{H}^{n-1}(\partial E) \leq \alpha \mathcal{L}^n(E).$$

This means that for a given intensity α, the more the ratio $\frac{\mathcal{H}^{n-1}(\partial E)}{\mathcal{L}^n(E)}$ is small, the more the set E is *good* for this energy.

The ratio $\frac{\mathcal{H}^{n-1}(\partial E)}{\mathcal{L}^n(E)}$ is optimal for the balls. The more a set E has an isotropic geometry, the less is this ratio. So, if we force the Mumford-Shah model to detect thin structures, it is impossible to discriminate the noise from the tubes with same section and intensity.

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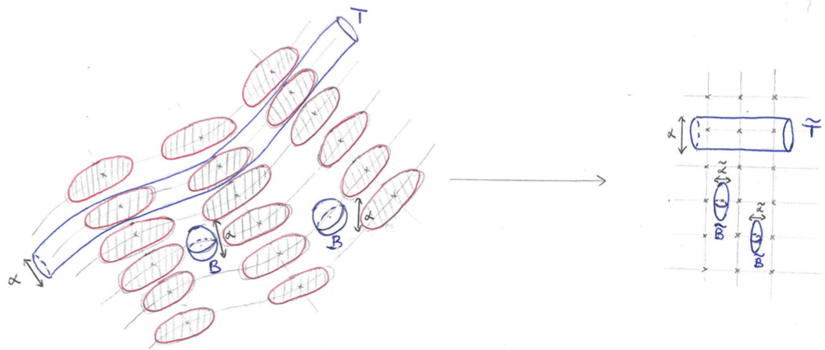
2. $\mathcal{H}^{n-1}(\partial E)$ comes from $\mathcal{H}^{n-1}(K)$, it is isotropic, we have to change it.

We will transform $\mathcal{H}^{n-1}(K)$ in such a way to incorporate a preference for sets *having a direction*. Our idea is to associate a new metric which is elongated in one direction (*anisotropic*). Moreover, this metric depends on the point of the domain (*inhomogeneous*).



For a $\mathcal{C}^1$-surface K, we replace $\mathcal{H}^{n-1}(K)$ by

$$\int_K \langle \mathbf{M}\nu, \nu \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

- ν is an unitary and orthogonal vector to K,
- $\mathbf{M}$ is a given riemannian metric, that is $\mathbf{M} : \Omega \to S_n^+$.

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The associated anisotropic energy is

$$\mathcal{E}_{\mathbf{M}}(u,K) = \int_{\Omega\setminus K} (u-g)^2 dx + \int_{\Omega\setminus K} |\nabla u|^2 dx + \int_K \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

where K is a compact $\mathcal{C}^1$-surface and $u \in W^{1,2}(\Omega \setminus K)$.

In this talk we will give answers to the questions:

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2) How construct a good approximation of this energy for numerics?

3) How construct the metric **M**?

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We recall that $u \in SBV(\Omega)$ if

i) $u \in L^1(\Omega)$,
ii) Du is a Radon measure,
iii) $Du = \nabla u \cdot \mathrm{d}x + (u^+ - u^-) \cdot d\mathcal{H}^{n-1} \llcorner J_u$.

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$SBV(\Omega)$ is a Banach space and J_u has the following structure.

Theorem

If $u \in SBV(\Omega)$, then J_u is $\mathcal{H}^{n-1}$-rectifiable, i.e. $J_u = N \cup C$ with

- $\mathcal{H}^{n-1}(N) = 0$,
- $C = \bigcup_{i=1}^{\infty} \Gamma_i$ is a countable union of $\mathcal{C}^1$ compact surface.

We give the relaxed formulation

$$\mathcal{E}_{\mathbf{M}}(u,K) = \int_{\Omega\setminus K}(u-g)^2 dx + \int_{\Omega\setminus K}|\nabla u|^2 dx + \int_K \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

$$\downarrow$$

$$\mathcal{E}^{\mathrm{r}}_{\mathbf{M}}(u) = \int_{\Omega}(u-g)^2 dx + \int_{\Omega}|\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

where $u \in SBV(\Omega)$ and ν_u is unitary and orthogonal to J_u.

We use the *direct method* of calculus of variations.

Théorème [Ambrosio, '95, lower semicontinuity]

If $(u_k)_k \subset SBV(\Omega)$ satisfies

$$\sup_k \left\{ \int_\Omega |\nabla u_k|^2 dx + \mathcal{H}^{n-1}(J_{u_k}) \right\} < \infty$$

and $(u_k)_k$ converge $\star$-weakly to $u \in SBV(\Omega)$, then $(\nabla u_k)_k$ converge to ∇u in $L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 dx,$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}).$$

Theorem [Ambrosio, '95, compactness]

Let $(u_k)_k \subset SBV(\Omega)$ be as in the previous theorem. If $\|u_k\|_\infty$ is uniformely bounded in k, then there exists a subsequence of $(u_k)_k$ which converges $\star$-weakly to $u \in SBV(\Omega)$.

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If **M** satisfies

i) ellipticity:

$$\exists \lambda > 0, \exists \Lambda > 0, \forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \lambda |\mathbf{v}|^2 \leq \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v} \rangle \leq \Lambda |\mathbf{v}|^2,$$

ii) Hölder-regularity:

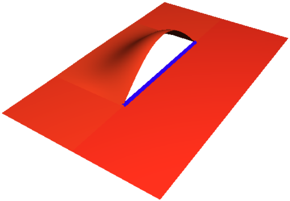
$$\exists \alpha > 0, \exists C \geq 0, \forall (x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq C|x - y|^\alpha.$$

then we adapt the two previous theorems and prove that there exits a minimizer $u \in SBV(\Omega)$ of $\mathcal{E}^{\mathrm{r}}_{\mathbf{M}}$.

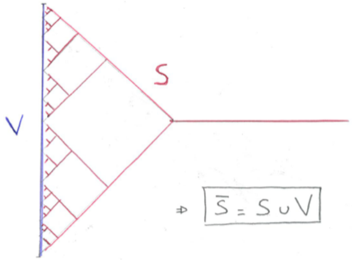
We have

$$\mathrm{Min}\mathcal{E}^{\mathrm{r}}_{\mathbf{M}} \leq \mathrm{Min}\mathcal{E}_{\mathbf{M}},$$

$$\left.\begin{array}{c} u \text{ is a minimizer of } \mathcal{E}^{\mathrm{r}}_{\mathbf{M}} \\ \mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0 \end{array}\right\} \Rightarrow (u, \overline{J_u}) \text{ is a minimizer of } \mathcal{E}_{\mathbf{M}}.$$



In general $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$ is not true. For example, we may construct a function *SBV* such that $J_u = S$ and $\overline{J_u} = S \cup V$ as below



To prove that a minimizer of the relaxed problem provides a minimizer of the initial problem, it remains to prove that this situation is impossible.

The *bad situation* for J_u is due to the fact that the *complexity* of S is increasing at its boundary. More precisely, we introduce the local entropy as follows

$$I_x(\rho) = \frac{1}{\rho^{n-1}} \left(\mathcal{H}^{n-1}(J_u \cap \overline{B_\rho(x)}) + \int_{B_\rho(x)} |\nabla u|^2 dx \right).$$

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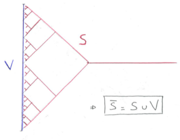
We have the typical behavior



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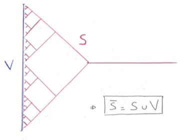

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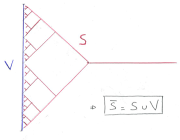


- if $x \in S$ then $I_x(\rho)$ is decreasing when $\rho \to 0$,
- if $x \in V$ then $I_x(\rho)$ tends to $+\infty$ when $\rho \to 0$.

A minimizer u of the relaxed problem satisfies the following.

Definition

A function $u \in SBV(\Omega)$ is said (Λ, c)-quasi minimizer of a free boundary problem if for any $x \in \Omega$ and $v \in SBV(\Omega)$ such as $\{u \neq v\} \subset B_\rho(x)$, we have

$$\begin{aligned} \int_{B_\rho(x)} |\nabla u|^2 dx &+ \mathcal{H}^{n-1}(J_u \cap \overline{B_\rho(x)}) \\ \leq \int_{B_\rho(x)} |\nabla v|^2 dx &+ \Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B_\rho(x)}) + c\rho^n. \end{aligned}$$

Theorem [Bucur, Luckhaus, '14, Monotonicity formula]

Let $u \in SBV(\Omega)$ a (Λ, c)-quasi minimizer of a free boundary problem, then

$$\rho \to I(\rho) \wedge \frac{c\Lambda^{2-n}}{n-1} + (n-1)c\rho$$

is decreasing in a neighborhood of 0^+.

This yields

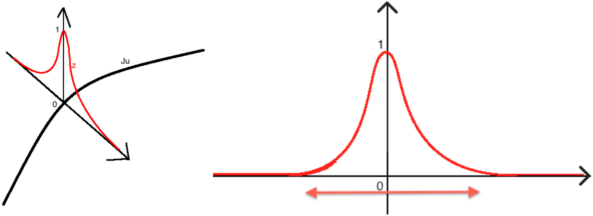
Theorem

Let $u \in SBV(\Omega)$ a (Λ, c)-quasi minimizer of a free boundary problem, then $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$.

So, a solution of the relaxed problem provides a regular solution.

To give a formulation of this problem more suitable for numerics we replace the $\mathcal{C}^1$ surface $K \subset \Omega$ by a smooth function $z : \Omega \to [0;1]$. It is a sort of *diffusion* of the surface in the following sense:

$$z(x) \approx \begin{cases} 1 & \text{if } x \text{ is closed to } K, \\ 0 & \text{otherwise,} \end{cases}$$



We approximate

$$\mathcal{E}_{\mathbf{M}}(u,K)=\int_{\Omega}(u-g)^2dx+\int_{\Omega\setminus K}|\nabla u|^2dx+\int_K\langle\mathbf{M}\nu,\nu\rangle^{\frac{1}{2}}d\mathcal{H}^{n-1},$$

by

$$\mathcal{E}_{\varepsilon}(u,z)=\int_{\Omega}\left((u-g)^2+|\nabla u|^2(1-z)^2+\varepsilon\langle\mathbf{M}\nabla z,\nabla z\rangle+\frac{z^2}{4\varepsilon}\right)dx.$$

This functional is defined on $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ and it is elliptic. So, it may be minimized by a non linear heat equation.

To justify this approximation, we prove the following result.

Theorem (Γ-convergence)

If $(\varepsilon_k)_k$ converges to 0^+ then the functionals $(\mathcal{E}_{\varepsilon_k})_k$ Γ-converges to $\mathcal{E}_{\mathbf{M}}$.

If (u_k, z_k) is a minimizer of $\mathcal{E}_{\varepsilon_k}$, then there exists a subsequence of $(u_k, z_k)_k$ which converges almost everywhere to $(u, 0)$ and u is a minimizer of $\mathcal{E}_{\mathbf{M}}$.

The proof of Γ-convergence is divided in two steps.

Theorem (lim-inf of Γ-convergence)

If $(\varepsilon_k)_k$ converges to 0^+ and $(u_k, z_k)_k$ converges almost everywhere to $(u, 0)$, then

$$\liminf_{k\to\infty} \mathcal{E}_{\varepsilon_k}(u_k, z_k) \geq \mathcal{E}^{\mathsf{r}}_{\mathbf{M}}(u).$$

We reduce to dimension one by using a *Slicing Property* of *SBV* functions.

Approximation of Functionals Depending on Jumps by Elliptic Functionals via Gamma-Convergence,1990, Ambrosio-Tortorelli

The second step of the proof consists in showing that the previous lower bound is optimal. For that, we recall the upper and lower Minkowski contents of a set.

$$\mathcal{M}^{\star}(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega : \operatorname{dist}(x, S) < \rho\})}{2\rho},$$

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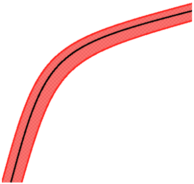
when it coincides, we call it the Minkowski contents of a set $\mathcal{M}(S)$.



We introduce an anisotropic and inhomogeneous version of the Minkowski content.

$$\mathcal{M}^{\star}_{\mathbf{M}}(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}_{\mathbf{M}}(x,S) < \rho\})}{2\rho}$$

where $\mathrm{dist}_{\mathbf{M}}$ is the distance associated to the dual metric of $\mathbf{M}$.

Theorem

If $(\varepsilon_k)_k$ converges to 0^+ and $u \in SBV(\Omega)$, then there exists $(u_k, z_k)_k$ converging to $(u, 0)$ a.e. such as

$$\limsup_{k \to \infty} \mathcal{E}_{\varepsilon_k}(u_k, z_k) \leq \int_{\Omega} (u-g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \mathcal{M}^{\star}_{\mathbf{M}}(J_u).$$

Theorem, Geometric Measure Theory, Federer

If W is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$, then

$$\mathcal{M}(W) = \mathcal{H}^{n-1}(W).$$

We generalize this result to the global setting of an anisotropic and inhomogeneous metric as follows.

Theorem

If $\mathbf{M}$ is elliptic and Holder regular and if W is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$, then

$$\mathcal{M}_{\mathbf{M}}(W) = \int_W \langle \mathbf{M}\nu, \nu \rangle \mathrm{d}\mathcal{H}^{n-1},$$

where ν is an unitary and orthogonal vector to W.

According to the Entropy Decay Property, we prove the following.

Theorem

If $u \in SBV(\Omega)$ is a quasi minimizer of a free boundary problem then

$$\mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle d\mathcal{H}^{n-1},$$

where ν is an unitary and orthogonal vector to W.

Theorem, Upper Γ-limit

For any $u \in SBV(\Omega)$, there exists $(u_k, z_k)_k$ converging a.e. to $(u, 0)$ a.e.

$$\limsup_{k\to\infty} \mathcal{E}_{\varepsilon_k}(u_k, z_k) \leq \int_\Omega (u-g)^2 + \int_\Omega |\nabla u|^2 + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle d\mathcal{H}^{n-1}.$$

We give a definition adapted to dimension 2. For that, we search for an unitary vector field $\mathbf{c} : \Omega \to \mathbb{S}^1$ following the direction of the tubes.

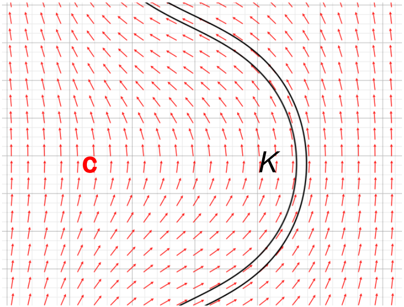


Figure : Vector field **c** along a tube K

We introduce

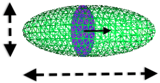
$$F(\mathbf{c}) = \int_\Omega \langle Dg, \mathbf{c}\rangle^2 \mathrm{d}x + \int_\Omega |D\mathbf{c}|^p \mathrm{d}x$$

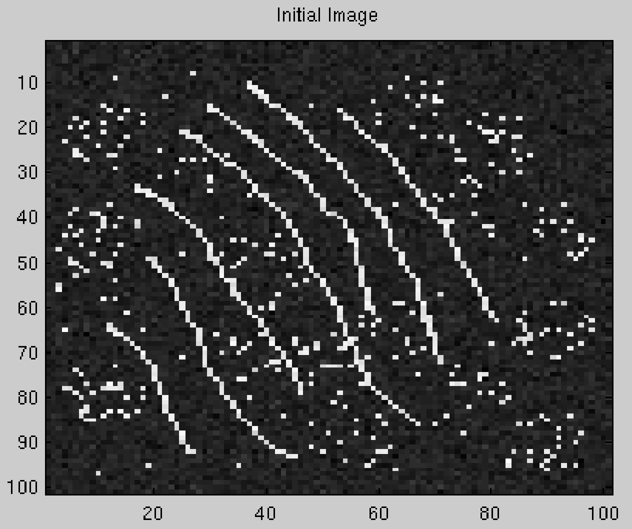
$$(\mathcal{P}_c): \quad \min\{F(\mathbf{c}) : \mathbf{c}(x) \in \mathbb{S}^1 \; a.e. \; x \in \Omega, \mathbf{c} \in W^{1,p}(\Omega)\}.$$

If we set $p > 2$ then, Sobolev embedding Theorem ensures that $\mathbf{c}$ is α-Holder regular with $\alpha = 1 - \frac{2}{p}$. It is easy to prove that a solution $\mathbf{c}_0$ of $(\mathcal{P}_c)$ exists and we set

$$\mathbf{M} = \mathrm{Id} + \mu\, {}^t\mathbf{c}_0 \mathbf{c}_0,$$

where $\mu > 0$ corresponds to the elongation of its unit ball.

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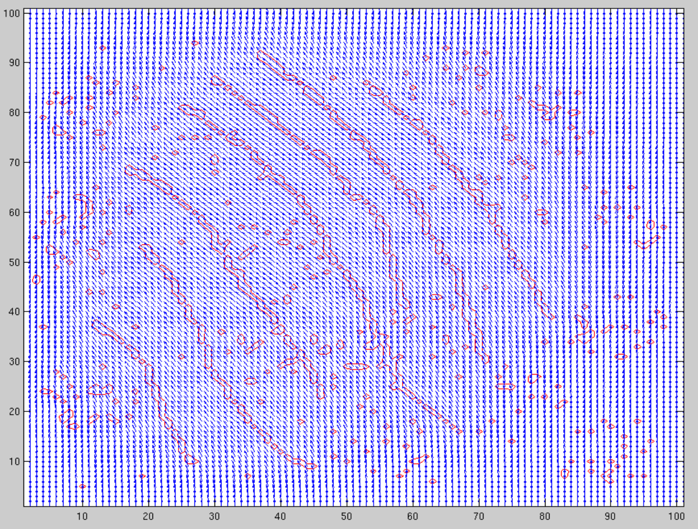
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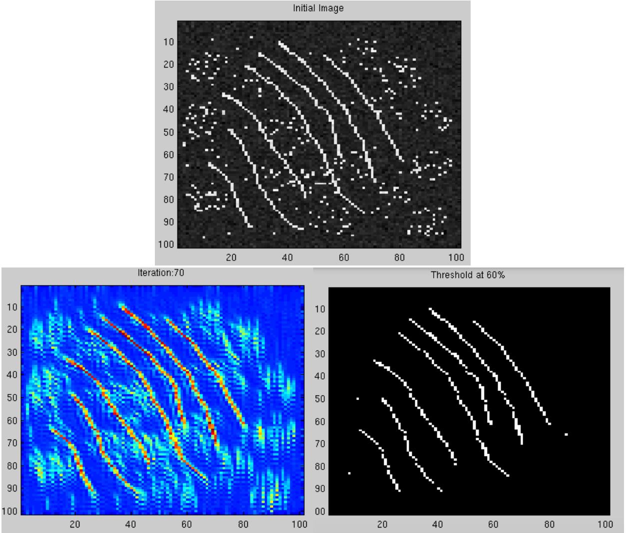
Initial Image
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20 40 60 80 100
Iteration:70
10 20 30 40 50 60 70 80 90 100
20 40 60 80 100
Threshold at 60%
10 20 30 40 50 60 70 80 90 00
20 40 60 80 100

In dimension 3, the previous approach is not adapted. In fact, a vector field can avoid *lateraly* a tube without penalizing the regularization term $\int_\Omega |D\mathbf{c}|^p dx$.

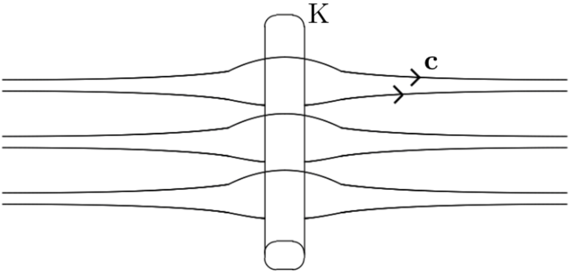

Figure : Vector field **c** avoiding lateraly a tube K

To overpass this problem, we introduce the second order derivative of **H** of g and the following minimization problem

$$F(\mathbf{M}) = \int_\Omega \|\mathbf{M} - \mathbf{H}\|^2 \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^p \mathrm{d}x$$

and the following minimization problem

$$(\mathcal{P}_H): \quad \min\{F(\mathbf{M}) : \mathbf{M} \text{ satisfies Ellipticity condition}, \mathbf{M} \in W^{1,p}(\Omega)\}.$$

If we assume that $\mathbf{H} \in L^2(\Omega)$, then it easy to prove that this problem admits a solution $\mathbf{M}_0$. As for the 2D case, we assume that $p > 3$ and Sobolev embedding Theorem ensures that $\mathbf{M}_0$ satisfies Holder regularity.

To sum things up, this approach has been developed for

- Modica-Mortola Energy,

  Anisotropic Bimodal Energy for Segmentation of thin tubes and its approximation with Γ-convergence, Advances in Calculus of Variation, V., 2015

- Mumford-Shah Energy, (Thesis Manuscript (in writing), HAL-01132067 (for the first part of this talk)).

It may be extended to another variational problems. The principle is to include the geometric prior inside the regularization term of the energy.

Thank you for your attention!