Geometric Computation of Curvature driven Plane Curve Evolutions

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Abstract

We present a new numerical scheme for planar curve evolution with a normal velocity equal to $F(\kappa)$ where κ is the curvature and F is a nondecreasing function such that $F(0) = 0$ and either $x \mapsto F(x^3)$ is Lipschitz with Lipschitz constant less or equal to 1 or $F(x) = x^\gamma$ for $\gamma \geq 1/3$. The scheme is completely geometrical and avoids some drawbacks of finite difference schemes. In particular, no special parameterization is needed and the scheme is monotone (that is, if a curve initially surrounds another one, then this remains true during their evolution), which guarantees numerical stability. We prove consistency and convergence of this scheme in a weak sense. Finally, we display some numerical experiments.

1 Introduction

In this paper, we investigate the evolution of a closed smooth plane curve, when each point of the curve moves with a normal velocity depending on the curvature of the curve at this point. More precisely, we study evolution of a curve C obeying the equation

$$\frac{\partial C}{\partial t}(s,t) = F(\kappa(s,t))\,\mathbf{N}(s,t). \tag{1}$$

where $\mathbf{N}(s,t)$ is the inner normal vector to the curve at the point with parameter s at evolution time t. Equations of type (1) model phenomena in physics or material science. They also play an important role in digital image analysis. Indeed, it was proved in [1] that any image analysis process satisfying some reasonable properties and invariance (essentially causality, stability, invariance with respect to isometries and contrast changes) is described by an equation of type (1), or more precisely by the corresponding grey level evolution described by the scalar equation

$$\frac{\partial u}{\partial t} = |Du| F(\kappa(u)(x), t), \tag{2}$$

which has to be considered in the viscosity sense [6]. In this equation

$$\kappa(u)(x) = \frac{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2}{(u_x^2+u_y^2)^{\frac{3}{2}}}$$

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is the curvature of the level line passing through x. Equation (2) means that the level lines of u move with respect to Equation (1). The case $F(x) = x^{1/3}$ (with the convention $F(x) = x \cdot |x|^{\alpha-1}$, for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}$) is particularly interesting since it is the only contrast invariant equation that also commutes with affine transformations preserving the area (this set is named the special affine group, and is composed of the mappings of the type $x \mapsto Ax + b$, where $b \in \mathbb{R}^2$ and A is a linear transformation such that $\det A = 1$). The case $F(x) = x$ (Mean Curvature Flow) has extensively been studied ([8, 9]), and existence, regularity, and vanishing in finite time have been proved. The affine invariant case $F(x) = x^{1/3}$ has also been studied in [2, 22, 23]. For general $\gamma > 0$, some results have been exposed in [25]. The equivalence between curve motion and contrast invariant smoothing was proved in [7].

In this paper, we focus on a possible numerical algorithm solving (1). According to [7], if u is a continuous real-valued function in $\mathbb{R}^2$, we can solve Equation (2) by applying this algorithm to all level lines of u. Of course, this implies that these level lines must not intersect during their evolution. We thus require that, like Equation (1) the algorithm satisfies an inclusion principle, meaning that the order (with respect to inclusion) is respected. Among the first attempts to solve Equation (1), Osher and Sethian ([19, 24]) solve (2) by introducing the signed distance function to the curve at time $t = 0$. Unfortunately, this algorithm satisfies neither inclusion principle nor rotation invariance. In addition, the evolution *a priori* depends on the chosen distance function, since the scalar algorithm is not contrast invariant. Moreover, the data (and thus the CPU time) becomes rapidly huge when a high precision is requested. Some attempts on curves by finite difference schemes have also been made ([15, 16]) with interesting results, but the nongeometric nature of the scheme still prevents inclusion principle from being satisfied. Moreover, for evolution driven by a power a the curvature larger than 1, the discretization of the curves are enclined to becoming sparser around points with high curvature, thus preventing a good accuracy. On the contrary, a completely different scheme has been implemented in [17, 18] for the affine invariant case. It is fully geometrical and also satisfies inclusion principle (implying numerical stability). In [11], a theoretical algorithm for moving hypersurface by a power of Gauss Curvature has also been studied. We generalize and implement this algorithm in the plane for nonconvex curves and for more general functions of the curvature. We just mention that a numerical approach on curves has the advantage that the resolution is not limited by the pixel size, which allows a very high precision. Moreover, the computation time is far shorter than in a scalar approach. On the other hand, topology changes (*e.g.* a single curve breaking into two connected components) are automatically handled in a scalar approach. This should not be a real problem for plane curves evolution since it is likely that no topological changes occur. This has been proved at least in [9] for the Mean Curvature Flow and in [2] for Affine Curve evolution. On the contrary, it is known that topology changes do occur in higher dimension. In addition, it seems difficult (at least not trivial) to generalize our algorithm in higher dimensions even for hypersurfaces in a three dimensional space.

The paper is as follows. We first give in section §2 some preliminary definitions and introduce some operators on sets. This provides operators on curves which are boundaries of sets. These operators are consistent with curvature dependent differential operators. They satisfy some monotonicity and continuity properties, allowing to extend them to real valued functions. We prove some consistency results and give the proof of convergence without entering too much into details since we prefer to focus on numerical applications. In §3, we adapt the operator previously defined to the special case of evolution driven by a power of the curvature and in §4, we give an algorithm with the same scaling covariance properties as the curve Scale Space. We then show in §5 some numerical experiments in the convex case as well as in the nonconvex case.

2 Definition and Properties

We first give some notations (previously used in [18]). Let C a semi-closed curve in $\mathbb{R}^2$ (that is an oriented simple curve dividing the plane in exactly two connected components) and K the interior of C (which is the bounded component of $\mathbb{R}^2 \backslash C$ when C is closed). We suppose that C is oriented such that K lies on "the left" when C is positively described. More rigorously, if we assume that the plane is counterclockwise oriented, the inner normal is such that the tangent vector and the inner normal form an orthonormal direct basis. We assume that a smooth parameterization is defined on C (for example piecewise C^1).

A *chord* is a segment of the form $]C(s), C(t)[$ that does not intersect any point of C with parameter between s and t. A *chord set* $C_{s,t}$ is the connected set enclosed by a chord $]C(s), C(t)[$ and the curve $C(]s,t[)$. We say that $C_{s,t}$ is a *σ-chord set* and (s,t) is a *σ-chord* if $\mathcal{L}^2(C_{s,t}) = \sigma$ and if the area of any chord set strictly included in $C_{s,t}$ is strictly less than σ. We denote by $\mathcal{K}_\sigma(C)$ the set of σ-chord sets.

Let $\mathcal{C} = [C(s), C(t)]$ a σ-chord of K and $C_{s,t}$ the associated σ-chord set. We call *chord arc distance* of $\mathcal{C}$ (or of $C_{s,t}$) the number $\delta(C([s,t]), [C(s), C(t)])$ where δ is the Hausdorff semi-distance (in particular it is not commutative) defined by

$$\delta(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|.$$

For x in $\mathbb{R}^2$, we also denote by $\delta_{s,t}(x)$ the signed distance from x to the oriented line $]C(s), C(t)[$, that is

$$\delta_{s,t}(x) = \frac{[x - C(s), C(t) - C(s)]}{|C(t) - C(s)|}.$$

(If x, y are in $\mathbb{R}^2$, we denote by $[x, y]$ the determinant of the 2×2 matrix with columns x and y). We also denote by $\mathcal{K}_\sigma^+(C)$ the sets of positive σ-chords *i.e* the chord-sets $C_{s,t}$ satisfying

$$\forall x \in C(]s,t[), \quad \delta_{s,t}(x) \geq 0.$$

In the same way, we can define $\mathcal{K}_\sigma^-$ the set of negative chord sets. Remark that for positive chord sets, the chord-arc distance is nothing but $\sup \delta_{s,t}(x)$ for $x \in C([s,t])$ and for negative chord sets, the chord-arc distance is $-\inf \delta_{s,t}(x)$ for $x \in C([s,t])$.
To finish with notations, we set $\omega = \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}$.

We now define a mapping on the set of plane sets that we shall assume piecewise regular for sake of simplicity. Let G be a nondecreasing 1-Lipschitz function defined in $\mathbb{R}_+$ and such that $G(0) = 0$. Let K be a set in $\mathbb{R}^2$ whose oriented boundary $C = \partial K$ is smooth (say piecewise C^1). Let $\sigma > 0$, and assume that $\mathcal{L}^2(K) > \sigma$. For $C_{s,t}$ a chord set of K with chord-arc distance h, we write

$$\tau_\sigma(C_{s,t}) = \left\{ x \in C_{s,t} \quad / \quad \delta_{s,t}(x) > h - \omega\sigma^{2/3} G\left(\frac{h}{\omega\sigma^{2/3}}\right) \right\}. \tag{3}$$

In the sequel, we shall briefly say that $\tau_\sigma(C_{s,t})$ is a modified chord set. We remark that the right-hand term of the inequality above is nonnegative because of the Lipschitz assumption on G. Hence, a modified σ-chord set is always included in its associated σ-chord set. On Figure 1, we represent a σ-chord and its modified chord. The modified σ-chord set is filled.

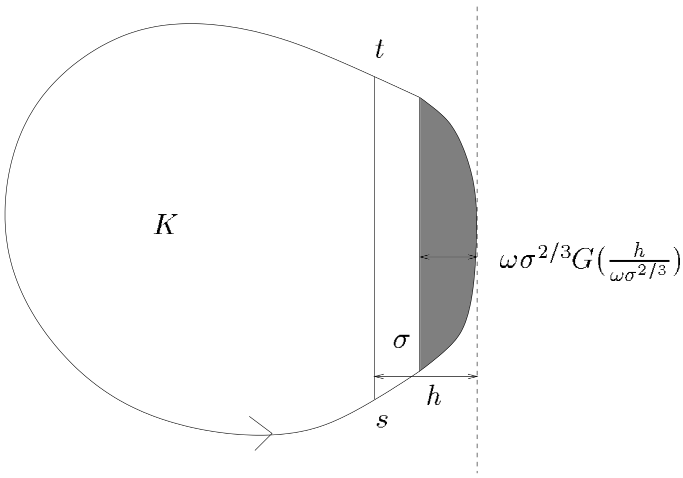

Figure 1: A σ-chord set before and after transform.

Definition 1 *Let $\sigma > 0$. Let K be the interior of a piecewise C^1 semi-closed curve. We define*

$$E_\sigma(K) = K \backslash \bigcup_{\substack{A \in \mathcal{K}^+_{\sigma'}(\partial K) \\ \sigma' \leq \sigma}} \tau_\sigma(A). \tag{4}$$

We will refer to E_σ as an *erosion operator* and to $E_\sigma(K)$ as the eroded of K at scale σ.
Remark 2. The algorithm in [18] corresponds to $G(x) = x$.
Remark 3. We can generalize Definition 1 to sets with several connected components by applying the erosion to each component.

Lemma 4 *Let K be a convex set of $\mathbb{R}^2$. Then $E_\sigma(K)$ is also convex.*

Proof. We can write

$$E_\sigma(K) = \bigcap_{\substack{A \in \mathcal{K}^+_{\sigma'}(\partial K) \\ \sigma' \leq \sigma}} K \backslash \tau_\sigma(A),$$

which proves that $E_\sigma K$ is convex as an intersection of convex sets (each of them is the intersection between K and a half plane). □

Lemma 5 *If K is a smooth compact set, then $E_\sigma(K)$ is a compact set.*

Proof. This is obvious since K is an intersection of compact sets. □

The proposition below is crucial in a theoretical point of view as well in a numerical one since it is necessary to obtain a stable numerical algorithm.

Proposition 6 (Inclusion Principle) *Let $K_1 \subset K_2$. Assume that G is nondecreasing and 1-Lipschitz. Then*

$$E_\sigma(K_1) \subset E_\sigma(K_2).$$

Proof. Assume that $x \in K_2$ and $x \notin E_\sigma(K_2)$. We prove that $x \notin E_\sigma(K_1)$. If $x \notin K_1$ then $x \notin E_\sigma(K_1)$ since $E_\sigma(K_1) \subset K_1$. Assume now that $x \in K_1$. By assumption, there exists a σ'-chord of K_2 (that we denote by $\mathcal{C}$) with $\sigma' \leq \sigma$ such that x belongs to the modified chord-set. The Lipschitz condition on G implies that x also belongs to the σ'-chord set. The same σ'-chord delimits in K_1 a unique chord-set containing x. Let σ'' be its area : we have $\sigma'' \leq \sigma'$ and thus $\sigma'' \leq \sigma$. It then suffices to prove that this chord excludes x from $E_\sigma(K_1)$. Consider the situation illustrated in Figure 2.

(i) h_1 and h_2 are the chord-arc distances of $\mathcal{C}$ in K_1 and K_2.

(ii) l_1 and l_2 are the chord-arc distances of the associated modified chords.

(iii) l is the difference of length between K_1 and K_2 in the direction that is orthogonal to the chords, *i.e.* $l = h_2 - h_1$.

It is enough to prove that $l + l_1 \geq l_2$. But, we know that $l_1 = \sigma^{2/3}\omega G(\frac{h_1}{\sigma^{2/3}\omega})$ and $l_2 = \sigma^{2/3}\omega G(\frac{h_2}{\sigma^{2/3}\omega})$. Since G is 1-Lipschitz, we conclude. □

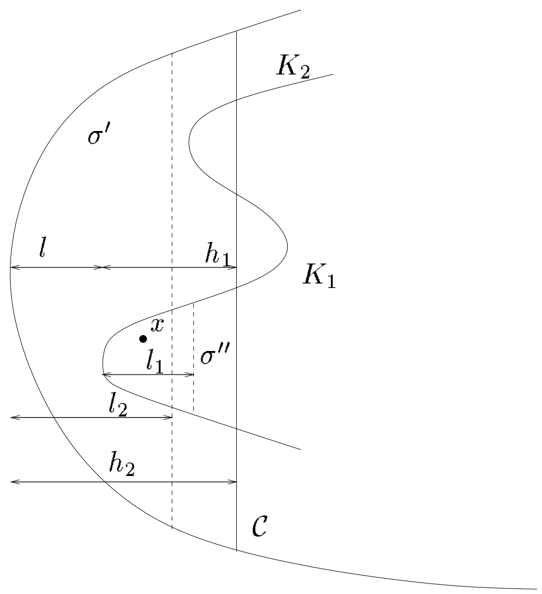

Figure 2: Inclusion Principle.

Proposition 7 *Assume that $C = \partial K$ is of class C^2. Let $M \in C$ such that the curvature of C at M is not equal to 0. Then*

$$\lim_{\sigma \to 0} \frac{d(M, E_\sigma(K))}{\omega \sigma^{2/3}} = G\left((\kappa^+(M))^{1/3}\right),$$

where κ^+ is the positive part of κ, defined by $\kappa^+ = \max(0, \kappa)$.

Proof. Assume that C is concave at M (that is the curvature at M is strictly negative in a neighborhood of M). Then for σ small enough, any σ-chord (s,t) such that $M \in C([s,t])$ is a (strictly) negative σ-chord. Hence $M \in E_\sigma(K)$ and the proposition follows from $G(0) = 0$. Assume now that C is strictly convex at M (thus in a neighborhood of M). Then for any chord parallel to the tangent at M enclosing a σ-chord set containing M, the chord-arc distance h satisfies

$$h = \omega \cdot \kappa^{1/3}(M) \cdot \sigma^{2/3} + O(\sigma^{4/3}) \qquad \text{as } \sigma \to 0, \tag{5}$$

which can be easily established for a parabola, then for any regular curve by approximation (see [18] for example). Then the result follows from 3 and 4, which imply

$$d(M, E_\sigma(K)) = \omega\sigma^{2/3}\, G\left(\frac{h}{\omega\sigma^{2/3}}\right),$$

and the fact that G is continuous. □

The following property is a continuity property allowing to extend E_σ to any plane set, and then to define an operator acting on functions with real values.

Proposition 8 (Continuity) *Let K_n a sequence of compact smooth sets. Let $K = \cap_n K_n$. Assume that K is also a smooth set. Then*

$$E_\sigma(\bigcap_n K_n) = \bigcap_n E_\sigma(K_n). \tag{6}$$

Proof. Since $K \subset K_n$ for any n, by monotonicity we have also $E_\sigma(K) \subset E_\sigma(K_n)$, implying the first part of the equality. In order to prove the reverse inclusion, we can assume that the family K_n is nonincreasing. Without loss of generality, we also suppose that (K_n) converges to K for the Hausdorff distance between compact sets. Assume that $x \notin E_\sigma(K)$. By definition, there is a chord (s,t) with area not more than σ such that the modified chord excludes x. Since $E_\sigma(K)$ is closed, its complementary is an open set; thus we can assume that the area of $C_{s,t}$ is strictly less than σ. In K_n it also defines a chord and for n large enough, the area of this chord is also less than σ (by using convergence of measures). Moreover, as K_n tends to K for the Hausdorff distance, the chord-arc distance also converges. Since G is continuous, this implies that the chord excludes x in K_n for n large enough. Hence $x \notin \cap E_\sigma(K_n)$ and this ends the proof. □

Remark 9. The compactness assumption in the proposition above is far from necessary. It suffices for example that the boundary of the sets is locally convex or concave. This ensures that the erosion is local when σ is small. We can then conclude by the same kind of arguments. Note also that the continuity property allows to define the erosion on any closed set by approximating closed sets by smooth closed sets.

We can now extend E_σ to real-valued functions in $\mathbb{R}^2$. First, if $u : \mathbb{R}^2 \to \mathbb{R}$, we define as usual the level set with value λ, the set

$$\chi_\lambda(u) = \{x \in \mathbb{R}^2, u(x) \geq \lambda\}.$$

Applying a theorem by Matheron (see [14]) yields the following

Proposition 10 *Let $\mathcal{F}$ a set of real valued functions in $\mathbb{R}^2$ such that the level sets of the elements of $\mathcal{F}$ are compact and smooth. Then, we can extend E_σ to elements of $\mathcal{F}$ by setting*

$$E_\sigma(u)(x) = \sup\{\lambda \text{ s.t. } x \in E_\sigma(\chi_\lambda(u))\}. \tag{7}$$

It is equivalent to define $E_\sigma(u)$ by its level sets,

$$\chi_\lambda(E_\sigma(u)) = E_\sigma(\chi_\lambda(u)). \tag{8}$$

This uniquely defines a monotone, translation invariant operator commuting with nondecreasing continuous functions.

We also define a dual operator D_σ (called dilation operator) by

$$D_\sigma(K) = (E_\sigma(K^c))^c \tag{9}$$

the subscript designing the complementary set in $\mathbb{R}^2$. This operator satisfies the same properties as E_σ except the consistency result where the positive part of the curvature has to be replaced by the negative part.

By standard arguments ([4], [10], [20]), we can derive the following consistency result on the operator acting on functions.

Proposition 11 *Let u be a C^3 function and x a point such that $Du(x) \neq 0$ and $\kappa(u)(x) \neq 0$. Then*

$$E_\sigma(u)(x) = u(x) - \omega\sigma^{2/3}|Du|G((\kappa(u)^-)^{1/3}) + o(\sigma^{2/3}). \tag{10}$$

$$D_\sigma(u)(x) = u(x) + \omega\sigma^{2/3}|Du|G((\kappa(u)^+)^{1/3}) + o(\sigma^{2/3}). \tag{11}$$

Proof. The whole proof is not very difficult but a bit technical and long. Thus, we do not enter into all details and we shall skip some points. The aim is to prove that $E_\sigma(u)$ only depends on local features of u. Choose $r = \sigma^\alpha$ such that $\sigma^{1/3} = o(r)$ and $r^3 = o(\sigma^{2/3})$ when σ tends to 0 (note that there is no incompatibility; choose $r = \sigma^{1/4}$ for instance). By using translation invariance and contrast invariance, we assume that $x = 0$ and $u(x) = 0$. The key of the proof is the following. If r is small enough, the curvature of the level lines of u has a strict sign in $D(0, r)$. As a consequence, for any σ-chord, we can estimate the chord-arc distance by Equation (5). Moreover the same kind of approximation (made on a circle or a parabola) shows that the length of the chord is of order $(\frac{\sigma}{\kappa})^{1/3}$. Hence, any σ-chord intersecting $D(0, \frac{r}{2})$ is asymptotically included in $D(0, \frac{r}{2}(1 + r^\varepsilon))$ for some $\varepsilon > 0$ (because of the choice of r). Assume first that $\kappa(u)(0) > 0$. Define u_+ and u_- by

$$\forall x \in D(0, r) \quad u_+(x) = u_-(x) = u(x)$$

and $u_+(x) = +\infty$, $u_-(x) = -\infty$ elsewhere (we can replace the infinite value by very large numbers). The global inequalities $u_- \leq u \leq u_+$ yield $E_\sigma(u_-) \leq E_\sigma(u) \leq E_\sigma(u_+)$. If σ is small enough, the level lines of u are uniformly strictly concave in $D(0, r)$. Thus, there is no positive σ-chord of level sets of u_- and u_+ intersecting $D(0, \frac{r}{2})$ (figure 3). Hence

$$\forall x \in D(0, \frac{r}{2}) \quad u(x) = E_\sigma(u_+) = E_\sigma(u) = E_\sigma(u_+).$$

Assume now that $\kappa < 0$. Define

$$v(x) = Du(0) \cdot x + \frac{1}{2}D^2u(0)(x, x) + kr^3 \tag{12}$$

in $D(0, r)$ and $v(x) = -\infty$ elsewhere. Define also

$$w(x) = Du(0) \cdot x + \frac{1}{2}D^2u(0)(x, x) - kr^3 \tag{13}$$

in $D(0, r)$ and $w(x) = +\infty$ elsewhere. The constant k is chosen such that for σ small enough, we have

$$\forall x \in \mathbb{R}^2 \quad v(x) \leq u(x) \leq w(x).$$

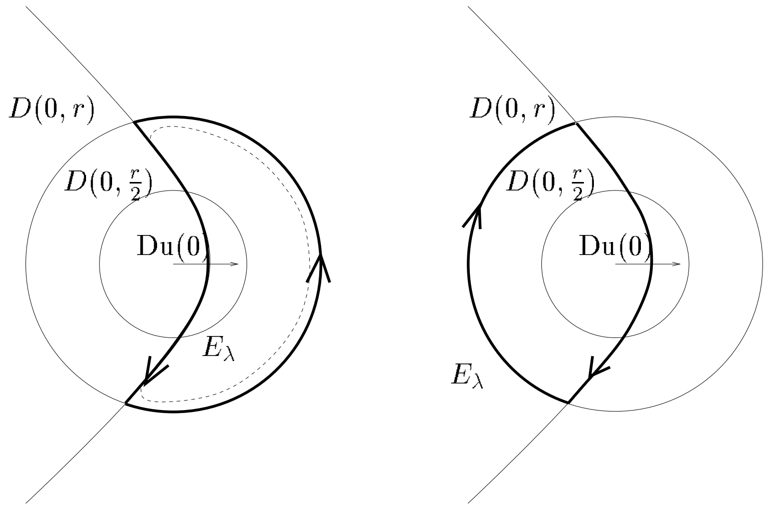

Figure 3: Case $\kappa > 0$. On the left, a level set of u_-. The oriented boundary is the bold line. The level set is the bounded connected component delimited by the curve. The dashed line is the boundary of the eroded set. On the right, a level set of u_+: the boundary is the bold oriented line and the level set is the unbounded component. In both cases, there is no positive σ-chord set intersecting $D(0, \frac{r}{2})$. Hence, the erosion has no effect.

This is possible since we assumed that u is C^3. Monotonicity yields

$$\forall x \in \mathbb{R}^2 \quad E_\sigma(v)(x) \leq E_\sigma(u)(x) \leq E_\sigma(w)(x).$$

As v and w have trivial level sets out of $D(0,r)$, it is quite easy to estimate their image by E_σ. We now use the consistency result (Proposition 7). The only trick is that the level lines of u and v are not parabola in the canonical form. Nevertheless, with some few arguments we can be led back to this situation ([4, 10, 20]). The computation of the eroded level sets of v and w is drawn on Figure 4. From this, it is no longer difficult to prove that

$$E_\sigma(v)(0) = -\omega\sigma^{2/3}|Du(0) + k_1 r|G((\kappa^-)^{1/3}) + o(\sigma^{2/3}), \tag{14}$$

and

$$E_\sigma(w)(0) = -\omega\sigma^{2/3}|Du(0) + k_2 r|G((\kappa^-)^{1/3}) + o(\sigma^{2/3}). \tag{15}$$

where k_1 and k_2 are constants depending on $D^2u(0)$. We can apply the same result to the dilation operator D_σ to obtain the second part of the proposition. □

For the next proposition, we first extend G to make it odd, that is if $x < 0$, we set $G(x) = -G(-x)$.

Proposition 12 *Let u be a C^3 function. Suppose that $Du(x) \neq 0$ and $\kappa(u)(x) \neq 0$. Then*

$$D_\sigma \circ E_\sigma(u)(x) = u(x) + \omega\sigma^{2/3}|Du|G((\kappa(u))^{1/3})(x) + o(\sigma^{2/3}). \tag{16}$$

Proof. This follows from the fact that : 1) E_σ and D_σ are monotone and commute with addition of constants, 2) near a point with gradient and curvature different from zero, the arguments developed in the previous proposition are uniform. □

In order to be complete, we just give without proof (it is simple when introducing the affine erosion operator ([18])) a lemma controlling the behavior of E_σ at critical points.

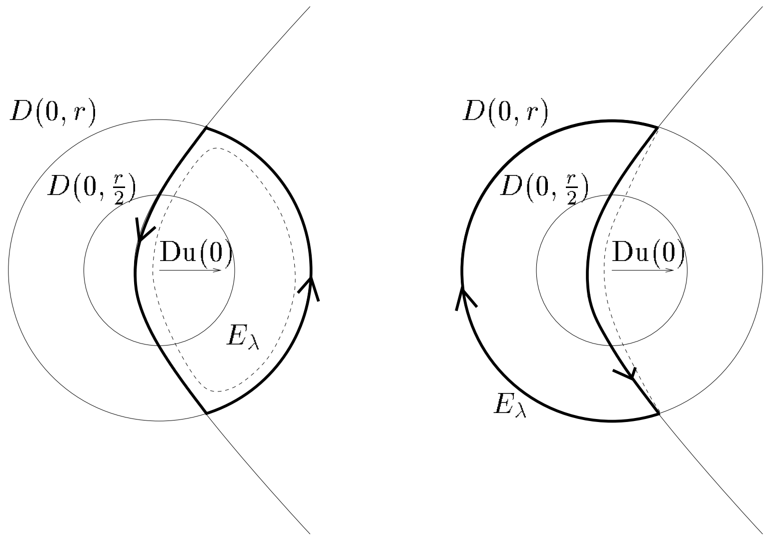

Figure 4: Case $\kappa < 0$. On the left, a level line of v (oriented bold line). The eroded line is the dashed line. On the right, the same thing with w.

Lemma 13 *Let $u \in C^3$ with $Du(x) = 0$ and $D^2u(x) = 0$. Then*

$$\lim_{\substack{y \to x \\ \sigma \to 0}} \frac{E_\sigma(u)(y) - u(y)}{\sigma^{2/3}} = 0, \tag{17}$$

the limit being taken when y and σ tend to 0 independently.

This and the consistency result above allow to deduce the convergence result we now enounce (see [21]). We denote $T_\sigma = D_\sigma \circ E_\sigma$.

Theorem 14 *Let u_0 bounded and uniformly continuous in $\mathbb{R}^2$. For $\sigma > 0$, let define u_σ : $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ by*

$$\forall t \in [n\omega\sigma^{2/3}, (n+1)\omega\sigma^{2/3}) \quad u_\sigma(x,t) = (T_\sigma)^n(u)(x).$$

Then, when σ tends to 0, u_σ converges locally uniformly to the unique viscosity solution of the equation

$$\frac{\partial u}{\partial t} = |Du|F(\kappa(u)), \tag{18}$$

$$u(\cdot, 0) = u_0(\cdot), \tag{19}$$

where $F(x) = G(x^{1/3})$.

The usual definition of viscosity solution ([6]) is no longer valid here because of the possible singularity of the operator at critical points. An extended notion of solution is defined in [3] and the solution still exists and is unique.

3 Curvature Power

3.1 Approximation of power functions

In this section, we show that the previous study can be adapted to the particular case of power functions $F(x) = x^\gamma$ for any $\gamma > 1/3$. Since $x \mapsto F(x^3)$ is not 1-Lipschitz on the whole real line, we define

$$G(x) = x^{3\gamma} \text{ if } x \leq \alpha_\gamma \tag{20}$$
$$= \alpha_\gamma^{3\gamma} + (x - \alpha_\gamma) \text{ if } x > \alpha_\gamma. \tag{21}$$

where

$$\alpha_\gamma = \left(\frac{1}{3\gamma}\right)^{\frac{1}{3\gamma-1}}$$

is the largest positive number at which the power function $x^{3\gamma}$ has a derivative less than 1. As G is 1-Lipschitz, we can then apply E_σ to this function. We could think that this scheme is not consistent with motion by curvature power. Indeed, if the curvature is too large for fixed σ then it may happen that the chord arc distance is also very large and the erosion is then given by the linear part of G. Nevertheless, we shall see that by an adequate scaling, G is not evaluated in its linear part. From now on, we slightly change the notations in the definition of E_σ. This will simplify the statements in the case of power functions. For a σ-set $C_{s,t}$, we now set

$$\tau_\sigma(C_{s,t}) = \{x \in C_{s,t} \,/\, \delta_{s,t}(x) > h - G(h)\} \tag{22}$$

and we still define the erosion operator by

$$E_\sigma(K) = K \backslash \bigcup_{\substack{A \in \mathcal{K}^+_{\sigma'}(\partial K) \\ \sigma' \leq \sigma}} \tau_\sigma(A). \tag{23}$$

When the scale tends to 0, h also tends to 0 and G is not taken in its linear part. The fact that we get an operator consistent with a power function is due to the homogeneity properties of power functions (this implies that except for power functions, this new definition of the erosion operator makes no sense). We can adapt the proof of the inclusion principle 6 to prove that the modified erosion operator E_σ still satisfies this inclusion principle. Consistency on curves (circles is enough !) is easy to establish. Continuity is not a problem as well. Thus, by using Matheron's Theorem, we can extend this erosion operator to an operator acting on functions. The following proposition asserts that this operator is consistent with a power of the curvature.

Proposition 15 *Let u be a C^3 function. Let $\mathbf{x}$ be such that $Du(\mathbf{x}) \neq 0$ and $\kappa(u)(0) \neq 0$. Then*

$$E_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) - \omega^{3\gamma}\sigma^{2\gamma}(\kappa^-)^\gamma + o(\sigma^{2\gamma}),$$

$$D_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) + \omega^{3\gamma}\sigma^{2\gamma}(\kappa^+)^\gamma + o(\sigma^{2\gamma}).$$

Moreover, consistency is locally uniform.

Proof. As usual, we assume that $\mathbf{x} = 0$ and $u(0) = 0$. In a first time, assume that $\kappa(u)(0) > 0$. We use the same locality argument as in the first proof of consistency in this chapter. Since the curvature of u in a small ball with radius $r > 0$ is bounded from below by a positive constant, say $\kappa - \eta$ with $\eta > 0$, the level sets of u have no positive σ-chord in $D(0, r)$. We set $u_-(\mathbf{x}) = u(\mathbf{x})$ for $\mathbf{x} \in D(0, r)$ and $u_(\mathbf{x}) = -\infty$ elsewhere. We use the same locality argument as in the first proof of consistency as above: For σ small enough, the level sets of u_- have no positive σ-chord in $D(0, \frac{r}{2})$. Thus the erosion has no effect upon u_- in $D(0, \frac{r}{2})$. By using monotonicity, we have

$$0 = u_-(0) = E_\sigma(u_-)(0) \leq E_\sigma(u)(0) \leq u(0) = 0$$

and the result is proved in the case $\kappa > 0$.

Let us now come to the most difficult case: Assume that $\kappa(u)(0) < 0$. Let also $Q(\mathbf{x}) = px + ax^2 + by^2 + cxy$ be the Taylor expansion of u at the origin with $b < 0$. Let $\varepsilon > 0$ be a small parameter and let

$$v(\mathbf{x}) = px + (a - \frac{p\varepsilon}{2})x^2 + (b - \frac{p\varepsilon}{2})y^2 + 2cxy = Q(\mathbf{x}) - \frac{p\varepsilon}{2}(x^2 + y^2).$$

If r is chosen small enough, we have $v(\mathbf{x}) \le u(\mathbf{x})$ in $D(0,r)$. By extending v by $-\infty$ out of $D(0,r)$ this remains true everywhere. Moreover, we can assume that the curvature of the level lines of v is still strictly negative. Indeed, its value is $\kappa - \varepsilon + O(r)$ where κ is the value at the origin. We want to approximate $E_\sigma(v)(0)$. Let now $\eta > 0$ be also small, such that the curvatures of the level lines of v is larger than $\kappa - \varepsilon - \eta$ (remind that in this part of the proof the curvatures are all negative, hence a circle with a small curvature will also have a small radius). We can again invoke the same locality property of the erosion in the case of a curve with a strictly negative curvature: We know that the chord-arc distance is equal to $O(\sigma^{2/3}|\kappa - \varepsilon - \eta|^{1/3})$ and the length of the chord is a $O(|\kappa - \varepsilon - \eta|^{-1/3}\sigma^{1/3})$. We deduce from this that the σ-chord sets containing 0 must be included in a ball with radius $O(\sigma^{1/3})$. A small computation shows that the modified corresponding σ-chord sets are included in a ball with radius $O(\sigma^\gamma)$. The constant in these terms are clearly uniform because the curvature is bounded from above by a negative constant. In particular, they do not depend on ε and η. Let now $\mathbf{x}$ be in a $D(0,r)$. We call $C_{\kappa-\varepsilon-\eta}(\mathbf{x})$ the disk of curvature $\kappa - \varepsilon - \eta$ that is tangent to the level line of v at $\mathbf{x}$ and that is in the same side as $\chi_{v(\mathbf{x})}(\mathbf{x})$. Because of the comparison of the curvatures, we can still assume that r is small enough such that we have the inclusion

$$D(0,r) \cap C_{\kappa-\varepsilon-\eta}(\mathbf{x}) \subset D(0,r) \cap \chi_{v(\mathbf{x})}(v).$$

Now, since the erosion operator is local, we also have

$$D(0,\frac{r}{2}) \cap E_\sigma(C_\eta(\mathbf{x})) \subset D(0,\frac{r}{2}) \cap E_\sigma(\chi_{v(\mathbf{x})}(v)).$$

Assume that $E_\sigma(v)(0) \le \lambda$. By definition, this means that $0 \notin E_\sigma(\chi_\lambda(u))$. By using inclusion principle, we deduce that $0 \notin E_\sigma(C_{\kappa-\varepsilon-\eta}(\mathbf{x}))$ for any point such that $v(\mathbf{x}) = \lambda$. On the level line of v with the same value λ, we can find a unique point $\mathbf{x}_\lambda$ such that $\mathbf{x}_\lambda$ and the normal at $\mathbf{x}_\lambda$ are colinear (this is due to the strict convexity of the level lines and the theorem of intermediary values). This point is also characterized by the fact that the distance between the origin and the tangent to the level line is minimal. Hence, the modified σ-chord set of $C_{\kappa-\varepsilon-\eta}(\mathbf{x}_\lambda)$ at $\mathbf{x}_\lambda$ also contains the origin. Let (x_λ, y_λ) be the coordinates of $\mathbf{x}_\lambda$. A simple calculation gives

$$Dv(\mathbf{x}_\lambda) = (p + 2(a - \frac{p\varepsilon}{2})x_\lambda + 2cy_\lambda, 2(b - \frac{p\varepsilon}{2})y_\lambda + 2cx_\lambda).$$

Since $\mathbf{x}_\lambda$ and $Du(\mathbf{x}_\lambda)$ are colinear, we have

$$|\mathbf{x}_\lambda| = -\mathbf{x}_\lambda \cdot \frac{Dv}{|Dv|}(\mathbf{x}_\lambda). \tag{24}$$

By using consistency on disks, we have $|\mathbf{x}_\lambda| \leq \omega^{3\gamma}\sigma^{2\gamma}(-\kappa+\varepsilon+\eta)^\gamma(1+o(1))$. But we also have

$$\begin{aligned}
|\mathbf{x}_\lambda| &= -\mathbf{x}_\lambda \cdot \frac{Dv}{|Dv|}(\mathbf{x}_\lambda) \\
&= -\frac{1}{|Dv(\mathbf{x}_\lambda)|}(px_\lambda + 2(a-\frac{p\varepsilon}{2})x_\lambda^2 + 2(b-\frac{p\varepsilon}{2})y_\lambda^2 + 4cx_\lambda y_\lambda) \\
&= -\frac{1}{|Dv(\mathbf{x}_\lambda)|}(v(x_\lambda) + O(\sigma^{4\gamma})) \\
&= -\frac{1}{|Dv(\mathbf{x}_\lambda)|}(\lambda + O(\sigma^{4\gamma}))
\end{aligned}$$

From this, we finally deduce

$$\lambda > -p\omega^{3\gamma}(|\kappa-\varepsilon-\eta|)^\gamma\sigma^{2\gamma}(1+o(1)),$$

where we have approximated $|Dv(\mathbf{x}_\lambda)|$ by p up to a $O(\sigma^{2\gamma})$ term. This analysis can be performed for any $\varepsilon > 0$ and $\eta > 0$, since eventhough the constant were not explicited, we already stress that they do not depend upon ε and η. We then deduce that

$$E_\sigma(u)(0) \geq -p\omega^{3\gamma}|\kappa|^\gamma\sigma^{2\gamma}(1+o(1)).$$

Let now search an upper bound to $E_\sigma(u)(0)$. We do not repeat all the arguments since there will be some similarity with the research of a lower bound. We approximate u by its Taylor expansion and define $w(\mathbf{x}) = px + (a+\frac{p\varepsilon}{2})x^2 + (b+\frac{p\varepsilon}{2})y^2 + 2cxy$ such that $u \leq w$ is a small ball of radius r with $0 < \frac{p\varepsilon}{2} < -b$. For $\eta > 0$ small enough, the curvature of the level lines of w is smaller than $\kappa-\varepsilon-\eta$ which can also be chosen negative if r is small enough. The locality of the σ-chords still holds. We now define $C_{\kappa+\varepsilon+\eta}(\mathbf{x})$ as above; its radius is equal to $|\kappa+\varepsilon+\eta|^{-1}$. If r is small enough, for any $\mathbf{x}$ the level set $\chi_{w(x)}(w)$ is included in $C_{\kappa+\varepsilon+\eta}(\mathbf{x})$ inside $D(0,r)$. The rest of the proof is still an application of comparison principle and the asymptotic behavior on erosion on disks. Assume that $E_\sigma(w)(0) \geq \lambda$. This means that $0 \in E_\sigma(\chi_\lambda(w))$. In particular $0 \in E_\sigma(C_{\kappa+\varepsilon+\eta}(\mathbf{x}_\lambda)$ where $\mathbf{x}_\lambda$ is a above. This implies that $\mathbf{x}_\lambda \geq \omega^{3\gamma}|\kappa+\varepsilon+\eta|^\gamma + \sigma^{2\gamma}(1+o(1))$. We use the characterization (24) of $\mathbf{x}_\lambda$ and deduce that we must have

$$\lambda \leq -p\omega^{3\gamma}|\kappa+\varepsilon+\eta|^\gamma + \sigma^{2\gamma}(1+o(1)).$$

Since, this is true for any $\varepsilon > 0$ and $\eta > 0$, we also obtain

$$E_\sigma(u)(0) \leq -p\omega^{3\gamma}|\kappa|^\gamma\sigma^{2\gamma}(1+o(1)).$$

Again, we can pass to the limit since the $o(1)$ term is uniform in ε and η, since all the curvatures may be taken bounded by above by a strictly negative constant. Thus, if $\kappa < 0$, we have

$$E_\sigma(u)(0) - u(0) = -|Du|\omega^{3\gamma}(\kappa^-)^\gamma\sigma^{2\gamma}(1+o(1)).$$

With the case $\kappa \geq 0$, this gives the result.
The case of the dilation can be deduced by the relation $D_\sigma(u) = -E_\sigma(-u)$. The uniform consistency follows from the fact that the σ-chords are uniformly bounded in some ball with radius $O(\sigma^{2/3})$ and the constants of these terms are bounded as soon as the curvature have an absolute value strictly more that a positive constant. □

Uniform consistency yields consistency for the alternate operator.

Corollary 16 *Let $u \in C^3$ with $Du(\mathbf{x}) \neq 0$. Then*

$$D_\sigma \circ E_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) + \omega^{3\gamma}\kappa^\gamma\sigma^{2\gamma} + o(\sigma^{2\gamma}).$$

At critical points, we need to describe the behavior of the erosion in the following manner. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that $f(0) = f'(0) = f''(0)$, $f''(r) > r$ for all $r > 0$ and $f'(r) = o(r^\gamma)$ when r tends to 0.

Lemma 17 *Let $u(\mathbf{x}) = -f(|\mathbf{x}|)$. Then, for any sequence of points $\mathbf{x}_n$ tending to 0, we have*

$$\lim_{\substack{n\to+\infty \\ \sigma\to 0}} \frac{T_\sigma u(\mathbf{x}_n) - u(\mathbf{x}_n)}{\sigma^{2\gamma}} = 0,$$

where the limit is taken as n and σ tends to their respective limit independently and T_h designs either E_σ or D_σ or $D_\sigma \circ E_\sigma$.

This lemma can be easily established by finding some estimates on the radius of the circle after erosion. We can then prove the following convergence theorem.

Theorem 18 *Let $\gamma > \frac{1}{3}$. Let $T_\sigma = D_\sigma \circ E_\sigma$ the alternate dilation-erosion for the curvature power function x^γ. Let u_0 in $BUC(\mathbb{R}^2)$ and define u_σ by*

$$\forall t \in [n\omega^{3\gamma}\sigma^{2\gamma}, (n+1)\omega^{3\gamma}\sigma^{2\gamma}) \qquad u_\sigma(\mathbf{x}, t) = (T_\sigma)^n u_0(\mathbf{x}).$$

Then, when σ tends to 0, u_σ tends locally uniformly to the unique viscosity solution of the equation

$$\frac{\partial u}{\partial t} u(\mathbf{x}, t) = |Du|(\operatorname{curv} u)^\gamma$$

with initial value u_0.

As soon as the power γ is more than 1, then the usual notion of viscosity solution is not appropriate since the elliptic operator $|Du|(\operatorname{curv} u)^\gamma$ is singular at critical points. Ishii and Souganidis proved in [12] that existence and uniqueness where still true if test functions were restricted to a class of functions with flat critical point. This flatness is given by the same conditions in the previous lemma. This point apart, the scheme of the proof is standard so we do not explicit it. The only new point is the previous lemma in the case where test functions are stationnary. In this case, the lemma directly gives the solution and we leave the rest of the proof to the reader.

3.2 Scale Covariance

We denote by H_λ the dilation with ratio λ, that is $H_\lambda(x) = \lambda x$. Let S_t be the evolution semi-group of

$$\frac{\partial C}{\partial t} = \kappa^\gamma \mathbf{N}, \tag{25}$$

that is, $C(t) = S_t(C)$ is the curve evolving according to Equation (25) above (the solution exists and is unique at least for short times for smooth initial data (see [25]). The semi-group S_t satisfies the relation

$$S_t \circ H_\lambda = H_\lambda \circ S_{\frac{t}{\lambda^{\gamma+1}}}. \tag{26}$$

Indeed, let $C_1(t)$ the evolving curve defined by

$$C_1(t) = H_\lambda C\left(\frac{t}{\lambda^{\gamma+1}}\right).$$

Then, it is simple to check that C_1 satisfies Equation (25) with initial condition $H_\lambda C(0)$. This is exactly what Equation (26) asserts. The erosion operator E_σ does not satisfy the same covariance property. We thus define a modified operator

$$O_t = H_a \circ E_{\frac{\sigma}{a^2}} \circ H_{a^{-1}}, \tag{27}$$

where a and σ are positive parameters depending upon t (and possibly on C). We want O_t to satisfy covariance Property (26), like S_t. This will be true as soon as

$$a(H_\lambda C, t) = \lambda a\left(C, \frac{t}{\lambda^{\gamma+1}}\right) \quad \text{and} \quad \sigma(H_\lambda C, t) = \lambda^2 \sigma\left(C, \frac{t}{\lambda^{\gamma+1}}\right). \tag{28}$$

We also want O_t to be consistent with S_t, that is, for any convex set K with C^3 boundary,

$$O_t(K) = S_t(K) + o(t),$$

the term $o(t)$ being measured with the Hausdorff distance (we also use an abusive notation by denoting $S_t(K)$ the set whose boundary is $S_t(\partial K)$). By using consistency result above, we see that a, σ and t must be linked by the relation

$$t = \omega^{3\gamma} a^{1-3\gamma} \sigma^{2\gamma}. \tag{29}$$

Assume that $\sigma > 0$ is fixed. If a is chosen large enough such that, for any σ chord of K with chord-arc distance h the inequality

$$\frac{h}{a} \leq \alpha_\gamma \tag{30}$$

holds, then the modified chord-arc distance is then given by G near the origin, that is Equation (21) is not involved. Precisely, for $M \in \partial K$ (with smooth boundary) consistency writes down

$$\begin{aligned} d(M, O_t K) &= aG\left(\frac{h}{a}\right) \\ &= a\left(\frac{h}{a}\right)^{3\gamma} \\ &= t\kappa^\gamma + o(t). \end{aligned} \tag{31}$$

Notice that it is interesting to take the smallest possible value of a (given by the case of equality in (30)) in order to get the largest possible scale step t from (29). We can summarize these results in the following

Proposition 19 *Let $h(A)$ denote the chord-arc distance of a chord A. Then, the operator O_t defined by (27) with*

$$a = \frac{1}{\alpha_\gamma} \sup_{\substack{A \in \mathcal{K}^+_{\sigma'}(\partial K) \\ \sigma' \leq \sigma}} h(A) \quad \text{and} \quad \sigma = \left(t\omega^{-3\gamma} a^{3\gamma-1}\right)^{1/2\gamma} \tag{32}$$

is consistent with (25) and satisfy the same scaling property as S_t in (26).

Remark 20. (Error Analysis) In order to obtain consistency, it is not necessary that $\frac{h}{a} \leq \alpha_\gamma$ for all (σ, a); for instance, if we fix a and define O_t by (27) and (29), we still have consistency since inequality (30) holds for σ small enough. However if a and σ do not satisfy (30), the difference between O_tC and S_tC is of the same magnitude as t (since Equation(21) is involved). On the contrary, this difference is $O(t^2)$ if a and σ satisfy (30) (because of Consistency 5).

4 Algorithm

4.1 General method

Each iteration of the operator defined above involves three parameters: the scale step t (that can be viewed as a *time* step), the erosion area σ and the saturation length a. These three quantities have to satisfy (32), which leaves only one degree of freedom. A usual numerical scheme would consider t as the free parameter (the time step, related to the required precision), and then define a and σ from t. In the present case, this would not be a good choice for two reasons. First, a is defined as an explicit function of σ but as an implicit function of t, which suggests that σ may be a better (or at least simpler) free parameter than t. Second, t has no geometrical interpretation in the scheme we defined, contrary to σ which corresponds in some way to "the scale at which we look at the curve". In particular, σ is constrained by the numerical precision at which the curve is known: roughly speaking, if C is approximated by a polygon with a precision ε (corresponding, for example, to the Hausdorff distance between the both of them), then we *must* have $\sigma \gg \varepsilon^2$ in order that the effect of the erosion at each iteration overcomes the effect of the spatial quantization. For all these reasons, we choose to fix σ as the free parameter, and then compute t and a using (32). If the scale step t obtained this way is too large, we can simply adjust it by reducing σ while keeping the same value of a. We propose the following algorithm for the evolution of a convex set K at final scale T with area precision σ.

1. Let $t = 0$ and $K_0 = K$.
2. While $t < T$
   - For each σ-set of K_t, compute the chord-arc distance.
   - Set a to the maximal value of these distances.
   - Let $\delta t = \omega^{3\gamma} a^{1-3\gamma} \sigma^{2\gamma}$.
   - If $\delta t > T - t$, take $\delta t = T - t$ and decrease σ in order to keep the previous equality.
   - Apply operator O_t to K_t, yielding $K_{t+\delta t}$.
   - Increment t by δt.

In practice, it is of course impossible to deal with all the σ-chords. In fact, the curve is a polygonal line and we take the chords with an end point equal to a vertex of the polygon.

4.2 Computation of the erosion

The boundary of $O_t(K)$ is included in the envelop of all the modified chords. To obtain an approximation of this set, we explicitly determine the position of the unique point of each chord belonging to the envelop. This result is a generalization of the middle point property exposed in [18],

Lemma 21 (Middle point Property) *Let K be a strictly convex set. Let A_1 be a σ-set of K with σ-chord $\mathcal{C}_1$. Let A_2 be another σ-set, and let $\mathcal{C}_2$ be its σ-chord. Then, when $d_H(A_1, A_2)$ tends to 0, the intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ tends to the middle point of $\mathcal{C}_1$.*

Proof. (quoted from [18]). Let θ be the geometrical angle between $\mathcal{C}_1$ and $\mathcal{C}_2$. If A_1 and A_2 are close enough, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at a point that we call $I(\theta)$. We also call $r_1(\theta)$ and $r_2(\theta)$ the length of the part of the chord $\mathcal{C}_2$ on each side of $I(\theta)$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are σ-chords, we

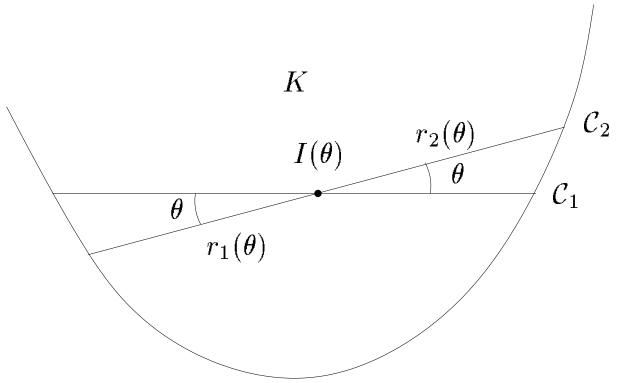

Figure 5: The middle point property.

have

$$\frac{1}{2}r_1^2(\theta)\cdot\theta = \frac{1}{2}r_2^2(\theta)\cdot\theta + o(\theta).$$

This implies that $\lim(r_1(\theta) - r_2(\theta)) = 0$ when θ tends to 0. □

The boundary of $O_t(K)$ is included in the envelop of the "modified" σ-chords. In the case $\gamma = 1/3$, these chords are the σ-chords themselves. For all other values of γ, this is no longer true and we have to compute the position of the intersection of closer and closer chords. This is the purpose of

Proposition 22 *Let K be a strictly convex set, and $\mathcal{C} = C_{s,t}$ a σ-chord of K. Consider P the farthest point of $C([s,t])$ from $\mathcal{C}$. If (L, h) are the coordinates of P in the direct orthonormal referential whose origin is the middle point of $\mathcal{C}$ and whose first axis is directed by C(t)-C(s), then the contribution of the modified chord arising from $\mathcal{C}$ is either void or the unique point with coordinates*

$$L\left(1 - 3\gamma\left(\frac{h}{a}\right)^{3\gamma-1}\right),\ h\left(1 - \left(\frac{h}{a}\right)^{3\gamma-1}\right) \tag{33}$$

in the same referential.

Proof. Examine the situation on Figure 6. Let $\mathcal{C}_\theta$ the σ-chord making an angle θ with $\mathcal{C}$. We search the coordinates of the intersection point of the modified chord of $\mathcal{C}$ and $\mathcal{C}_\theta$ when θ tends to 0. We set $x(\theta)$ the abscissa of the point $\mathcal{C}\cap\mathcal{C}_\theta$. By the middle point property, we know that $x(\theta) \to 0$ as $\theta \to 0$. Let $\mathcal{C}'_\theta$ the modified chord of $\mathcal{C}_\theta$. The distance between these two chords is $H(\theta) = h - a(\frac{h}{a})^{3\gamma}$ where $h = h(\theta)$ is the chord-arc distance of $\mathcal{C}_\theta$. Let $(L(\theta), H(0))$ be the coordinates of the common point of $\mathcal{C}'$ and $\mathcal{C}'_\theta$. Elementary but a bit fastidious geometry proves

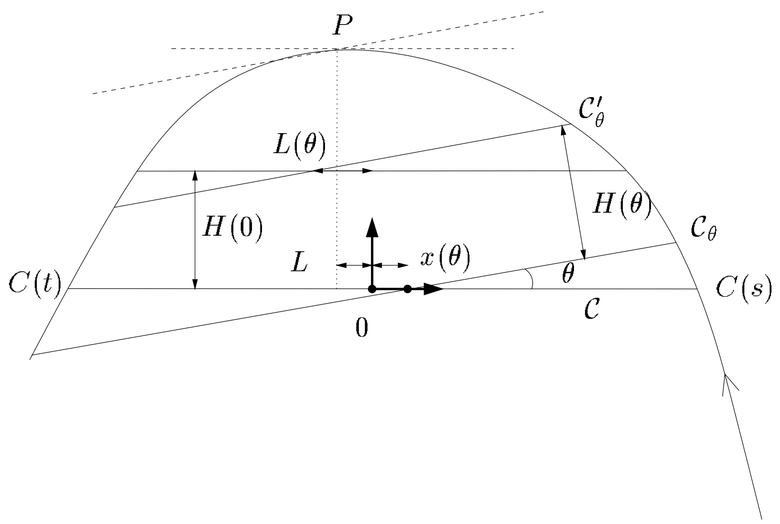

Figure 6: The modified middle point property.

that

$$L(0) = L\left(1 - 3\gamma\left(\frac{h}{a}\right)^{3\gamma-1}\right), \tag{34}$$

implying that the limit point we are looking for has coordinates given by (33). □

Remark 23. In the case of the general function of the curvature and without introducing the scaling preventing saturation phenomenon, the coordinates of the point are

$$L\left(1 - G'\left(\frac{h}{\omega\sigma^{2/3}}\right)\right),\ h - \omega\sigma^{2/3}G\left(\frac{h}{\omega\sigma^{2/3}}\right).$$

As precised in the proposition, the limit point may not belong to the boundary of $O_t(K)$. Indeed $\partial O_t(K)$ is in general strictly included in the envelop of the modified σ-chords. In general, this envelop is not even the boundary of a convex set! Nevertheless, if we know that C is a convex curve then it is simple to decide whether a point has to be kept or not by comparing its position with adjacent modified chord. Hence we can remove the bad points and obtain a convex set. For example, on Figure 7, we display the envelop of the modified σ-chords of a square. If the set is a “corner”, the explicit computation can be made and shows the same behavior in the corner as in Figure 7. The eroded set is obtained by removing the parts with cusps in the corners of the square.

5 Numerical Experiments

To finish, we display numerical experiments, first in the case of convex sets. By a change of scale variable, we implemented an approximation of the equation

$$\frac{\partial C}{\partial t} = (t\kappa)^\gamma. \tag{35}$$

For this rescaling, scale and space are homogeneous (precisely, if T_t maps C to $C(t)$ by this equation, we check that $T_t \circ H_\lambda = H_\lambda \circ T_{t/\lambda}$).

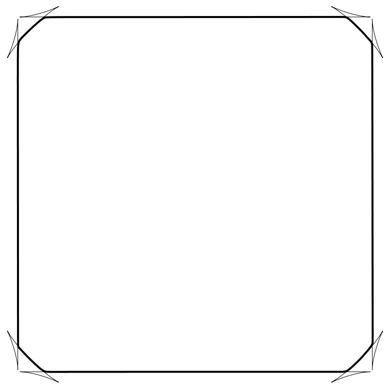

Figure 7: Envelop of the modified σ-chords for a square ($\gamma = 2$). The result is not convex. The bold line is what is to be kept.

5.1 Closed convex curves

The first example is the the case of circles. The radius is explicitly computable for the scale space since

$$R(t) = \left(R(0)^{\gamma+1} - t^{\gamma+1}\right)^{\frac{1}{\gamma+1}} . \tag{36}$$

Remark that the extinction scale for a circle with initial radius $R(0)$ is $R(0)$ for any γ. On Figure 8, we display the evolution of a circle with radius 10, for $t = 9$.

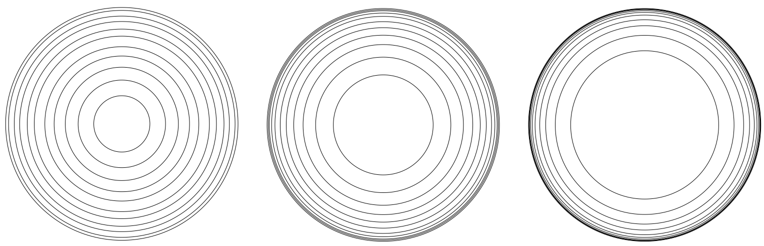

Figure 8: Evolution of circles for $\gamma = 0.4$, 1, 2 displayed at scale 0, 1,..., 9 (fast computation).

In Table 1, we give the theoretical and computed radius for several values of γ and for a circle with initial radius equal to 10. We performed two sets of experiments with different precisions, and for each of them we give the CPU time on a Pentium II 366MHz, the number of performed iterations and the final obtained radius. In Figures 9 and 10, we display the evolution of convex closed polygons. On each figure, the display scales are the same for all the different values of γ.

5.2 Unclosed curves evolution

Until now, we have only studied the evolution of compact convex sets. The boundary of such sets is a closed convex curve. It is possible to make nonclosed convex curves evolve by fixing their end points as in [5]. This is equivalent to symmetrize and periodize the curve. The steady state to this evolution is a segment if the two ends are disjoint. If they are equal, then a singularity

γ	R_{theo}	fast computation			slow computation		
		R_{comp}	# iter	CPU (s)	R_{comp}	# iter	CPU (s)
0.34	2.20	2.25	50	0.66	2.21	234	20
1.0	4.35	4.30	112	1.30	4.35	388	27
2.0	6.47	6.39	112	1.22	6.45	421	27
3.0	7.65	7.56	108	1.38	7.64	400	28
10.0	9.66	9.62	100	1.19	9.66	214	15

Table 1: The shortening at scale 9 of an initial circle with radius 10 is considered. For different values of γ, we give the theoretical value of the final radius (R_{theo}), the corresponding values obtained with different precisions (fast or slow computation), the number of performed iterations and the used CPU time.

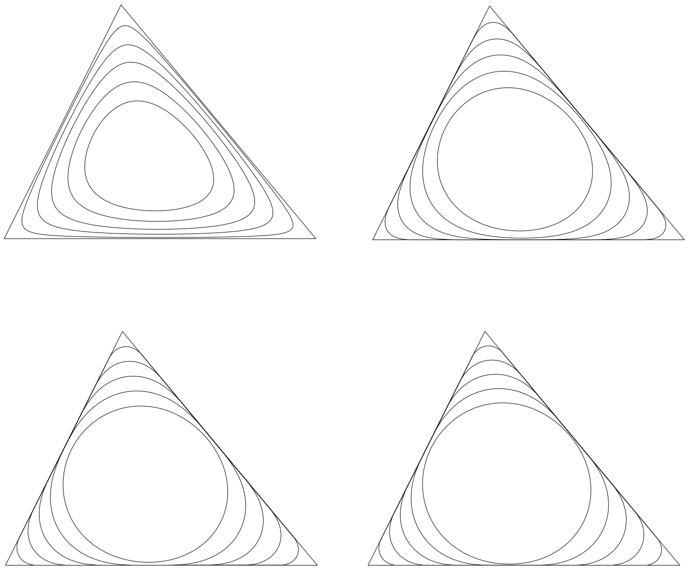

Figure 9: Evolution of a triangle. Up-left: $\gamma = 0.34$, CPU=6s; top-right $\gamma = 1$, CPU=20s, bottom-left: $\gamma = 2$, CPU=43s, bottom-right $\gamma = 3$, CPU=68s.

occurs in finite time. Such an evolution is displayed on Figure 11 for a nonclosed convex curve for several values of γ.

As can be derived from [8, 9, 2], if C moves by Equation 25 and is locally the graph of a function, y then y satisfies the equation

$$y_t = (y'')^\gamma (1 + (y')^2)^{\frac{1}{2}(1-3\gamma)}. \tag{37}$$

We use this equation to determine whether a convex curve with distinct fixed end points becomes a straight line in finite time.

Proposition 24 *Let u_0 be a strictly convex function on $[-1, 1]$ with $u_0(0) = u_0(1) = 0$. Let u*

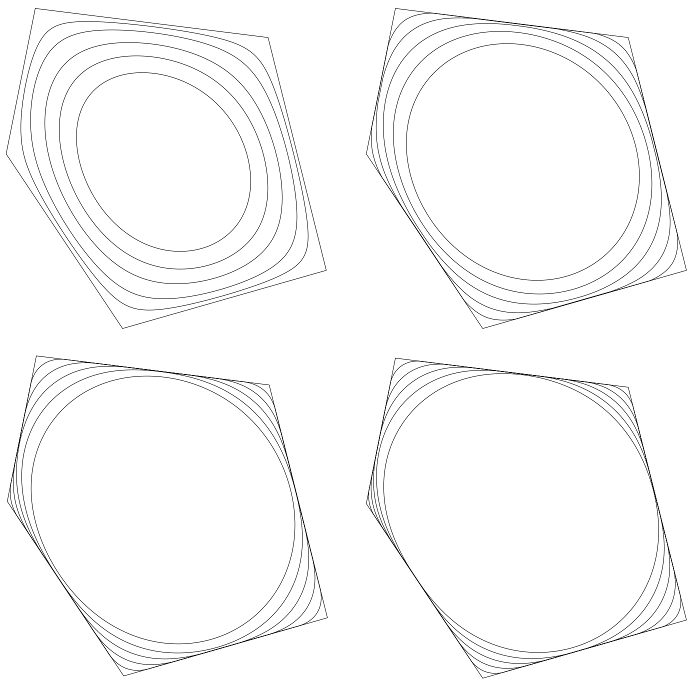

Figure 10: Evolution of a pentagon. Up-left: $\gamma = 0.34$, CPU=0.5s; top-right $\gamma = 1$, CPU=1.2s, bottom-left: $\gamma = 2$, CPU=10s, bottom-right $\gamma = 3$, CPU=14s.

be the solution of

$$\begin{aligned} u_t &= (u'')^\gamma (1+(u')^2)^{\frac{1}{2}(1-3\gamma)} \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned}$$

Then, if $\gamma < 1$, u becomes identically zero in finite time. If $\gamma \geq 1$, then the steady state is attained for infinite time.

Proof. From Equation (37), we deduce that u is subsolution of the following equation

$$u_t = (u'')^\gamma. \tag{38}$$

On the other hand, we can derive from Equation (37) an equation for u'. This equation is also parabolic. By maximum principle, the supremum of u' is attained at time $t = 0$. We conclude that u is supersolution of

$$u_t = C(u'')^\gamma \quad \text{with } C = (1 + (\sup u_0')^2)^{\frac{1}{2}(1-3\gamma)}. \tag{39}$$

For both Equations (38), (39), we can compute separable solutions of the type $g(t)f(x)$. Both functions f and g then satisfy an ordinary differential equation that is explicitely solvable for g and can be expressed in terms of elliptic functions for f. We can then check that the time components becomes null in finite time if and only if $\gamma < 1$. In order to conclude, it suffices to bound u_0 fromabove if $\gamma < 1$ and from below if $\gamma \leq 1$ by a adequate f (the spatial part of the separable solution) and to apply maximum principle. □

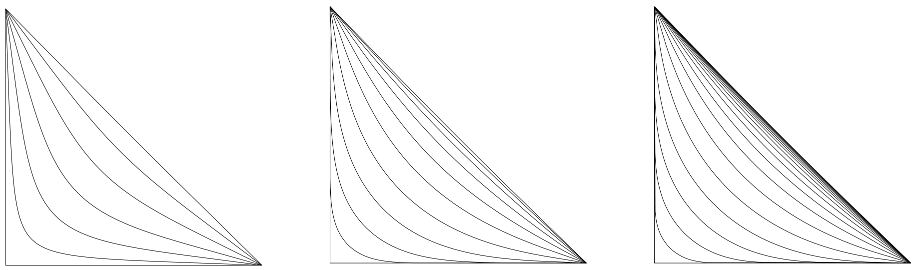

Figure 11: Evolution of an angle for different powers of the curvature (left:$\gamma = 0.34$, middle: $\gamma = 1.0$, right: $\gamma = 2.0$). The displayed curves correspond to scales that are integer multiples of a same fixed value.

Let us interpret this experiment in terms of image processing. Imagine that we process the filtering of a grey level image by applying (25) to its level lines, as done in [13] for $\gamma = 1/3$. This will have smoothing effects, and in particular one can expect to remove pixellization effects. The periodic structure corresponding to the initial state of Figure 5.2 is a "staircase" line that corresponding to the discretization of a perfect straight line (oriented at 45°). As the evolution scale increases, this infinite staircase is smoothed and eventually becomes a perfect straight line in finite time if $\gamma < 1$. A natural question now arises: how choose γ in order to smooth these staircase effects with the smallest possible damaging effects on the image? In other terms, what power of the curvature regularizes discrete lines in the shortest time? The previous proposition asserts that only powers smaller than 1 can straigthen a staircase in finite time. Experiments of Figure 5.2 corroborate this result and indicate that the straightening time increases with γ. But this result has to be counterbalanced in the following. By smoothing an image, we would like to remove small undesirable details while keeping the rest of the image unchanged. The results and experiments on circles (Figure 8) tend to prove that large powers can do this in a better way than small powers (*i.e.* when a circle with initial radius 9 disappears at scale $t = 9$, a circle with radius 10 is less changed for large values of γ).

5.3 Generalization to nonconvex sets

An algorithm for nonconvex curves has been proposed in [18] for the affine erosion $\gamma = 1/3$. We apply the same method in the general case. It consists in splitting a curve into its convex and concave components. This decomposition is unique and well defined. By this method, we do not find inflexion points but inflexion segments and we define an inflexion point as the middle point of an inflexion segment. We then apply the erosion operator to all the convex components by fixing the end points (which are the inflexion points defined above). Once this is done we gather all the parts to form a new curve. We reapply the decomposition and the erosion to this new curve. Notice that near an inflexion point, the ending segments of the joining convex components do not stay parallel in general; thus, inflexion points have have no reason to stay still. Practically, this is what we observe. Moreover, as γ increases, they seem to move more and more slowly, which also seems logical. On Figures 12 and 13, we display the result of the algorithm on nonconvex curves. Nevertheless, we do not have any strict justification for the convex component decomposition and the displacement of inflexion points should be studied more carefully.

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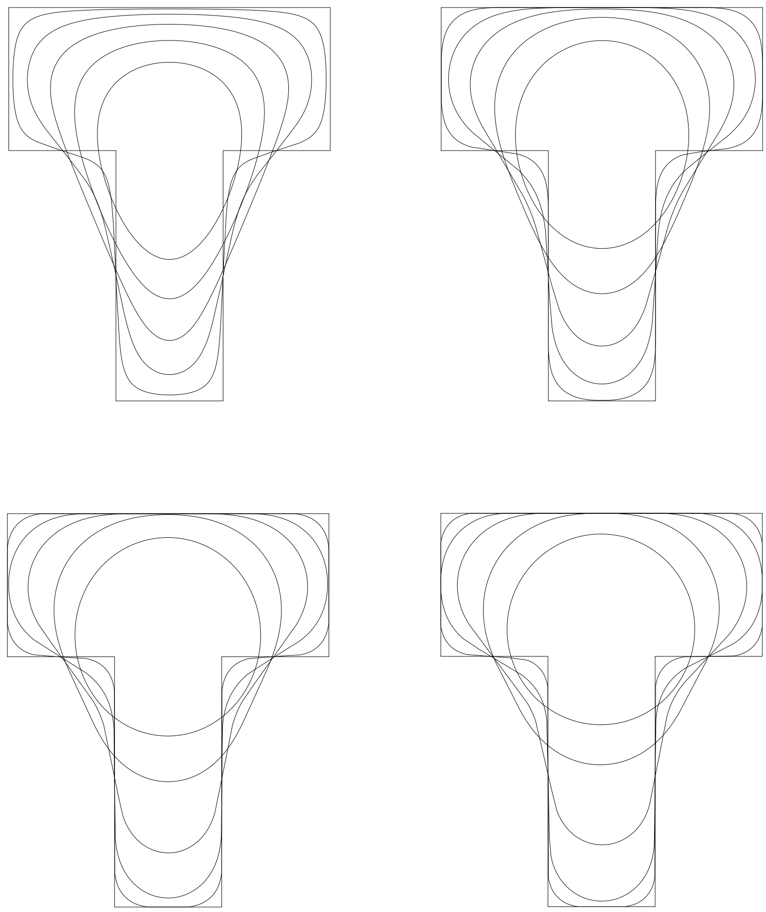

Figure 12: Scale space of a “T” shape: the curves are displayed at the same evolution scale. From left to right and top to bottom: γ =0.4, 1, 2, 3. CPU times are respectively 5, 9, 23 and 42 seconds.

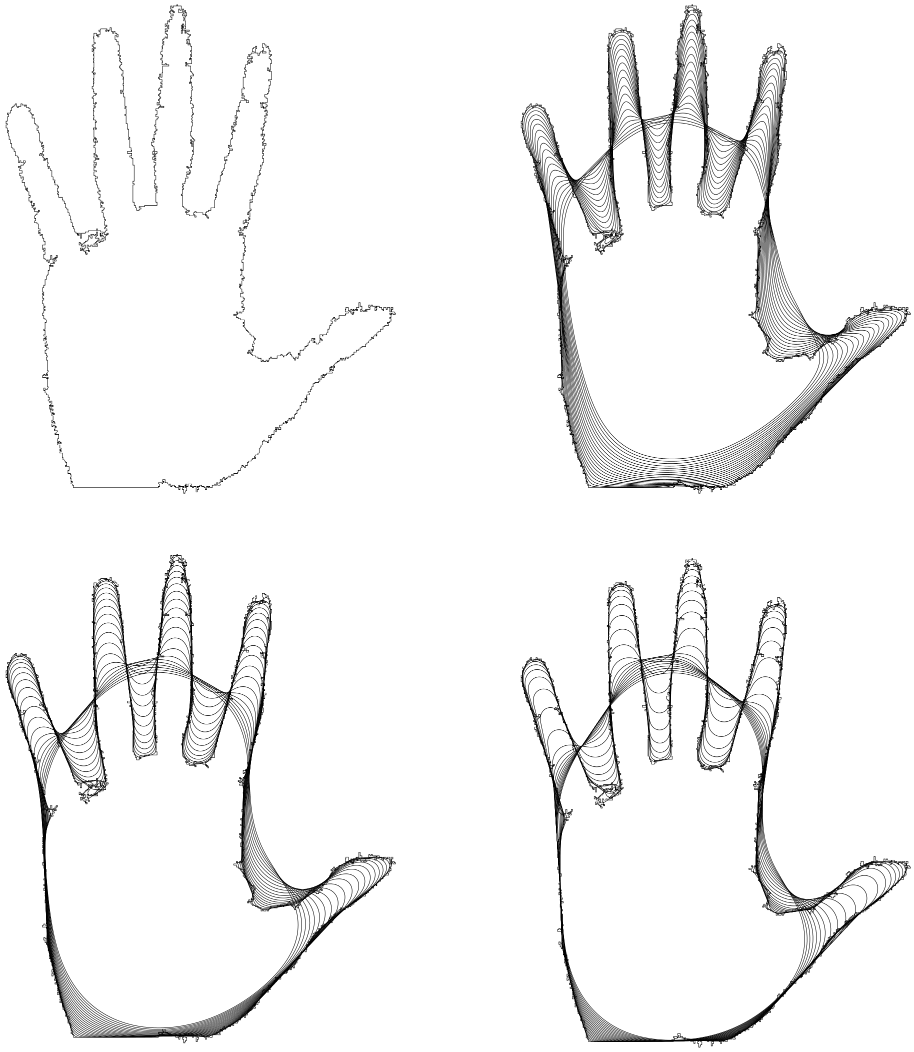

Figure 13: Scale space of a hand curve: the curves are displayed at the same evolution scale. From left to right and top to bottom: initial curve,γ =0.4, 1, 2.. CPU times are respectively 6, 28 and 91 seconds.

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