A Robust Signal Quantization
Based on Error Correcting Codes

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Abstract

We propose a quantization method that makes a signal correctable after being corrupted by noise. The principle is to replace each bit level in the binary representation of each vector of time or time-frequency samples by a codeword of the same length, provided by an error-correction coder. Hence, assuming that the amount of errors does not exceed the correction capabilities, decoding each bit level of the noisy signal can suppress the noise.

We show that the required error correction capability for each bit level can be determined from the signal and noise probability density functions and the targeted binary error probability. The codewords are chosen from codebooks with the adequate error correction capabilities so as to minimize the quadratic error between the original and the coded signals. For this purpose, we use a modified matching pursuit algorithm.

Applying this quantization method on a speech signal transmitted over a noisy channel demonstrates that it is possible to choose a set of coders so that the noise resulting from the coding-channel-decoding chain is less annoying than the noise of the channel alone.

Keywords: quantization; error correcting codes; denoising; matching pursuit

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1. Introduction

We consider the transmission of a signal over any noisy channel, be it analog or digital. Our goal is to find a signal representation that makes it robust to the channel noise, while staying in the original space of representation. For instance, for a speech signal, the channel can consist of the whole communication chain between the microphone and the earpiece in a mobile communication, equating the global impairment to an additive noise.

This approach differs from the classical source coding [11] plus channel coding approach, where the final result is a binary stream [17]. It differs also from the joint source-channel coding, which converts the signal into a representation in the modulation space [18], [4], [19]. Here, a speech signal (for instance) remains a speech signal.

Our proposal is however inspired both by source and channel coding. As in source coding, the representation will be based on a projection into a reduced set of atoms, with respect to some objective criterion of fidelity to the original signal. While source coding uses a code-book based on the data that minimizes a mean square error criterion, we will use a code-book that maximizes the distance between its elements, so that the latter are robust to noise. For this purpose, the code-book will be based on error correction codes [2].

The principle is the following. According to the classical binary decomposition, each vector of n samples is represented as a weighted sum of binary vectors of dimension n, where the weights are successive powers of 2. Our coding consists in replacing each binary vector by a code word of length n so that the new vector of n samples is as close as possible to the original one.

We will describe this principle in Section 2. Section 3 is devoted to the choice of the adequate error correction coders according to the signal and noise respective distributions. In Section 4, we will specify how to approximate the signal using these code-books and the decoding process to cancel the channel noise. The whole method is illustrated in Section 5 by an example of speech corrupted by impulsive noise.

2. Principle

2.1. Binary representation of the signal

Let $x = [x_0, x_1, \ldots, x_{n-1}]$ be a block of n samples of a numerical signal in a given space of representation (e.g. time samples or frequency coefficients). We consider $x \in A^n$ with $A = [\![-2^L + 1, 2^L - 1]\!] \subset \mathbb{Z}$, so that x can be written as a linear combination of L binary vectors:

$$x = XS_x, \quad \text{with}\, X = \sum_{i=0}^{L-1} 2^i X_i, \tag{1}$$

where $X_i \in \mathbb{F}_2^n = (\mathbb{Z}/2\mathbb{Z})^n$ for $0 \leq i \leq L-1$ and S_x is a diagonal matrix containing the signs of x, that is $(S_x)_{ii} = \text{sign}(x_{i-1}), i = 1, \ldots, L$.

2.2. Coding and decoding

The principle of the proposed quantization is to replace each vector X_i by a codeword C_i generated by an error correcting coder $\mathcal{C}(n, k_i)$, where k_i is the code dimension, yielding a new samples vector $\tilde{x}$:

$$\tilde{x} = \tilde{X}S_x, \quad \text{with}\, \tilde{X} = \sum_{i=0}^{L-1} 2^i C_i, \tag{2}$$

so that the new vector $\tilde{x}$ is as close as possible to x. The definition of "close" will be discussed in Section 4. Note that this quantization lets the signs unchanged.

Now let us suppose that the quantized signal $\tilde{x}$ is corrupted by an additive noise. The resulting signal y can be written as in Equation (1):

$$y = YS_y, \quad \text{with} \quad Y = \sum_{i=0}^{L-1} 2^i Y_i, \tag{3}$$

The original codewords $C_i, 0 \leq i \leq L-1$, can possibly be retrieved from the vectors Y_i using classical decoding methods, thus allowing to cancel the noise and recover $\tilde{x}$, assuming $S_y = S_x$. This decoding process will be specified in Section 4.

2.3. Block coded modulations

The choice of coder set will be discussed in detail in Section 3. One can intuitively foresee that the least significant bits are more sensitive to errors than the most significant bits. From this perspective, this quantization can be compared to the block coded modulation (BCM [13]), which jointly optimizes the coding and the modulation.

The principle of BCM is illustrated by Fig. 1. Considering a M-ary modulation with $M = 2^m$, this coding exploits the fact that the most significant bits are less vulnerable to the noise of the transmission channel. To transmit $k_1 + k_2 + \ldots + k_m$ bits, one codes each word of k_i bits by a block code of length $n \geq k_i$. The resulting $m \times n$ binary matrix is then transmitted as n M-ary symbols. The higher the bit weight, the lower the error rate. Therefore several codes of decreasing error-correction capabilities are used. Codes with higher error-correction capabilities encode the lines of least significant bits, while the lines of most significant bits are encoded by codes with low error-correction capabilities. In other terms, we have $k_1 \leq k_2 \leq \ldots \leq k_m$.

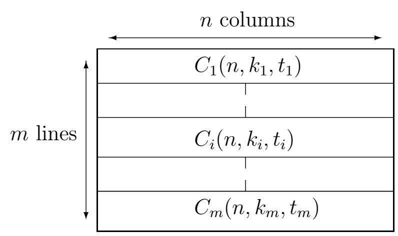

Figure 1: Principle of a block coded modulation.

While the classical coding process transforms words of length k into words of length $n > k$, our quantization keeps the binary vectors in the same dimension n, making it similar to a decoding process.

2.4. Comparison with vector quantization

The proposed scheme is a form of vector quantization: instead of quantizing each sample separately, we code sample vectors. Our method differs however from the classical approach of vector-quantization [10], [9], [16] in two ways.

Firstly, while vector-quantizing x would be an application from A^n to a subset of A^n, our quantization consists of L applications from $\mathbb{F}_2^n$ to subspaces $\mathbb{D}_i$ of dimensions k_i of $\mathbb{F}_2^n$, which can be independent from each other.

Secondly, while vector quantization relies on code-books built from the data by minimizing a mean square error criterion, the code-books here are independent from the data and are built according to the criterion of maximization of the Hamming distance between two code-words.

3. Choosing the error correction coders

3.1. Setting the adequate correction capability

Given code-words of length n corrupted with j binary errors, we denote subsequently by $N_{decod}(j,n)$ the corresponding mean number of binary errors after decoding. Obviously, $N_{decod}(j,n)$ depends on the selected coder.

Proposition 1. *Consider code-words of length n generated by an error correcting coder with a correcting capability t and transmitted by a binary symmetric channel with a binary error probability P_e. Then the binary probability of error after decoding is given by:*

$$P_t^{decod} = \frac{1}{n} \sum_{j=t+1}^{n} N_{decod}(j,n) \binom{n}{j} P_e^j (1-P_e)^{n-j}, \tag{4}$$

Proof. Let A_j = "j erroneous bits in the word before decoding" and B = "erroneous bit after decoding". The probability P_t^{decod} can be written as:

$$P_t^{decod} = \sum_{j=0}^{n} \Pr(B|A_j) \Pr(A_j) \tag{5}$$

where:

$$\Pr(A_j) = \binom{n}{j} P_e^j (1-P_e)^{n-j} \tag{6}$$

and

$$\Pr(B|A_j) = \frac{1}{n} N_{decod}(j,n) \tag{7}$$

Since $N_{decod}(j,n) = 0$ for $j \leq t$, it comes (4). □

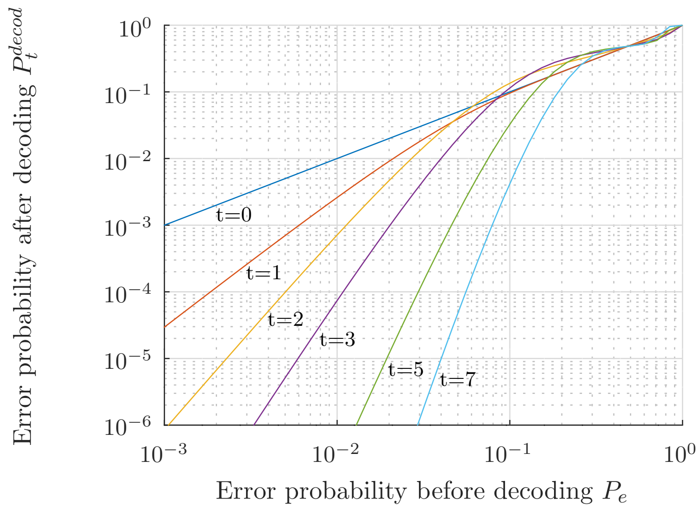

Figure 2: For BCH codes of length 31 with various correction capabilities, error probability after decoding *vs* error probability before decoding.

Figure 2 illustrates the relationship given by Proposition 1 for BCH codes [3] of length 31. Hence, knowing the error probability for a given bit level in the binary representation of the signal allows one to set the adequate correction capability of the coder according to the targeted error probability after decoding. For this purpose, the error probability before decoding must be determined for each bit level. These errors depend on the probability density functions of the signal and of the channel noise.

3.2. Error probabilities at the bit level

We consider here the addition of a signal sample s and a noise sample v, in the sign plus absolute value representation on L bits, that is $\pm s_{L-1} \dots s_0$ and $\pm v_{L-1} \dots v_0$, respectively, where s_i and $v_i \in \{0,1\}$ for $0 \leq i \leq L-1$. If their signs are different, we consider the subtraction $||s| - |v||$. Let c_i denote the carry at range i resulting from the elementary addition or subtraction at range $i-1$ for $i \geq 1$, and set $c_0 = 0$. Then an error occurs on the i^{th} bit if $v_i \neq c_i$. The carry c_{i+1} depends on s_i, v_i and c_i, in a manner that depends on the signs of s and v, and on their relative absolute values, as indicated in Tables 1 and 2.

c_i \ $s_i v_i$	00	01	11	10
0	0	0	1	0
1	0	1	1	1

Table 1: Carry c_{i+1} if s and v have the same sign.

c_i \ $a_i b_i$	00	01	11	10
0	0	1	0	0
1	1	1	1	0

Table 2: Carry c_{i+1} if s and v have different signs. $a = \max(|s|, |v|)$ and $b = \min(|s|, |v|)$.

Proposition 2. *Consider a signal sample s corrupted by a noise v. In a sign-plus-absolute-value binary representation:*

1. *The error probability on the i^{th} bit s_i is given by:*

$$P_e(i) = \beta_i + \frac{1}{2}(\rho_i^+ + \rho_i^-)(1 - 2\beta_i), \tag{8}$$

where

$$\beta_i \triangleq \Pr(v_i = 1) = \Pr(2^i \le |v| < 2^{i+1}) + \frac{1}{2}\Pr(|v| \ge 2^{i+1}) \tag{9}$$

$$\rho_i^+ \triangleq \Pr(c_i = 1 \mid \text{sign}(v) = \text{sign}(s)) \tag{10}$$

$$\rho_i^- \triangleq \Pr(c_i = 1 \mid \text{sign}(v) \neq \text{sign}(s)); \tag{11}$$

2. *The conditional probabilities ρ_i^+ and ρ_i^- are 0 for $i = 0$ and, for $i \ge 0$, they satisfy*

$$\rho_{i+1}^+ = \rho_i^+(\alpha_i + \beta_i - 2\alpha_i\beta_i) + \alpha_i\beta_i \tag{12}$$

$$\begin{aligned}\rho_{i+1}^- = {} & \Pr\left(b \ge 2^i \mid a \ge 2^i, \text{sign}(v) \neq \text{sign}(s)\right) \\ & \times \{(1 - \gamma_i - \delta_i + 2\gamma_i\delta_i)\theta_{i,i} - (1 - \gamma_i)\delta_i\}\end{aligned} \tag{13}$$

where, we have set $a \triangleq \max(|v|, |s|)$ and $b \triangleq \min(|v|, |s|)$,

$$\alpha_i \triangleq \Pr(s_i = 1) = \Pr(2^i \le |s| < 2^{i+1}) + \frac{1}{2}\Pr(|s| \ge 2^{i+1}) \tag{14}$$

$$\gamma_i = \Pr(a_i = 1 \mid a \ge 2^i, \text{sign}(v) \neq \text{sign}(s)) \tag{15}$$

$$\delta_i = \Pr(b_i = 1 \mid b \ge 2^i, \text{sign}(v) \neq \text{sign}(s)) \tag{16}$$

and where $\theta_{i,j}$ is defined for $j \ge i$, by $\theta_{0,j} = 0, \forall\, j$ and for $j \ge i \ge 1$,

$$\theta_{i,j} = \frac{1}{2}(\theta_{i-1,j} + \delta_{i-1})\Pr(b \ge 2^{i-1} \mid a \ge 2^j, \text{sign}(v) \neq \text{sign}(s)) \tag{17}$$

The proof of Proposition 2 is given in Appendix, with more details for the expressions of γ_i, δ_i and $\theta_{i,j}$.

From this proposition, knowing the signal and noise probability density functions allows to compute the error probability at each bit weight. As an example, let us compute it for Laplacian signal and noise, which has the advantage of allowing easy calculations (reminding that the probabilities involved in Proposition 2 imply bi-dimensional integration).

Let:

$$\begin{aligned} \mathrm{f} : & \mathbb{R} \times \mathbb{N} \to \mathbb{R} \\ & (\sigma, i) \mapsto \exp\left(-\frac{2^i\sqrt{2}}{\sigma}\right) \end{aligned} \tag{18}$$

Considering a signal and a noise with zero mean and standard deviations σ_s and σ_v, respectively, one can easily derive from Proposition 2 and basic integral calculus the following formulas:

$$\alpha_i = \mathrm{f}(\sigma_s, i) - \frac{1}{2}\mathrm{f}(\sigma_s, i+1) \tag{19}$$

$$\beta_i = \mathrm{f}(\sigma_v, i) - \frac{1}{2}\mathrm{f}(\sigma_v, i+1) \tag{20}$$

$$\gamma_i = 1 - \frac{1}{2}\frac{\mathrm{f}(\sigma_s, i+1) + \mathrm{f}(\sigma_v, i+1) - \mathrm{f}(\sigma_{sv}, i+1)}{\mathrm{f}(\sigma_s, i) + \mathrm{f}(\sigma_v, i) - \mathrm{f}(\sigma_{sv}, i)} \tag{21}$$

$$\delta_i = 1 - \frac{\mathrm{f}(\sigma_{sv}, i+1)}{2\mathrm{f}(\sigma_{sv}, i)} \tag{22}$$

$$\begin{aligned} \theta_{i\geq 1, j\geq i} = & (\theta_{i-1,j} + \delta_{i-1}) \\ & \times \frac{1}{2}\frac{\mathrm{f}(\sigma_v, i-1)\mathrm{f}(\sigma_s, j) + \mathrm{f}(\sigma_s, i-1)\mathrm{f}(\sigma_v, j) - \mathrm{f}(\sigma_{sv}, j)}{\mathrm{f}(\sigma_s, j) + \mathrm{f}(\sigma_v, j) - \mathrm{f}(\sigma_{sv}, j)} \end{aligned} \tag{23}$$

$$\rho^-_{i+1} = \mathrm{f}(\sigma_{sv}, i)\left\{(1 - \gamma_i - \delta_i + 2\gamma_i\delta_i)\theta_{i,i} - (1 - \gamma_i)\delta_i\right\} \tag{24}$$

where $1/\sigma_{sv} = 1/\sigma_s + 1/\sigma_v$.

The resulting error probability is represented on Fig. 3. The independence assumptions on which the proof of Proposition 2 is based are not perfectly verified (especially a_i and b_i independence), which explains the small difference between the theoretical and the empirical curves.

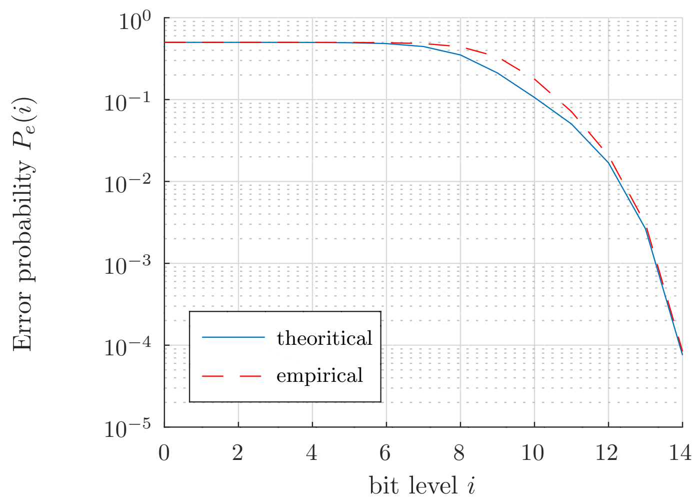

Figure 3: Error probability at each bit weight for a Laplacian signal of standard deviation 0.1×2^{15} corrupted by a Laplacian noise of standard deviation 0.01×2^{15}. The empirical probabilities are obtained from a simulation on 10^6 samples.

4. Coding and decoding

In [7], each binary vector X_i was replaced by the codeword C_i minimizing the Hamming distance, as in a classical channel decoding. While this criterion is very adequate in a purely binary setting, the ℓ_2 norm is more appropriate to measure real-valued signal distortion. Accordingly, the set of codewords $\{C_i\}_{0 \le i \le L-1}$ in (2) is determined in this paper so as to minimize the quadratic error $\|x - \tilde{x}\|_2^2$, which also reads as $\|X - \tilde{X}\|_2^2$. Note that for audio or image signals, perceptual criteria could be chosen, involving however more complexity.

To proceed, let us consider a set of error correcting coders $\{\mathcal{C}(n, k_i)\}_{0 \le i \le L-1}$, where k_i is the code dimension. Let $\{D_0, D_1, \ldots, D_{L-1}\}$ be the corresponding set of code-books, where each code-book D_i is a $2^{k_i} \times n$ binary matrix. Let H be the vertical concatenation of $D_0, 2D_1, \ldots 2^i D_i \ldots 2^{L-1} D_{L-1}$ and set

$$\Gamma = \{\alpha = (\alpha_0, \ldots, \alpha_{L-1}) \,|\, \alpha_i \in \mathbb{F}_2^{2^{k_i}} \text{ and } \|\alpha_i\|_1 = 1, \forall\, i \in [0, L-1]\}.$$

Then the optimal vector $\tilde{X}$ is:

$$\tilde{X}_{opt} = \hat{\alpha} H \tag{25}$$

where

$$\hat{\alpha} = \underset{\alpha \in \Gamma}{\arg\min} \|X - \alpha H\|_2^2 \tag{26}$$

The high dimension of this optimization problem makes it difficult to solve. We therefore use a classical greedy algorithm, as the basic matching pursuit [5, 12] which we have adapted to our case in Algorithm 1 below. At the end of the algorithm, the final residue gives $\|R_L\| = \|X - \tilde{X}\|$. Unfortunately, when running this algorithm, the norm of the residue $\|R_i\|$ increases with each iteration i. This can be explained by the following expression of the quantization error:

$$\begin{aligned} \|X - \tilde{X}\|^2 &= \|X\|^2 - 2\langle X|\tilde{X}\rangle + \|\tilde{X}\|^2 \\ &= \|X\|^2 - 2\sum_{i=0}^{L-1} \langle X|2^i C_i\rangle + \|\sum_{i=0}^{L-1} 2^i C_i\|^2. \end{aligned} \tag{27}$$

Maximizing the scalar product $\langle 2^{L-i} C_i | R_{i-1}\rangle$ favors codewords with higher Hamming weights, which increases the third term of (27). Consequently, the scalar product should be penalized by the Hamming weight of the codeword.

Algorithm 1: Modified matching pursuit algorithm for the quantization. The notation $\langle U|V\rangle$ stands for the scalar product between vectors U and V.

```
Data: the vector X to be quantized
Result: the quantized vector X̃
X̃_0 ← 0_{1×n} ; R_0 ← X
for i ← 1 to L do
    C_{L−i} ← arg max_{C∈D_{L−i}} ⟨2^{L−i} C | R_{i−1}⟩
    R_i ← R_{i−1} − 2^{L−i} C_{L−i}
    X̃_i ← X̃_{i−1} + 2^{L−i} C_{L−i}
end
return X̃ = X̃_L
```

This penalization can be seen when minimizing $\|X - \tilde{X}\|^2$ incrementally, *i.e.* by replacing in Algorithm 1 the maximization of $\langle 2^{L-i}C|R_{i-1}\rangle$ by the minimization of the norm of the new residue R_i:

$$\begin{aligned}\|R_i\|^2 &= \|R_{i-1}\|^2 - 2\langle R_{i-1}|2^{L-i}C\rangle + \|2^{L-i}C\|^2 \\ &= \|R_{i-1}\|^2 - 2^{L-i+1}\big(\langle R_{i-1}|C\rangle - 2^{L-i-1}w_C\big),\end{aligned} \tag{28}$$

where w_C denotes the Hamming weight of C. Hence, minimizing the norm of the residue R_i at each iteration means maximizing the scalar product $\langle R_{i-1}, C\rangle$ penalized by the Hamming weight of C.

This incremental decrease of the quantization error does not necessarily guarantee the global minimization of $\|X-\tilde{X}\|^2$. Consequently, we propose to replace in Algorithm 1 the maximization of $\langle 2^{L-i}C|R_{i-1}\rangle$ by the maximization of:

$$J_{\lambda,i}(C, R_{i-1}) = \langle C|R_{i-1}\rangle - \lambda 2^{L-i}w_C. \tag{29}$$

In other words, the previous idea of incremental minimization of the residue is a particular case of this approach, with $\lambda = 1/2$. The factor 2^{L-i} in (29) is used to put w_C to the same scale as $\langle C, R_{i-1}\rangle$.

The value of λ minimizing the final quantization error resulting from the modified Algorithm 1 can be empirically found: for a given set of coders, one has to code a large number of random vectors from A^n with the modified matching pursuit algorithm with various values of λ around 1/2. As an example, we consider $n = 31$ and a set of BCH coders with dimension 21 for the four most significant bits, and 16 for the other bit weights. For various probability density functions, we ran the modified Algorithm 1 on 10^3 vectors for λ between 0 and 4, and for each value of λ we computed the average quadratic quantization error, from which we deduced the signal to coding noise ratio (SNR). As illustrated by Fig. 4, the maximum SNR is reached for $\lambda = 1$.

Decoding the quantized signal corrupted by an additive noise can be done using the same matching pursuit algorithm as for coding, but this does not explicitly use the properties of the error correcting codes. Another approach consists in decomposing each received block y of n samples according to (3),

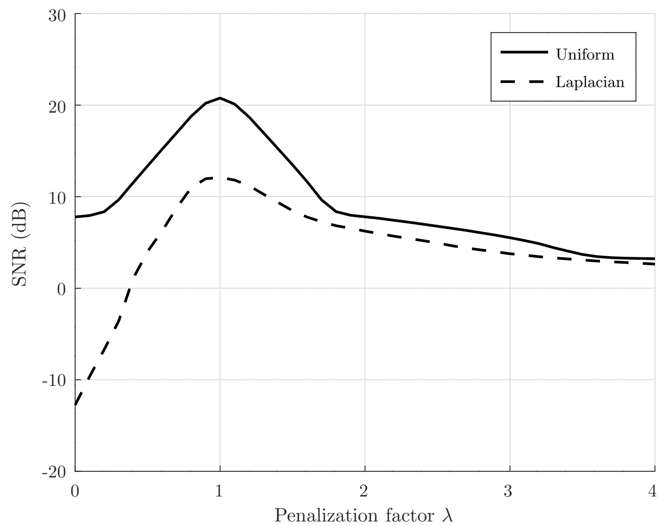

Figure 4: Signal to coding noise ratio (SNR) according to the value of the penalization factor λ in the modified matching pursuit algorithm, for two distributions of the signal.

and retrieving the original codewords $C_i, 0 \leq i \leq L-1$, from the vectors Y_i using classical decoding methods. This approach may benefit from the low computational complexity of these methods. Using the formalism of Subsection 2, the estimated block $\hat{x}$ is given by:

$$\hat{x} = \hat{X} S_y, \quad \text{with}\, \hat{X} = \sum_{i=0}^{L-1} 2^i \hat{X}_i, \tag{30}$$

where

$$\forall\, 0 \leq i \leq L-1, \quad \hat{X}_i = \mathsf{decod}(Y_i), \tag{31}$$

where $\mathsf{decod}(\cdot)$ denotes an error decoding function providing the closest codeword.

Note that only the absolute value of each sample can be corrected this way, since the signs of the noisy block y are kept. If a signal sample is corrupted by a stronger noise sample having an opposite sign, this may create a peaky error. That is why we propose the following sign correction. For each sample that was erroneous in y, we correct the sign according to an adaptive linear prediction from the neighboring past and future samples. If the opposite of the corrected sample is closer from its predictor than the corrected sample itself, we replace the latter by its opposite. See Algorithm 2 for details.

Algorithm 2: Joint adaptive prediction and sign correction for samples which absolute value was corrected. We use the normalized least mean square algorithm [15] to identify the prediction coefficients a.

Data: the absolute-value-corrected samples $\hat{x}(n)$

Result: the samples $\hat{x}(n)$ with corrected signs

Set adaptation step μ and model order p

$a \leftarrow 1_{2p\times 1}/2p$

for $i \leftarrow 0$ *to* N **do**
  $V \leftarrow [\hat{x}(n-p)\dots\hat{x}(n-1)\hat{x}(n+1)\dots\hat{x}(n+p)]$
  $\hat{x}_{pred} \leftarrow Va$
  if $\hat{x}(n) \neq y(n)$ **then**
    if $|\hat{x}_{pred} + \hat{x}(n)| < |\hat{x}_{pred} - \hat{x}(n)|$ **then**
      $\hat{x}(n) \leftarrow -\hat{x}(n)$
    end
  end
  $e \leftarrow \hat{x}(n) - \hat{x}_{pred}$
  $a \leftarrow a + \mu e V^{\top}/\|V\|^2$
end

5. Application to speech denoising

As indicated by Fig. 4, the coding SNR can be low, depending on the signal distribution and on the chosen set of coders. A trade-off must be found between the coding noise and the ability to correct errors, so that the signal after the coding-channel-decoding chain is better than after the channel alone. According to the experiments of [6], it seems difficult to have a global SNR of the coding-channel-decoding chain better than that of the channel alone. That is why the proposed technique is practically appropriate for two cases.

The first case is that of artificial signals built from the code-books, to which the notion of fidelity to an original signal does not strictly apply. This was almost the case studied in [7]: although the coded alarms where built from recordings of real alarms, with a low coding SNR, they respected the expected features of such alarms, namely their time-frequency signatures.

The second case concerns sounds and images, and consists of a channel which impairment is perceptually more annoying than that of the coding-channel-decoding chain, although it may be better in terms of SNR. In this section, we will present experimental results for this case: we will consider speech over a channel adding sporadic noise, occurring by bursts.

The signal is a 4s speech signal from a male speaker, sampled at 8000 Hz, which distribution can be modeled by a Laplacian law [8]. We consider a Laplacian noise occurring with a probability of 5%, having the same variance as the speech signal. Using Proposition 2 with formulas (18) to (24), we computed the theoretical error probability at each bit level. Then, from these probabilities, using Proposition 1, we deduced the theoretical error probability after decoding for each bit level and each correction capability, when using BCH codes of length 31. These probabilities are represented by Fig. 5. From these results, we chose error correcting capabilities in order to have an error probability around 10^{-3} for all bit weights: from the most significant bit to the least significant bit, the error correcting capabilities are $0, 1, 2, 3, \ldots, 3$.

In the experiment, we considered a noise occurring by bursts with random length following a Gaussian law of mean 10 samples and standard deviation 2. Consequently, to scatter the error, we used interleaving with a 20×31 matrix. The complete process is summarized by Fig. 6.

The resulting theoretical and empirical error probabilities before and after decoding are represented on Fig. 7. Thanks to these low binary error probabilities after decoding, the channel noise is almost canceled: only rare isolated samples are erroneous. Considering the global transmission chain however, the coding-channel-decoding process replaced the sporadic annoying noise of the channel by a continuous quantization noise. We estimated the perceptual effect

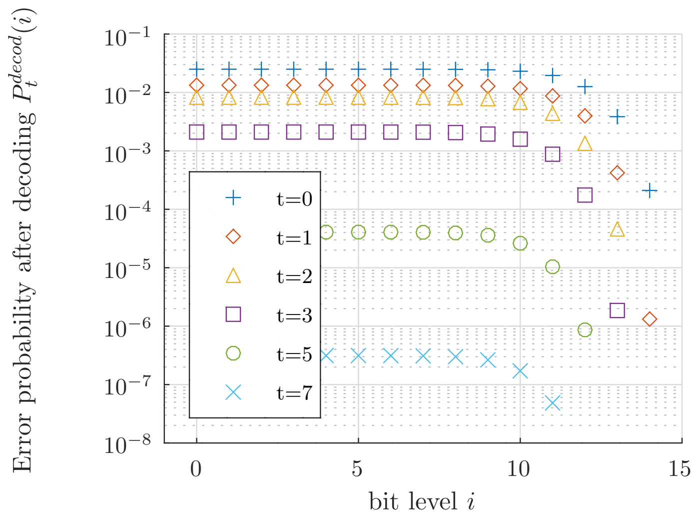

Figure 5: For a speech signal corrupted by a Laplacian noise with the same variance occurring with a probability of 5%, theoretical error probability after decoding for each bit level and each correction capability t, using BCH codes of length 31. The probability before decoding corresponds to the correction capability 0.

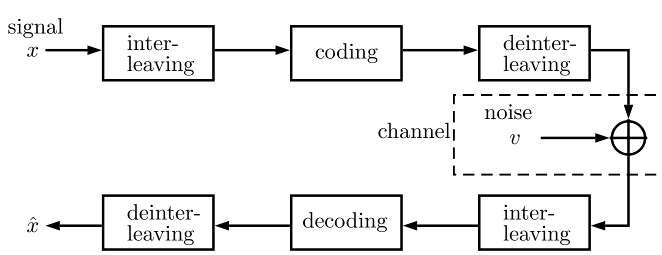

Figure 6: Communication chain through a noisy channel, including the proposed coding and interleaving.

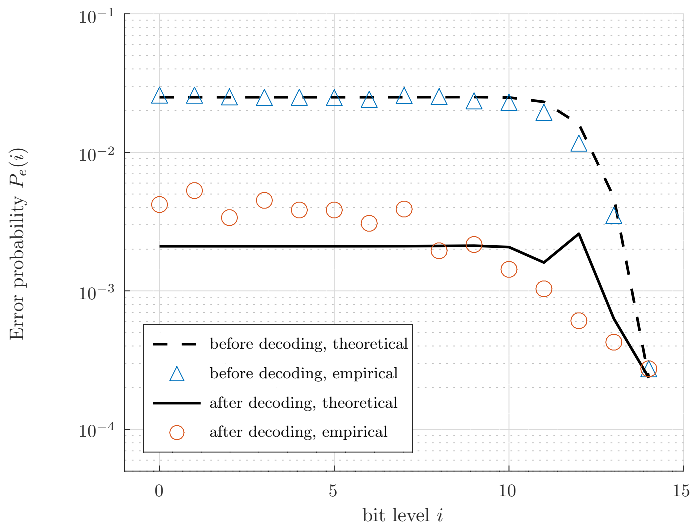

Figure 7: For a speech signal corrupted by a Laplacian noise with same variance and probability of occurrence 5%, error probabilities before and after decoding, using BCH codes of length 31 and error correcting capabilities $0, 1, 2, 3, \ldots, 3$ from the most significant bit to the less significant bit.

through the instrumental quality evaluation provided by PESQ [1]. As indicated by the middle line of Table 3, the quality of the coded speech is poor, but, considering the global communication chain, introducing the coding-decoding process enhances the audio quality of the received speech. Note that the rare errors remaining after decoding are sufficient to noticeably impair the decoded speech relatively to the coded speech.

The perceptual quality of the quantized speech can be improved through a spectral reshaping of the quantization noise. Since the latter is a white noise, quantization noise shaping can be performed by filtering the signal to be coded by a flattening filter and applying to the coded signal a coloring filter having the inverse frequency response, so that the quantization noise has a power spectral density with the same shape as that of the signal [14]. The same processing have to be done before and after the decoder. Fig. 8 summarizes this proposition. The flattening filter should ideally adapt to the signal, based on a block-by-block

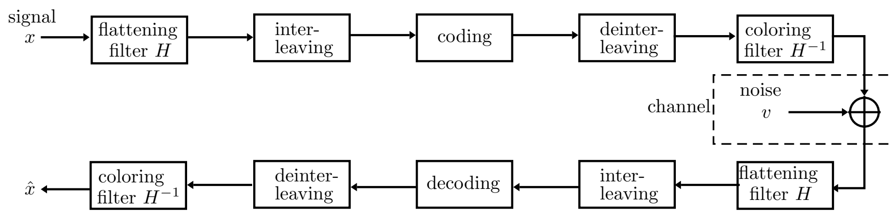

Figure 8: Communication chain through a noisy channel, including the proposed coding, interleaving and noise spectral shaping.

spectral analysis or on an adaptive model identification. On the other hand, it must be identical in the coding and the decoding parts. Since the signal to be decoded is corrupted by the channel noise, fulfilling these two features is not possible without regularly transmitting the filter coefficients. Consequently, we used a fixed flattening filter roughly adapted to the long-term spectrum of speech, of transfer function $1 - 0.8z^{-1}$.

Since this filtering modifies the signal and noise distributions, even with a gain factor preserving the variance, the theoretical model used to build Fig. 5 and to choose the error correction capabilities does not hold anymore, so that we chose the latter from the empirical error probabilities before decoding and from Fig. 2, so as to get an error probability around 10^{-3} after decoding, as in the previous case. Fig. 9 represents the error probabilities before and after coding. The resulting mean opinion scores estimated by PESQ are given by the last line of Table 3. The proposed rough noise shaping yields a clear enhancement of the coded speech. Since the decoder does not correct the errors as well as in the previous case, the overall quality enhancement of the whole communication chain is however not so high.

The audio files can be heard at:
`https://helios2.mi.parisdescartes.fr/~mahe/Recherche/robustSignals/`

6. Conclusion

We have proposed a quantization method that makes a signal correctable after being corrupted by noise, by approximating it as a concatenation of weighted

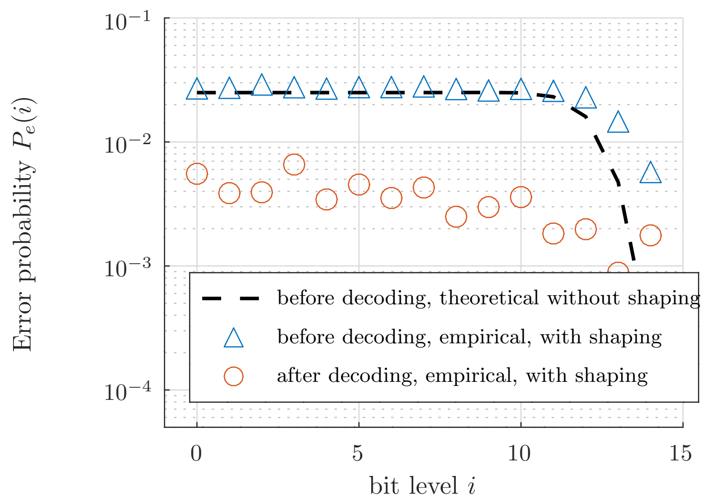

Figure 9: For a speech signal corrupted by a Laplacian noise with same variance and probability of occurrence 5%, error probabilities before and after decoding, in the communication chain of Fig. 8, using BCH codes of length 31 and error correcting capabilities $1, 3, \ldots, 3$ from the most significant bit to the less significant bit.

	$\tilde{x}$ *vs.* x	$\hat{x}$ *vs.* $\tilde{x}$	$\hat{x}$ *vs.* x	$x + v$ *vs.* x
without shaping	1.98	2.99	1.83	1.55
with shaping	2.51	2.78	2.14	1.55

Table 3: Mean Opinion Score estimations from PESQ for the coded signal $\tilde{x}$, the decoded signal $\hat{x}$ and the signal + noise $x + v$ without coding-decoding.

sums of error correcting codes. The required error correction capability can be fully determined by the signal and noise probability density functions and the targeted binary error probability of each bit weight. A trade-off has to be found between the error correction power and the quantization noise created by coding: the higher the error correction capabilities, the lower the signal to coding noise ratio.

We have shown on an example of speech transmission over a noisy channel that it is possible to choose a set of coders so that the noise resulting from the coding-channel-decoding chain is less annoying than channel noise alone.

The barrier to breakdown is the trade-off between the error correction capability of the proposed system and the noise generated by the quantization, in order to guarantee that the signal at the output of the coding-channel-decoding chain is less noisy (or has a better quality according to a given criterion) than the signal only undergoing the noisy channel. For audio or image signals, perceptual criteria could be used when choosing the codewords. Another possible track is to build codebooks having the same properties as those provided by error correcting coders, but being adapted to the data to be coded, as in source coding.

The Octave scripts used for this article can be downloaded from:
`https://helios2.mi.parisdescartes.fr/~mahe/Recherche/robustSignals/`

Appendix: proof of Proposition 2

Proof. Let us first prove (8):

$$\begin{aligned} P_e(i) &= \Pr(v_i \neq c_i) && (32) \\ &= \Pr(v_i = 0)\Pr(c_i = 1) + \Pr(v_i = 1)\Pr(c_i = 0) \\ &\quad \text{assuming } v_i \text{ and } c_i \text{ independent} \\ &= \beta_i + \Pr(c_i = 1)(1 - 2\beta_i) && (33) \end{aligned}$$

Assuming that the probability having v and s with the same sign is the same as having v and s with different signs, it comes (8).

If $|v| < 2^i$, $v_i = 0$; if $2^i \leq |v| < 2^{i+1}$, $v_i = 1$; if $|v| \geq 2^{i+1}$, $v_i = 1$ with a probability of 1/2. This leads to (9). The same rationale holds with s for (14).

Equation (12) derive directly from Table 1, assuming that s_i, v_i and c_i are mutually independent when v and s have the same sign.

Let us now prove (13). In the following, we recall that we have set $a = \max(|s|, |v|)$ and $b = \min(|s|, |v|)$, and we introduce the notation

$$\mathrm{Pr}_c(\cdot) = \mathrm{Pr}(\cdot \mid \mathrm{sign}(s) \neq \mathrm{sign}(v)),$$

to simplify the expressions. From Table 2, it comes:

$$\begin{aligned} \rho^-_{i+1} &= \mathrm{Pr}_c(c_i = 1, s_i = v_i) + \mathrm{Pr}_c(a_i b_i = 01) \\ &= \mathrm{Pr}_c(c_i = 1, s_i = v_i \mid a \geq 2^i)\mathrm{Pr}_c(a \geq 2^i) \\ &\quad + \mathrm{Pr}_c(a_i b_i = 01 \mid a \geq 2^i)\mathrm{Pr}_c(a \geq 2^i) \end{aligned} \tag{34}$$

In the first term, we discarded the case $a < 2^i$, because it makes impossible $c_i = 1$. For the second term, we only consider the case $a \geq 2^i$, since otherwise $b < 2^i$ and thus $b_i = 0$. We assume that, knowing $\{\mathrm{sign}(v) \neq \mathrm{sign}(s), a \geq 2^i\}$, the events $\{c_i = 1\}$ and $\{s_i = v_i\}$ are independent. Note that, although this could intuitively seem always true whenever $\{\mathrm{sign}(v) \neq \mathrm{sign}(s)\}$, we empirically found that it is false without the condition $\{a \geq 2^i\}$. In this case,

$$\begin{aligned} \rho^-_{i+1} = \mathrm{Pr}_c(c_i = 1 \mid a \geq 2^i)\ \mathrm{Pr}_c(s_i = v_i \mid a \geq 2^i)\ \mathrm{Pr}_c(a \geq 2^i) \\ + \mathrm{Pr}_c(a_i b_i = 01 \mid a \geq 2^i)\mathrm{Pr}_c(a \geq 2^i) \end{aligned} \tag{35}$$

Let us consider the first term of (35) and let

$$\theta_{i,j} \triangleq \mathrm{Pr}_c(c_i = 1 \mid a \geq 2^j) \quad \text{for } j \geq i \tag{36}$$

For $i = 0$, $\theta_{i,j} = 0$, and, following the calculation of (34) and (35) for $i \geq 0$,

$$\begin{aligned} \theta_{i+1,j} &= \mathrm{Pr}_c(c_i = 1, s_i = v_i \mid a \geq 2^j) + \mathrm{Pr}_c(a_i b_i = 01 \mid a \geq 2^j) \\ &= \theta_{i,j}\mathrm{Pr}_c(s_i = v_i \mid a \geq 2^j) + \mathrm{Pr}_c(a_i b_i = 01 \mid a \geq 2^j), \end{aligned} \tag{37}$$

under the assumption that knowing $\{\mathrm{sign}(v) \neq \mathrm{sign}(s), a \geq 2^j, j \geq i\}$, the events $\{c_i = 1\}$ and $\{s_i = v_i\}$ are independent.

To achieve the calculation of (35) and (37), we need to compute $\mathrm{Pr}_c(s_i = v_i \mid a \geq 2^j)$ and $\mathrm{Pr}_c(a_i b_i = 01 \mid a \geq 2^j)$, for $j \geq i$. In the following, we assume that knowing $\{\mathrm{sign}(v) \neq \mathrm{sign}(s), a \geq 2^j, j \geq i\}$, a_i and b_i are independent. Then, $\mathrm{Pr}_c(s_i = v_i \mid a \geq 2^j) = \mathrm{Pr}_c(a_i = b_i = 0 \mid a \geq 2^j) + \mathrm{Pr}_c(a_i = b_i = 1 \mid a \geq 2^j)$ may be rewritten for $j \geq i$, as

$$\begin{aligned} &\mathrm{Pr}_c(s_i = v_i \mid a \geq 2^j) = \\ &\quad (1 - \mathrm{Pr}_c(a_i = 1 \mid a \geq 2^j))(1 - \mathrm{Pr}_c(b_i = 1 \mid b \geq 2^i))\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j) \\ &\qquad + \mathrm{Pr}_c(a_i = 1 \mid a \geq 2^j)\mathrm{Pr}_c(b_i = 1 \mid b \geq 2^i)\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j) \end{aligned} \tag{38}$$

For $j > i$, $\mathrm{Pr}_c(a_i = 1 \mid a \geq 2^j) = \mathrm{Pr}_c(b_i = 1 \mid b \geq 2^j) = 1/2$. For $j = i$, let

$$\gamma_i \triangleq \mathrm{Pr}_c(a_i = 1 \mid a \geq 2^i) \tag{39}$$

$$\delta_i \triangleq \mathrm{Pr}_c(b_i = 1 \mid b \geq 2^i) \tag{40}$$

Equation (38) can be re-written:

$$\mathrm{Pr}_c(s_i = v_i \mid a \geq 2^j) = \begin{cases} (1 - \gamma_i - \delta_i + 2\gamma_i\delta_i)\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^i) & \text{for } j = i \\ \frac{1}{2}\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j) & \text{for } j > i \end{cases} \tag{41}$$

Let us now calculate, for $j \geq i$:

$$\begin{aligned} \mathrm{Pr}_c(a_i b_i = 01 \mid a \geq 2^j) &= (1 - \mathrm{Pr}(a_i = 1 \mid a \geq 2^j))\mathrm{Pr}_c(b_i = 1 \mid b \geq 2^i) \\ &\qquad \times \mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j) \\ &= \begin{cases} (1 - \gamma_i)\delta_i \mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^i) & \text{for } j = i \\ \frac{1}{2}\delta_i \mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j) & \text{for } j > i \end{cases} \end{aligned} \tag{42}$$

Substituting formulas (41) and (42) in (35) and (37), we get:

$$\rho^-_{i+1} = \mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^i)\big\{(1-\gamma_i-\delta_i+2\gamma_i\delta_i)\theta_{i,i}+(1-\gamma_i)\delta_i\big\} \tag{43}$$

$$= (\mathrm{Pr}_c(2^i \leq |v| < |s|) + \mathrm{Pr}_c(2^i \leq |s| < |v|) \tag{44}$$

$$\times \big\{(1-\gamma_i-\delta_i+2\gamma_i\delta_i)\theta_{i,i}+(1-\gamma_i)\delta_i\big\}$$

$$\theta_{i+1,j} = \frac{1}{2}\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j)(\theta_{i,j}+\delta_i) \tag{45}$$

Hence we got Eq. 13 and 17, respectively.

It remains to compute γ_i, δ_i and $\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j)$:

$$\begin{aligned}\gamma_i &= \frac{\mathrm{Pr}_c(a_i = 1)}{\mathrm{Pr}_c(a \geq 2^i)} \\ &= \frac{\mathrm{Pr}_c(|v| < |s|, s_i = 1) + \mathrm{Pr}_c(|s| < |v|, v_i = 1)}{\mathrm{Pr}_c(|v| < |s|, |s| \geq 2^i) + \mathrm{Pr}_c(|s| < |v|, |v| \geq 2^i)} \\ &= \frac{\left\{\begin{array}{c}\mathrm{Pr}_c(|v| < |s|, 2^i \leq |s| < 2^{i+1}) + \frac{1}{2}\mathrm{Pr}_c(|v| < |s|, |s| \geq 2^{i+1}) \\ +\mathrm{Pr}_c(|s| < |v|, 2^i \leq |v| < 2^{i+1}) + \frac{1}{2}\mathrm{Pr}_c(|s| < |v|, |v| \geq 2^{i+1})\end{array}\right\}}{\mathrm{Pr}_c(|v| < |s|, |s| \geq 2^i) + \mathrm{Pr}_c(|s| < |v|, |v| \geq 2^i)},\end{aligned} \tag{46}$$

using the same rationale as for (14) and (9).

Similarly,

$$\begin{aligned}\delta_i &= \frac{\mathrm{Pr}_c(b_i = 1)}{\mathrm{Pr}_c(b \geq 2^i)} \\ &= \frac{\mathrm{Pr}_c(|v| < |s|, v_i = 1) + \mathrm{Pr}_c(|s| < |v|, s_i = 1)}{\mathrm{Pr}_c(|v| < |s|, |v| \geq 2^i) + \mathrm{Pr}_c(|s| < |v|, |s| \geq 2^i)} \\ &= \frac{\left\{\begin{array}{c}\mathrm{Pr}_c(|v| < |s|, 2^i \leq |v| < 2^{i+1}) + \frac{1}{2}\mathrm{Pr}_c(|v| < |s|, |v| \geq 2^{i+1}) \\ +\mathrm{Pr}_c(|s| < |v|, 2^i \leq |s| < 2^{i+1}) + \frac{1}{2}\mathrm{Pr}_c(|s| < |v|, |s| \geq 2^{i+1})\end{array}\right\}}{\mathrm{Pr}_c(|v| < |s|, |v| \geq 2^i) + \mathrm{Pr}_c(|s| < |v|, |s| \geq 2^i)},\end{aligned} \tag{47}$$

Finally,

$$\begin{aligned}&\mathrm{Pr}_c(b \geq 2^i \mid a \geq 2^j) \\ &= \frac{\mathrm{Pr}_c(b \geq 2^i, a \geq 2^j)}{\mathrm{Pr}_c(a \geq 2^j)} \\ &= \frac{\mathrm{Pr}_c(|v| < |s|, |v| \geq 2^i, |s| \geq 2^j) + \mathrm{Pr}_c(|s| < |v|, |s| \geq 2^i, |v| \geq 2^j)}{\mathrm{Pr}_c(|v| < |s|, |s| \geq 2^j) + \mathrm{Pr}_c(|s| < |v|, |v| \geq 2^j)}\end{aligned} \tag{48}$$

Now we have all elements for the calculation of Eq. 13 and 17. □

Acknowledgments

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