Affine plane curve evolution : a fully consistent scheme

Lionel Moisan

Abstract— ***We present an accurate numerical scheme for the affine plane curve evolution and its morphological extension to grey-level images. This scheme is based on the iteration of non-local, fully affine invariant and numerically stable operator, which can be exactly computed on polygons. The properties of this operator ensure that a few iterations are sufficient to achieve a very good accuracy, unlike classical finite difference schemes which generally require a lot of iterations. Convergence results are provided, as well as theoretical examples and experiments.***

Keywords— ***curve evolution, affine invariance, scale space, numerical scheme.***

I. Introduction

THE affine scale space has been discovered simultaneously a few years ago in its geometrical and image formulation. If we represent a grey-level image as a function $u_0 : \mathbb{R}^2 \to \mathbb{R}$, the affine morphological scale space of u_0 (shortly written AMSS) is the collection of images $(\boldsymbol{x} \mapsto u(\boldsymbol{x},t))_{t \geqslant 0}$ defined by

$$\frac{\partial u}{\partial t} = |Du|\kappa(u)^{\frac{1}{3}}. \tag{1}$$

with initial condition $u(\cdot,0) = u_0$. Here, Du means the gradient of u with respect to $\boldsymbol{x}$ and the second order operator $\kappa(u) = \mathrm{div}(\frac{Du}{|Du|})$ can be interpreted when $|Du| \neq 0$ as the curvature of the level curve $\{\boldsymbol{y};\ u(\boldsymbol{y}) = u(\boldsymbol{x})\}$ at point $\boldsymbol{x}$. Equation (1) makes sense for continuous images according to the theory of viscosity solutions (see [7]) ; its geometrical interpretation is that all level curves of u evolve according to the equation

$$\frac{\partial C}{\partial t}(p,t) = \kappa(p,t)^{\frac{1}{3}} \boldsymbol{N}(p,t), \tag{2}$$

where $\kappa(p,t)$ and $\boldsymbol{N}(p,t)$ are respectively the curvature and the normal vector of the curve $C(\cdot,t)$ in $C(p,t)$. As shown G. Sapiro and A. Tannenbaum in [14], using the affine arclength parameterization s reduces Equation (2) to the nonlinear heat equation $\partial C/\partial t = \partial^2 C/\partial s^2$.

The AMSS has been characterized in [1] as the only regular semigroup $T_t : u_0 \mapsto u(\cdot,t)$ which satisfies the following invariance properties :

[Monotonicity] : $u \leqslant v \ \Rightarrow\ T_t(u) \leqslant T_t(v)$

[Morphology] : For any monotone scalar function g,
$T_t(g \circ u) = g \circ T_t(u)$

[Affine invariance] : For any affine map ϕ,
$T_t(u \circ \phi) = \left(T_{t.|det\,\phi|}(u)\right) \circ \phi.$

Here, $\det \phi$ means the determinant of the linear part of ϕ, i.e. $\det \phi = \det A$ where $\phi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ and $(A, \boldsymbol{b}) \in GL(\mathbb{R}^2) \times \mathbb{R}^2$.

CMLA, Ecole Normale Supérieure de Cachan, France.
E-mail: moisan@cmla.ens-cachan.fr

Several algorithms have been proposed to implement numerically Equations (1) or (2), but none of them manages to satisfy numerically the previous properties. In 1993, L. Alvarez and F. Guichard proposed a quasilinear scheme where a 3x3 neighbourhood is used in each point of the image to compute its evolution (see [10]). Of course, such a local scheme cannot be affine invariant for the neighbourhood size is fixed in advance.

An inf-sup operator was also proposed in [11] to implement the affine morphological scale space. Inspired by mathematical morphology operators, this inf-sup scheme uses an affine invariant basis of structuring elements. Its Euclidean analog had been treated by F. Catté and F. Dibos in [5]. However, the full morphological invariance (no new grey-level is created on the image) and the grid discretization create difficulties. Indeed, a level curve is constrained to move by entire speeds : either it does not move, or it jumps over at least one pixel (see [6]).

For the affine scale space of curves, all geometrical schemes that have been proposed suffer from the space discretization of the curves (see [10]), which prevents the monotonicity property. The main difficulty comes from the fact that there is no a priori relation between the number of vertices of a polygon and the number of the vertices of a discretization of its affine shortening (this number should increase drastically for a triangle, but decrease as much for a very irregular curve). Thus, an algorithm based on a local point-by-point evolution cannot implement successfully the affine scale space.

A numerical scheme for the affine scale space becomes a scheme for the AMSS when applied to the level curves of an image. Conversely, S. Osher and J. Sethian (see [13], [16]) computed the affine scale space of the boundary of a set S by applying the AMSS to its distance image $u(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})\mathrm{dist}(\boldsymbol{x}, \partial S)$, where $\varepsilon(\boldsymbol{x}) = -1$ if $\boldsymbol{x} \in S$, 1 otherwise. This approach permits complicated curve evolutions, but inherits the drawbacks of numerical schemes on images.

In this paper, we first define a geometrical operator —called *affine erosion*—, whose tangent operator spans the positive affine scale space. Then, we extend it to grey level images using the level set decomposition, and prove its consistency using the Matheron's characterization theorem of morphological operators and a general consistency theorem for inf-sup operators from [11]. We also prove that the iterated alternated operator converges towards the affine scale space. Last, an exact algorithm is described and experiments are given. We skiped some of the technical proofs (which can be found in [12]), but tried to maintain most relevant arguments.

II. Affine erosion of sets

We are going to define a geometrical operator E_σ (σ is a nonnegative scale parameter), called affine erosion, and acting on subsets of the plane. Since the geometrical definition of $E_\sigma(U)$ requires some regularity on the boundary of U, we first restrain our definition to a certain kind of sets U. We shall see further how to extend E_σ to other sets.

First, we need some geometrical definitions on curves. We choose to call a *simple curve* any subset of $\mathbb{R}^2$ homeomorphic to the circle $S^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$ (closed curve) or to an open interval of $\mathbb{R}$ (non closed curve). We shall often refer to a simple curve using the notation $C(I)$, which means implicitely that $C : I \to C(I)$ is a parameterization of the curve. We also define a semi-closed curve as an oriented simple curve $\mathcal{C}$ such that $\mathbb{R}^2 - \mathcal{C}$ has exactly two connected components (e.g. a parabola). A semi-closed curve can also be viewed as a simple oriented closed curve defined on the Alexandroff compactification of the plane $\mathbb{R}^2 \cup \{\infty\}$. Last, we say that a simple curve $C(I)$ is piecewise convex if there exists a finite subdivision $(s_1, s_2, \ldots s_n)$ of I such that each sub-curve $C(]s_i, s_{i+1}[)$ is a convex curve.

Let $C(I)$ be a simple curve. We say that $(s,t) \in I^2$ is a *chord* of C if and only if the piece of curve $C(]s,t[)$ and the open segment $]C(s)C(t)[$ are either disjoint, or equal. The connected closed set enclosed by $C(]s,t[)$ and the chord segment $]C(s)C(t)[$ is a *chord set* of C, written $C_{s,t}$. This definition is naturally extended to infinite segments when s or t belong to the boundary of I. If area $(C_{s,t}) = \sigma$, then (s,t) is called a σ-chord and $C_{s,t}$ a σ-chord set of C.

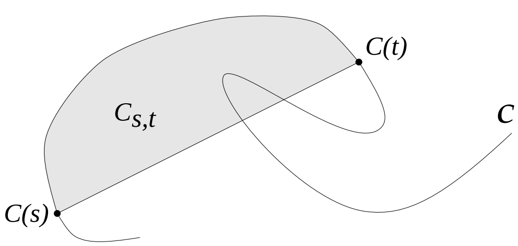

Fig. 1. A chord set of a simple curve. Notice that the chord segment $[C(s)C(t)]$ can intersect $\mathcal{C} - C([s,t])$.

If $\mathcal{C}$ is oriented and area $(C_{s,t}) \neq 0$, the orientation induced by C on the boundary of $C_{s,t}$ tells whether (s,t) is a positive or a negative chord. We take the convention that a 0-chord set is both positive and negative. Last, since the previous definition of chord does no depend on the parameterization of the curve, it makes sense to call (A,B) a chord of $\mathcal{C} = C(I)$ when $A = C(s)$, $B = C(t)$, and (s,t) is a chord of C.

Definition 1: A open subset S of the plane $\mathbb{R}^2$ is a C-set if

(i) it has a finite number of connected components

(ii) the boundary of any connected component is a finite disjoint union of semi-closed piecewise convex curves.

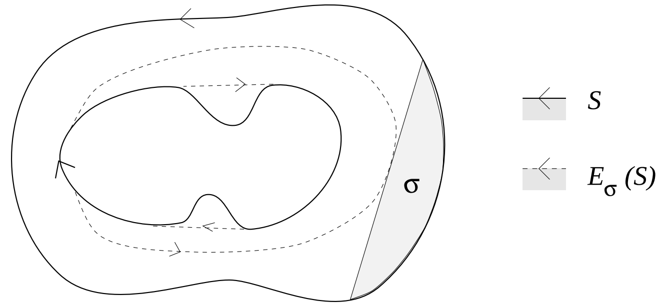

Fig. 2. Affine erosion of a C-set with 2 components

These oriented curves enclosing the connected components of S are called the components of ∂S.

Remark : The components of ∂S are not necessarily disjoint : if S is the inside part of two tangent disks, ∂S is connected but has two components.

This definition of C-sets is a compromise between regularity (the boundary of a C-set admits a tangent almost everywhere) and genericity (any finite union of convex sets is a C-set).

A. *Affine erosion of a C-set*

Definition 2: The σ-affine erosion of a C-set S is the set of the points of S which do not belong to any positive chord set —with area less than σ— of a component of ∂S.

$$E_\sigma(S) = S \quad - \bigcup_{\substack{\sigma' \leqslant \sigma \\ K \in \mathcal{K}^+_{\sigma'}(\partial S)}} K.$$

Here, $\mathcal{K}^+_{\sigma'}(\partial S)$ means the collection of all positive σ'-chord sets of all components of ∂S.

Proposition 1: The affine erosion of a C-set is a C-set.

The main argument is that if S is a C-set, the boundary of $E_\sigma(S)$ is made of concave pieces of ∂S and a finite number of new convex pieces. A complete proof is given in [12]. A consequence of Proposition 1 is that we can define the affine erosion of a piecewise convex semi-closed curve as a collection of such curves, using the natural correspondance between a C-set and its boundary.

A.1 Example : affine erosion of a corner

Proposition 2: The σ-affine erosion of a "corner" $W = \{x\, v_1 + y\, v_2,\ \ x > 0, y > 0\}$ is the inside convex part of the (half) hyperbola defined by

$$\{xy = \frac{\sigma}{2\ [v_1, v_2]},\ \ x > 0, y > 0\}.$$

in the affine basis (O, v_1, v_2).

The notation $[v_1, v_2]$ means the determinant of the two plane vectors v_1 and v_2, that is, the algebraic area of the parallelogram $(\boldsymbol{0}, v_1, v_2, v_1 + v_2)$.

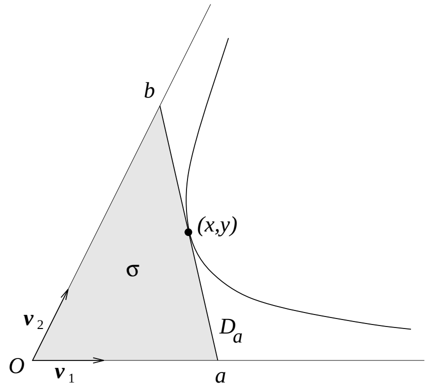

Fig. 3. The affine erosion of a "corner" is a hyperbola

Proof: First, we notice that only the σ-chord sets are significant to define the affine erosion of W because W is convex (a chord set with area less than σ can be enclosed in a σ-chord set). Secondly, the lines supporting the σ-chord segments of W have equation $x/a + y/b = 1$ (see Fig. 3) and are submitted to the area constraint $2\sigma = ab\,[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Consequently, the boundary of $E_\sigma(W)$ is obtained by the envelope of these lines, given by the system

$$\left\{ \begin{array}{l} D_a :\ \dfrac{x}{a} + \dfrac{a\,[\boldsymbol{v}_1, \boldsymbol{v}_2]\,y}{2\sigma} = 1 \\[2ex] D'_a :\ \dfrac{-x}{a^2} + \dfrac{[\boldsymbol{v}_1, \boldsymbol{v}_2]\,y}{2\sigma} = 0. \end{array} \right.$$

Then, eliminating a yields

$$xy = \frac{\sigma}{2\,[\boldsymbol{v}_1, \boldsymbol{v}_2]}.$$

■

B. Basic properties of the affine erosion

Proposition 3: $E_\sigma(S)$ is nonincreasing with respect with σ, and nondecreasing with respect with S, i.e.

$$\sigma_1 \leqslant \sigma_2 \quad \Rightarrow \quad E_{\sigma_2}(S) \subset E_{\sigma_1}(S), \quad \text{and}$$

$$S_1 \subset S_2 \quad \Rightarrow \quad E_\sigma(S_1) \subset E_\sigma(S_2).$$

Proof: (second assertion) Let S_1 and S_2 be two C-sets such that $S_1 \subset S_2$, and consider M a point of S_2. If M does not belong to $E_\sigma(S_2)$, there exists a σ'-chord segment D of a component of ∂S_2 such that $\sigma' \leqslant \sigma$ and M belongs to the associated chord set.

1. If $M \notin S_1$, then $E_\sigma(S_1) \subset S_1$ yields $M \notin E_\sigma(S_1)$.

2. If $M \in S_1$, consider the connected component A of S_1 containing M :

2.a. If $A \cap D = \emptyset$, then the external boundary of A encloses a subset of area less than σ', so that $E_\sigma(A) = \emptyset$ and $M \notin E_\sigma(S_1)$.

2.b. If $A \cap D \neq \emptyset$, then a subset of D defines a σ''-chord set of S_1 containing M and $S_1 \subset S_2$ implies $\sigma'' \leqslant \sigma'$, so that $M \notin E_\sigma(S_1)$.

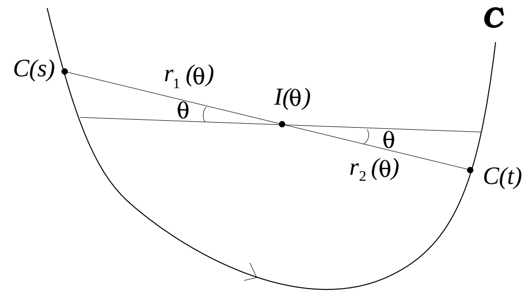

Fig. 4. The middle point property

Thus, $M \notin E_\sigma(S_2) \ \Rightarrow\ M \notin E_\sigma(S_1)$, which means that $E_\sigma(S_1) \subset E_\sigma(S_2)$. ■

The second part of Proposition 3 establishes the monotonicity of E_σ, which guarantees the numerical stability of this operator. The first part, easy to prove, justifies a posteriori the name "erosion" in a geometrical sense. Notice that E_σ is not an erosion on a lattice as defined by Serra (see [15]), because $E_\sigma(A \cap B) \neq E_\sigma(A) \cap E_\sigma(B)$ in general.

Proposition 4: The affine erosion is covariant with respect to the affine transformations of the plane, i.e for any affine map ϕ and any C-set S,

$$\phi\left(E_\sigma(S)\right) = E_{\sigma \cdot |\det \phi|}\left(\phi(S)\right).$$

C. Affine erosion of convex C-sets

So far, the definition of the affine erosion is not very practical, especially for curves, since we must consider the associated sets. In fact, for most convex curves $\mathcal{C}$, $E_\sigma(\mathcal{C})$ is generated directly by the middle points of the σ-chords of $\mathcal{C}$. The reason is roughly explained on Fig. 4 : given a σ-chord segment $[C(s)C(t)]$, another σ-chord segment intersects $[C(s)C(t)]$ in $I(\theta)$, and as $\theta \to 0$, the equi-area constraint forces

$$\frac{1}{2} r_1^2(\theta) \cdot \theta = \frac{1}{2} r_2^2(\theta) \cdot \theta + o(\theta),$$

so that $r_1(\theta) - r_2(\theta) \to 0$ and $I(\theta)$ converges towards the middle of $[C(s)C(t)]$.

If a convex semi-closed curve $\mathcal{C}$ is non trivial (that is to say, different from a straight line), the affine erosion of $\mathcal{C}$ is included in the set of the middle points of the σ-chords of $\mathcal{C}$. However, the reverse inclusion only happens up to a limiting scale (which can be either finite, infinite or zero according to $\mathcal{C}$). More precisely, we say that a chord (A, B) of $\mathcal{C}$ is *regular* if a measure α of the angle made by the left tangent in A and the right tangent in B satisfies $0 \leqslant \alpha < \pi$ (see Fig. 5). Then, we say that the scale $\sigma \geqslant 0$ is regular for $\mathcal{C}$ if any σ-chord of $\mathcal{C}$ is regular, and we note $\sigma_r(\mathcal{C})$ the supremum of the regular scales of $\mathcal{C}$.

Theorem 1 (middle point property) Let $\mathcal{C}$ be a non-trivial convex semi-closed curve. For any $0 < \sigma < \sigma_r(\mathcal{C})$, $E_\sigma(\mathcal{C})$ is exactly the set of the middle points of the σ-chord segments of $\mathcal{C}$, and this defines a natural homeomorphism between $\mathcal{C}$ and $E_\sigma(\mathcal{C})$.

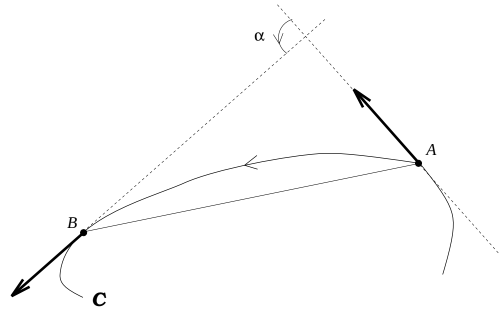

Fig. 5. (A, B) is a regular chord iff $0 \leqslant \alpha < \pi$

Obviously, this theorem is interesting only when $\sigma_r(\mathcal{C}) > 0$, which is not always the case, even for simple convex curves as polygons. However, one can see that

1. If $\mathcal{C}$ is a convex semi-closed curve of class C^1, then $\sigma_r(\mathcal{C}) > 0$.
2. If $\mathcal{C}$ is a convex polygon with vertices $P_0, P_1, \ldots P_{n-1}$, then $\sigma_r(\mathcal{C}) > 0$ if and only if $[P_iP_{i+1}, P_{i+2}P_{i+3}] > 0$ for all i (the indices being taken modulo n).

What happens for non-regular scales ? In general, the curve described by the middles of the σ-chord segments has "ghost parts" which must be removed to obtain the desired affine erosion. For instance, these "ghost parts" appear at any scale in the erosion of a triangle, for which $\sigma_r = 0$ (see Fig. 6). This phenomenon is very similar to the crossing of fronts for a flame propagation : the "ghosts parts" must then be removed according to the Huygens principle ; roughly speaking, *once a particle is burnt it stays burnt and cannot burn any more* (see [13]).

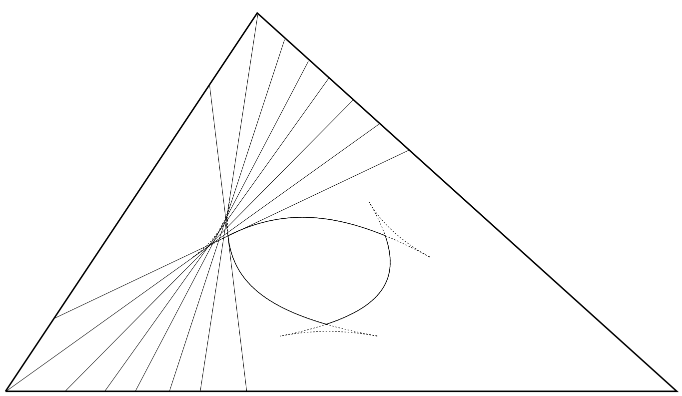

Fig. 6. "ghost parts" (dashed) always appear for triangles

D. Consistency

Theorem 2: let $\mathcal{C}$ be a semi-closed convex curve of class C^n with $n \geqslant 1$. Then for any $\sigma < \sigma_r(\mathcal{C})$, $E_\sigma(\mathcal{C})$ is a semi-closed convex curve of class C^n. Moreover, if $n \geqslant 3$, the infinitesimal evolution as $\sigma \to 0$ of a point $M \in \mathcal{C}$ where the curvature κ is nonzero is given by

$$M_\sigma = M + \omega \cdot \sigma^{\frac{2}{3}} \cdot \kappa^{\frac{1}{3}}\, \boldsymbol{N} + O(\sigma^{\frac{4}{3}}) \quad \text{with} \quad \omega = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{2}{3}},$$

where $\boldsymbol{N}$ is the normal vector to $\mathcal{C}$ in M.

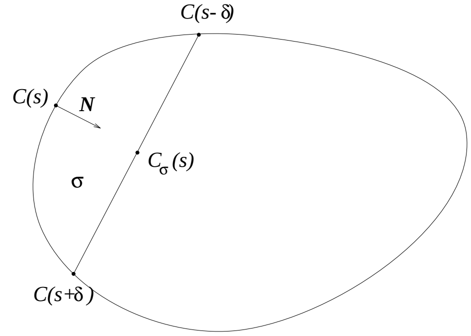

Fig. 7. Affine erosion of a convex semi-closed curve

Proof: Consider $s \mapsto C(s)$ an Euclidean length parameterization of $\mathcal{C}$ (i.e. $|C'(s)| = 1$ everywhere). Since $\mathcal{C}$ is convex, we know from Theorem 1 that $E_\sigma(\mathcal{C})$ is exactly made of the middle of the σ-chords of $\mathcal{C}$ as soon as $0 < \sigma < \sigma_r(\mathcal{C})$ (which makes sense because we know that $\sigma_r(\mathcal{C}) > 0$). Let $(s-\delta, s+\delta)$ be a σ-chord of C and $C_\sigma(s)$ the middle of the associated segment (see Fig. 7). Since C is of class C^1, we can use the Green formula to compute the area

$$\sigma = \frac{1}{2} F(s, \delta(s,\sigma)), \quad \text{where}$$

$$\begin{aligned} F(s,t) \;&=\; \int_{s-t}^{s+t} [C(u), C'(u)]\; dt \\ &+\; [C(s+t), C(s-t) - C(s+t)]\,. \end{aligned} \qquad (3)$$

Then, differentiating Equation (3) yields

$$\frac{\partial F}{\partial t}(s,t) = [C(s+t) - C(s-t), C'(s+t) - C'(s-t)]\,.$$

$\mathcal{C}$ being convex, we have, for any distincts points $C(a)$ and $C(b)$ of $\mathcal{C}$, the inequality

$$[C'(a), C(b) - C(a)] \geqslant 0,$$

and the equality holds iff the piece of curve $C([a,b])$ is a segment. Hence, $[C(s+t) - C(s-t), C'(s+t)]$ and $[C(s+t) - C(s-t), -C'(s-t)]$ are positive numbers and their sum cannot be zero unless $\sigma = 0$, which is not the case, or unless $C(s+t) = C(s-t)$, which is impossible as soon as $0 < t \leqslant \delta$. As a consequence, $\frac{\partial F}{\partial t}(s,\delta)$ never vanishes and the global inversion theorem allows us to claim that the map $s \mapsto \delta(s,\sigma)$ is of class C^n as well as $(s,t) \mapsto F(s,t)$.

We just proved that the function

$$s \mapsto C_\sigma(s) = \frac{1}{2}\left[C(s - \delta(s,\sigma)) + C(s + \delta(s,\sigma))\right]$$

is of class C^n. Moreover, since the vectors $C'(s - \delta(s,\sigma))$ and $C'(s + \delta(s,\sigma))$ cannot be colinear for $\sigma < \sigma_r(\mathcal{C})$, the derivative

$$2\frac{\partial}{\partial s} C_\sigma(s) = (1 - \frac{\partial \delta}{\partial s}) C'(s-\delta) + (1 + \frac{\partial \delta}{\partial s}) C'(s+\delta)$$

never vanishes. As a consequence, the curve C_σ is of class C^n in the geometrical sense (that is, C_σ is a regular parameterization).

Let us now suppose that C is of class C^3, so that the curvature $\kappa(s) = [C'(s), C''(s)]$ is well defined in $C(s)$. A simple expansion near $t = 0$ gives

$$\begin{aligned}\frac{\partial F}{\partial t}(s,t) &= \left[2tC'(s) + O(t^2), 2tC''(s) + O(t^2)\right] \\ &= 4t^2\kappa(s) + O(t^3),\end{aligned}$$

which can be integrated to obtain

$$2\sigma = \frac{4}{3}\delta^3\kappa(s) + O(\delta^4).$$

Thus, whenever $\kappa(s) \neq 0$ we have

$$\delta(s,\sigma) = \left(\frac{3\sigma}{2\kappa(s)}\right)^{\frac{1}{3}} + O(\sigma^{\frac{2}{3}}),$$

and finally

$$\begin{aligned}C_\sigma(s) &= \frac{1}{2}\left[C(s-\delta) + C(s+\delta)\right] \\ &= C(s) + \frac{\delta^2}{2}C''(s) + O(\delta^3) \\ &= C(s) + \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{2}{3}}\sigma^{\frac{2}{3}} \cdot \kappa^{\frac{1}{3}}(s)\,\boldsymbol{N}(s) + O(\sigma^{\frac{4}{3}}),\end{aligned}$$

where $\boldsymbol{N}(s)$ is the normal vector to $\mathcal{C}$ in $C(s)$. ■

Notice that the $O(\sigma^{\frac{4}{3}})$ makes the affine erosion an infinitesimal approximation of the affine shortening of order 2, so that we can expect the iterated affine erosion to converge quickly towards the affine scale space.

The consistency is also satisfied for non convex curves : precisely, the class of semi-closed piecewise convex curves of class piecewise C^n is stable under affine erosion, and the asymptotic estimation of Theorem 2 remains true when $n \geqslant 3$, provided that we replace the curvature κ by its positive part $\kappa^+ = \max(0,\kappa)$.

D.1 Example : Affine erosion and scale space of an ellipse

Proposition 5: The σ-affine erosion of an ellipse with area A_0 is an ellipse with same axes and excentricity and with area

$$A(\sigma) = A_0 \cos^2\frac{\theta(\sigma)}{2},$$

where $\theta(\sigma)$ is defined by

$$\theta(\sigma) - \sin\theta(\sigma) = \frac{2\pi\sigma}{A_0}.$$

In particular, for an infinitesimal erosion, we have the canonical expansion

$$A^{\frac{2}{3}}(t^{\frac{3}{2}}) = A_0^{\frac{2}{3}} - \sqrt[3]{\frac{2\pi^2}{3}}\,t + O(t^2),$$

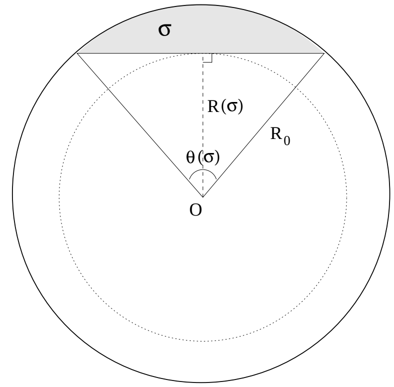

Fig. 8. Affine erosion of a circle

whereas the affine scale space (Equation 2) yields

$$A^{\frac{2}{3}}(t^{\frac{3}{2}}) = A_0^{\frac{2}{3}} - \frac{3}{4}\pi^{\frac{2}{3}}\,t.$$

Proof: Consider the parametrization of the ellipse

$$M(t) = \sqrt{\frac{A_0}{\pi}}(\cos t\,\boldsymbol{v}_1 + \sin t\,\boldsymbol{v}_2)$$

satisfying $[\boldsymbol{v}_1, \boldsymbol{v}_2] = 1$. We can find a linear map ϕ with determinant 1 which transforms the affine basis $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ into an orthogonal basis, in which $\phi(M(t))$ is the parameterization of a circle with same area A_0. Then, because the affine erosion commutes with the rotations, the affine erosion of a circle of radius R_0 necessarily is a circle with same center and with radius $R(\sigma) < R_0$. On Fig. 8 we can see that

$$\frac{R(\sigma)}{R_0} = \cos\frac{\theta(\sigma)}{2} \quad \text{and} \quad \frac{\sigma}{R_0^2} = \left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right).$$

Hence, as ϕ commutes with the affine erosion and with the homotheties, we deduce that on the ellipse as well as on the circle, the affine erosion acts as a homothety with ratio $\cos\frac{\theta(\sigma)}{2}$, which proves the first part of Proposition 5.

Let us now evaluate $A(\sigma) = A_0\cos^2\frac{\theta(\sigma)}{2}$ when σ tends towards 0. From

$$\theta - \sin\theta = \frac{2\pi\sigma}{A_0}$$

we deduce easily that

$$\theta(\sigma) = \left(\frac{12\pi\sigma}{A_0}\right)^{\frac{1}{3}} + O(\sigma).$$

This way, we obtain

$$A(\sigma) = A_0\left(1 - \sin^2\frac{\theta(\sigma)}{2}\right) = A_0 - A_0^{\frac{1}{3}}\left(\frac{3\pi\sigma}{2}\right)^{\frac{2}{3}} + O(\sigma^{\frac{4}{3}}),$$

and the "canonical" expansion of $A(\sigma)$ is

$$A^{\frac{2}{3}}(t^{\frac{3}{2}}) = A_0^{\frac{2}{3}} - \alpha \cdot t + O(t^2), \text{ with } \alpha = \frac{2}{3}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}} = \sqrt[3]{\frac{2\pi^2}{3}}.$$

As expected, $\frac{\alpha}{\omega} = \frac{3}{4}\pi^{2/3}$ (ω is the constant of Theorem 2). ∎

Fig. 9 shows the canonical representation of A, i.e. the graph of $A^{\frac{2}{3}}$ function of $\sigma^{\frac{2}{3}}$ up to normalization constants. For this representation, the action of the affine scale space on the ellipse is linear (dotted straight line on Fig. 9). As we can see, the action of the affine erosion on ellipses is very close to the one of its tangent operator, the (normalized) affine scale space, even for large scales. This suggests that we can build a fast scheme for the affine scale space by iterating the affine erosion with arbitrarily large time steps.

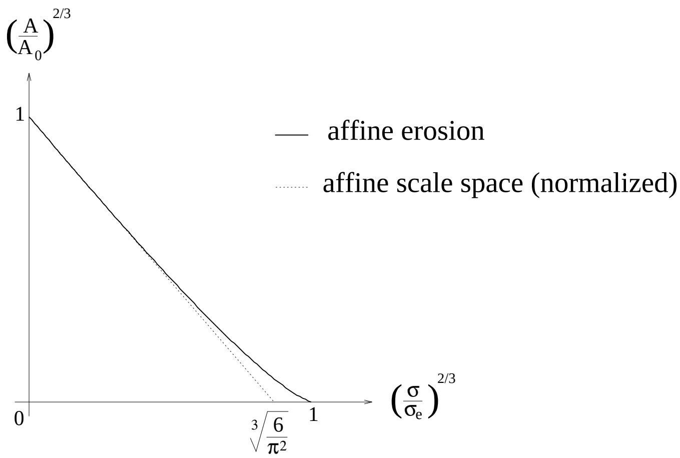

Fig. 9. Canonical area evolution for the affine erosion of an ellipse

III. Affine erosion of grey-level images

A. From sets to images

In terms of human vision, it is commonly accepted that an image (let us say a map $u : \mathbb{R}^2 \to \mathbb{R}$) carries more or less the same information as any image of the kind $g(u)$, where g is an arbitrary contrast change, that is to say an increasing scalar function. This principle is the basis of flat grey-scale kernels in mathematical morphology. From this point of view, an image u reduces to the decreasing collection of its level sets[1]

$$\chi_\lambda(u) = \{\boldsymbol{x} \in \mathbb{R}^2;\ u(\boldsymbol{x}) > \lambda\}.$$

Conversely, any image u can be recovered up to a contrast change from the family of its level sets by the relation

$$u(\boldsymbol{x}) = \sup\{\lambda;\ \boldsymbol{x} \in \chi_\lambda(u)\}.$$

Let us consider an nondecreasing operator T acting on open sets, and suppose that T is $\nearrow$-continuous, which means that for any nondecreasing sequence of open sets (X_n) we have

$$\bigcup_{n\in\mathbb{N}} E_\sigma(X_n) = E_\sigma\left(\bigcup_{n\in\mathbb{N}} X_n\right).$$

[1] For our study, it is more convenient to consider the "open" level sets rather than the "closed" classical ones defined by

$$\chi_\lambda(u) = \{\boldsymbol{x} \in \mathbb{R}^2;\ u(\boldsymbol{x}) \geqslant \lambda\}.$$

Then, applying T to the level sets of a l.s.c (read lower semicontinuous) image u defines a new image $\tilde{T}(u)$ which satisfies

$$\forall\lambda,\quad \chi_\lambda\left(\tilde{T}(u)\right) = T\left(\chi_\lambda(u)\right),$$

and this way we "extend" T to grey-level images. Notice that the monotonicity and the $\nearrow$-continuity of T are required due to the topological properties of a collection of level sets (see [12]).

B. Definition and basic properties

We would like to extend the affine erosion to grey-level images through the morphological level set decomposition. For that purpose, we first need to define the affine erosion of any subset of the plane. But the geometrical definition of the affine erosion (Definition 2) does not make sense for any subset of the plane, since in general its boundary is not a curve in a reasonable sense. This is the reason why we define the affine erosion of any set by completion with respect with the monotonicity property.

Definition 3: The σ-affine erosion of a set $U \subset \mathbb{R}^2$ is the open set

$$E_\sigma(U) = \bigcup_{S\ \text{C-set},\ S\subset U} E_\sigma(S).$$

This definition makes sense because if U is a C-set, we know that for any C-set S subset of U we have $E_\sigma(S) \subset E_\sigma(U)$. Moreover, the extended operator E_σ is clearly monotone, and one can check that it is also $\nearrow$-continuous. Hence, we can extend the affine erosion to grey-level images according to the level set decomposition.

Definition 4: The σ-affine erosion of a l.s.c image $u : \mathbb{R}^2 \to \mathbb{R}$ is the l.s.c image

$$E_\sigma(u) : x \mapsto \sup\{\lambda \in \mathbb{R};\ x \in E_\sigma(\chi_\lambda(u))\},$$

where $\chi_\lambda(u) = \{x;\ u(x) > \lambda\}$ is the λ-level set of u.

Proposition 6: E_σ defined on images is a monotone, morphological, and affine invariant operator, which means

[Monotonicity] : $u \leqslant v \Rightarrow E_\sigma(u) \leqslant E_\sigma(v)$

[Morphology] : For any increasing real function g,
$$E_\sigma(g \circ u) = g \circ E_\sigma(u)$$

[Affine invariance] : For any affine map ϕ,
$$E_\sigma(u \circ \phi) = \left(E_{\sigma\cdot|\det\phi|}(u)\right) \circ \phi.$$

Therefore, the affine erosion satisfies the same strong properties as the affine scale space, excepted, naturally, the semi-group property

$$T_t \circ T_{t'} = T_{t+t'},$$

which is *not* satisfied by the affine erosion even for any scale normalization of the kind $T_t = E_{f(t)}$. This is the reason why we need to iterate the affine erosion in order to approximate the affine scale space.

C. Asymptotic behaviour

There is a simple way to establish the consistency of the affine erosion. Indeed, the operator E_σ being translation invariant, monotone and morphological, the Matheron characterization theorem applies (see [11]) and we can write

$$E_\sigma(u)(\boldsymbol{x}) = \sup_{B\in\mathcal{B}_e} \inf_{\boldsymbol{y}\in B} u(\boldsymbol{x}+\sqrt{\sigma}\cdot\boldsymbol{y}),$$

where $\mathcal{B}_e = \{X \subset \mathbb{R}^2;\ 0 \in E_1(X)\}$. Thus, E_σ belong to the class of affine invariant inf-sup operators which have been studied in [11]. In particular, we have the following consistency theorem.

Theorem 3 (F. Guichard, J.M. Morel) Let $\mathcal{B}$ be a localizable set of plane closed nonempty bounded sets which is invariant by the special linear group $SL(\mathbb{R}^2)$. Then, there exists two constants c^+ and c^- depending on $\mathcal{B}$ such that, for any image u C^3 in a neighbourhood of $\boldsymbol{x}$,

$$\inf_{B\in\mathcal{B}_e} \sup_{\boldsymbol{y}\in B} u(\boldsymbol{x}+\sqrt{s}\cdot\boldsymbol{y}) = u(\boldsymbol{x})+s^{2/3}|Du|g\left(\kappa(u)\right)(\boldsymbol{x})+o(s^{2/3}),$$

$$\begin{aligned} \text{where}\quad g(r) &= c^+\, r^{\frac{1}{3}} \ \text{ if } r \geqslant 0 \\ &= c^-\,(-r)^{\frac{1}{3}} \ \text{ if } r<0. \end{aligned}$$

The only requirement we have to check is that the basis $\mathcal{B}_e$ is localizable in the following sense :

Proposition 7 (Localizability) The basis $\mathcal{B}_e$ associated with the affine erosion operator is localizable, i.e. there exists a constant $c>0$ such that

$$\forall r \geqslant \sqrt{c},\ \forall B\in\mathcal{B}_e,\ \exists B'\in\mathcal{B}_e,$$

$$B' \subset D(0,r) \quad\text{and}\quad \delta(B',B) \leqslant \frac{c}{r}.$$

Here, $D(0,r)$ means the open disk of radius r centered at the origin, and $\delta(B',B)$ denotes the Hausdorff semi-distance between B' and B, given by

$$\delta(B',B) = \sup_{\boldsymbol{x}'\in B'} \mathrm{dist}(\boldsymbol{x}',B) = \sup_{\boldsymbol{x}'\in B'} \inf_{\boldsymbol{x}\in B} |\boldsymbol{x}-\boldsymbol{x}'|.$$

Proof: 1. Given $r \geqslant 1$ and a set B element of $\mathcal{B}_e$, we have $0\in E_1(B)$ and from Definition of $E_1(B)$ we can find a C-set A included in B such that $0\in E_1(A)$ (i.e. $A\in\mathcal{B}_e$). We consider the $\frac{1}{r}$-Euclidean dilation of A restrained to the disk $D(0,r)$, i.e.

$$B' = \{\boldsymbol{x}\in D(0,r);\ \mathrm{dist}(\boldsymbol{x},A)\leqslant \frac{1}{r}\}.$$

B' is a C-set containing $A\cap D(0,r)$, contained in $D(0,r)$, and

$$\delta(B',B) \leqslant \delta(B',A)+\delta(A,B) \leqslant \frac{1}{r}+0.$$

Now we are going to prove that $B'\in\mathcal{B}_e$, that is to say that $0\in E_1(B')$.

Suppose that 0 belongs to D, a chord segment of B' associated to a chord set K of area σ (see Fig. 10). Two cases can be distinguished.

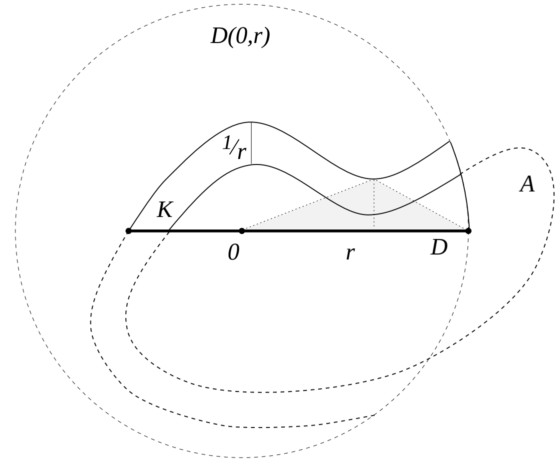

Fig. 10. Area of K is greater than 1

1.a. If $A\cap K \subset D(0,r)$, then a subset of K defines a chord set of A containing 0 and of area no more than σ. But since $A\in\mathcal{B}_e$, we necessarily have $\sigma>1$.

1.b. If $A\cap K$ is no a subset of $D(0,r)$, which means that $K\cap\partial D(0,r)$ is not empty, then we can easily inscribe in K a triangle of base larger than r and height $\frac{1}{r}$ (see Fig. 10), so that we get $\sigma = \mathrm{area}\,(K)>1$.

In both cases, 0 belongs to no 1-chord set of B', so that $B'\in\mathcal{B}_e$. Consequently, we proved that $\mathcal{B}_e$ is localizable with a constant $c=1$. ■

Hence, Theorem 3 applies to $\mathcal{B}_e$ and we have, for any image C^3 near x,

$$E_\sigma(u)(\boldsymbol{x}) = u(\boldsymbol{x})+\omega\cdot|Du(\boldsymbol{x})|\left[\kappa^-(u)(\boldsymbol{x})\right]^{\frac{1}{3}}\sigma^{\frac{2}{3}}+o(\sigma^{\frac{2}{3}}), \quad (4)$$

where $\omega = \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}$ as in Theorem 2, and $(\kappa^-)^{\frac{1}{3}}$ means $-(\max(0,-\kappa))^{\frac{1}{3}}$. If we want the exact consistency with the AMSS (i.e. κ instead of κ^-), we can consider the alternate operator $D_\sigma\circ E_\sigma$, the affine dilation D_σ being defined by

$$D_\sigma(u) = -E_\sigma(-u).$$

The consistency of E_σ (and of the associated operators D_σ and $D_\sigma\circ E_\sigma$) can also be deduced from the geometrical consistency proven in Theorem 2 (see [12]).

D. Convergence

As we know that the affine erosion of images is consistent with the AMSS, it is natural to wonder whether the iterated infinitesimal affine erosion spans exactly the affine morphological scale space. The answer is yes, and the proof is classical (see [3], [5] and [11]). We first define the step of a subdivision $s=(s_0,s_1,\dots s_n)$ as

$$|s| = \max_{1\leqslant i\leqslant n}(s_i-s_{i-1}).$$

Theorem 4: Let u_0 be a Lipschitz image, for any subdivision s of $[0,t]$ we define

$$u_s(\boldsymbol{x},0)=u_0(x) \quad\text{and}\quad u_s(\boldsymbol{x},s_{i+1}) = T_{s_{i+1}-s_i}(u_s(\cdot,s_i))(\boldsymbol{x}).$$

Then, as $|s| \to 0$, $u_s(\cdot,t)$ converges uniformly on every compact subset of the plane towards a function $x \mapsto u(\boldsymbol{x},t)$, the unique viscosity solution of

$$\frac{\partial u}{\partial t} = \omega \cdot |Du|\, \kappa(u)^{\frac{1}{3}},$$

subject to initial condition $u(\boldsymbol{x},0) = u_0(\boldsymbol{x})$, where

$$T_h = D_{h^{3/2}} \circ E_{h^{3/2}} \quad \text{and} \quad \omega = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{2}{3}}.$$

IV. Numerical Scheme

Many reasons lead to choose the polygonal representation to implement the affine erosion on curves, but the major advantage of this choice in our case is, as we shall see further, that we can compute *exactly* the affine erosion of a polygon. The lack of regularity of polygons (not C^1 everywhere) shall not be a problem, since most of the previous analyses apply to piecewise C^1 curves.

Obviously, neither the affine erosion nor the affine scale space of a polygon is a polygon. But since no simple dense set of parameterized curves satisfies this property (as far as we know), an approximation is always required to iterate the affine erosion. The main advantage of being able to compute exactly the affine erosion of a polygon is that *we can dissociate completely the two approximations required to compute the affine scale space : the scale quantization* (we have to iterate the affine erosion several times) and *the space quantization*, which is necessary to work on discrete data. Processing these two steps successively and independently, we avoid a classical trap of the implementation of scale space on curves which prevents algorithms from satisfying [Monotonicity] and [Affine invariance]. In particular, with our method there is no a priori relation between the number of vertices of a polygon and the number of vertices of the polygons resulting on the approximation of its affine scale space : as noticed in the introduction, this number can drastically increase (case of a polygon with very acute angles) or decrease as well (case of a very “noisy” curve). In other words, our algorithm processes a polygon *as a curve* and not as a set of points, and for that reason *it is not a point evolution scheme.*

A. Affine erosion of a convex polygon

Proposition 8: Let $\mathcal{P} = P_1P_2...P_n$ be a convex polygon, and $0 < \sigma < \sigma_r(\mathcal{P})$. The σ-affine erosion of $\mathcal{P}$ is a C^1 curve made of the concatenation of the pieces of hyperbolae $H_{i,k}$ defined by Equations 6 to 12, the couples (i,k) satisfying Equation 5 and being sorted in lexical order[2].

Proof: If $\mathcal{P} = P_1P_2...P_n$ is a (positively oriented) convex polygon and $0 < \sigma < \sigma_r(\mathcal{P})$, we know from Theorem 1 that $E_\sigma(\mathcal{P})$ is made exactly of the middle of the σ-chord segments of $\mathcal{P}$. Consider two non-parallel edges $[P_{i-1}P_i]$ and $[P_kP_{k+1}]$, then there exists σ-chords whose extremities lie on $[P_{i-1}P_i]$ and $[P_kP_{k+1}]$ if and only if

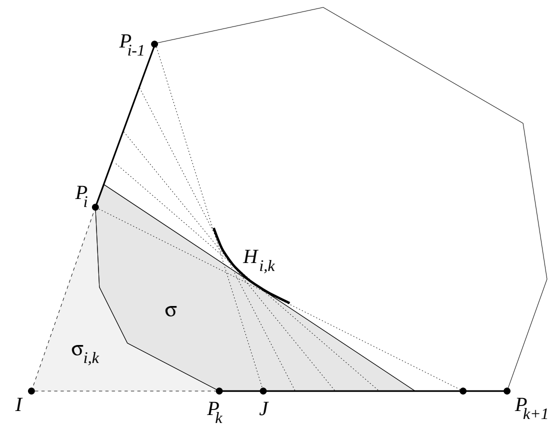

Fig. 11. Piece of hyperbola resulting from two edges.

$$\frac{1}{2}[IP_k, IP_i] \leqslant \sigma + \sigma_{i,k} \leqslant \frac{1}{2}[IP_{k+1}, IP_{i-1}], \tag{5}$$

where I is defined (see Fig. 11) by

$$I := (P_{i-1}P_i) \cap (P_kP_{k+1}) \text{ and } \sigma_{i,k} := \text{area}\,(IP_i...P_k). \tag{6}$$

In this case, we know from Proposition 2 that the middle points of the σ-chord segments whose endpoints lie on $[P_{i-1}P_i]$ and $[P_kP_{k+1}]$ span a piece of hyperbola

$$H_{i,k}: \quad M(t) = I + \lambda(e^t IP_k + e^{-t} IP_i), \quad t_1 \leqslant t \leqslant t_2 \tag{7}$$

whose apparent area is

$$\sigma + \sigma_{i,k} = 2\lambda^2 [IP_k, IP_i],$$

so that

$$\lambda = \sqrt{\frac{\sigma + \sigma_{i,k}}{2[IP_k, IP_i]}}. \tag{8}$$

As concerns the limit values t_1 and t_2, one checks easily that

$$\begin{aligned} t_1 &= -\ln \frac{IP_{i-1}}{2\lambda.IP_i} \quad \text{if area}\,(IP_{i-1}...P_k) > \sigma + \sigma_{i,k}, && (9)\\ &= -\ln(2\lambda) \quad \text{otherwise}, && (10)\\ t_2 &= \ln \frac{IP_{k+1}}{2\lambda.IP_k} \quad \text{if area}\,(IP_i...P_{k+1}) > \sigma + \sigma_{i,k}, && (11)\\ &= \ln(2\lambda) \quad \text{otherwise}. && (12) \end{aligned}$$

Last, we have to check that the admissible hyperbolae $H_{i,k}$ are encountered on $E_\sigma(\mathcal{P})$ in lexical order, that is,

$$H_{i,k} < H_{i',k'} \quad \Leftrightarrow \quad i < i' \text{ or } (i = i' \text{ and } k < k').$$

The reason is very simple : as we know that $E_\sigma(\mathcal{P})$ is convex, we must consider the σ-chord segments of $\mathcal{P}$ in such an order that the angles of their directions increase continuously on S^1. Thus, the previous assertion simply results from

$$i \leqslant j \leqslant k \quad \Rightarrow \quad \alpha(P_iP_j, P_iP_k) \leqslant \alpha(P_iP_j, P_jP_k),$$

[2] with the convention that $P_{k+n} = P_k$ and $i \leqslant k < i + n$ for each couple (i,k).

where $\alpha(\boldsymbol{v}_1, \boldsymbol{v}_2)$ measures on $[0, 2\pi[$ the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. ∎

Due to Theorem 1, the previous study only applies for $\sigma < \sigma_r(\mathcal{P})$. When $\sigma \geqslant \sigma_r(\mathcal{P})$ (this case cannot be avoided since $\sigma_r(\mathcal{P}) = 0$ for some polygons), we still have the inclusion

$$E_\sigma(\mathcal{P}) \subset \bigcup_{i,k} H_{i,k},$$

but the reverse inclusion can be false so that we have to remove the "ghost parts" of $\cup H_{i,k}$ to obtain $E_\sigma(\mathcal{P})$. We explain how to do it in the next section (step B).

B. Affine erosion of any polygon

When the polygon $\mathcal{P}$ is possibly non-convex, we proceed in two steps.

step A : we collect all the pieces of curves which can possibly be part of $E_\sigma(\mathcal{P})$. These pieces are of three kinds :

1. The valid pieces of hyperbola $H_{i,k}$ described previously, completed with their two half chord segments at their endpoints (see Fig. 12). The interval $[t_1, t_2]$ defining each piece of hyperbola (Equation 7) may have to be shortened because of internal occlusions ; however, the resulting admissible piece of hyperbola remains connected (that is, $[t_1, t_2]$ remains an interval).
2. The two limit σ-chord segments of each non-regular piece of hyperbola, i.e. resulting from non-regular chords (see Fig. 12).
3. The σ'-chord segments defined by two "inside" vertices, with $0 \leqslant \sigma' \leqslant \sigma$ (see Fig. 12).

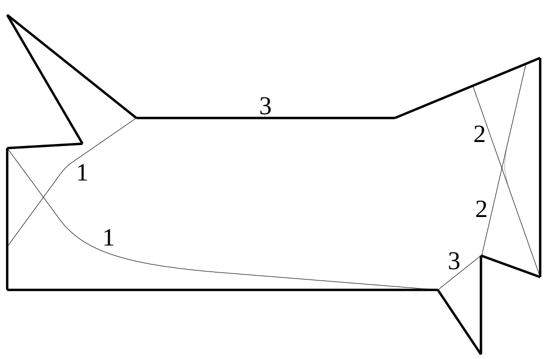

Fig. 12. The three kind of curves encountered in the computation of the affine erosion of a polygon

Fig. 13 shows what we obtain after step A for a reasonable polygon.

step B : we compute the intersections between the remaining pieces of curves (sorted with respect with their start number a). At this stage, we may have to compute intersections between two segments, between a segment and an hyperbola, or between two hyperbolae. The first two cases reduce to equations of degree 1 and 2 respectively. The last case (intersection of two hyperbolae) can be more difficult. If the two hyperbolae have a common axis, then the equation of the intersection is quadratic and can be solved easily. However, in more general cases (which happen), we can have two solve an algebraic equation of degree 4 ; if so, we use Newton's method, which converges in a few iterations.

Now, for each intersection, we remove from each of the two curves the part enclosed in the chord sets defined by the other one. We have to maintain —at least, formally— two data structures to process this step correctly : one is the original set of curves obtained from step A, the other is a copy of these curves, updated iteratively as we just explained.

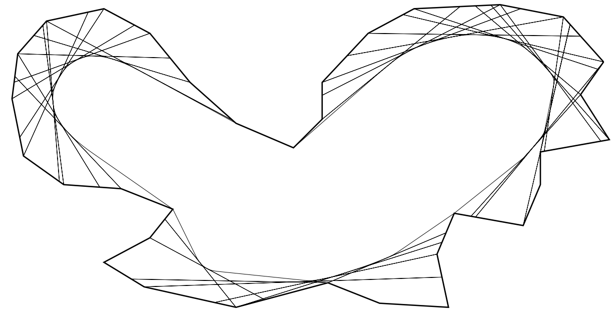

Fig. 13. Curves obtained after step A (the affine erosion is the envelope of these curves).

Finally, we obtain the affine erosion of the polygon as the concatenation (in the natural order) of the pieces of curves obtained from step B. This algorithm is a bit heavy (1600 lines of C source code), but not too slow for reasonable quantizations (a polygon with 100 vertices is processed in one second or so). Notice that the whole algorithm is much faster than classical ones for which the only way to guarantee numerical stability is to process numerous iterations with a very small value of the scale step δt. We must be careful when computing the intersections, because of the finite numerical precision of the computer (this can be done by considering point equalities modulo a relative error, for instance).

Another way to implement the affine erosion is to consider the polygon as a concatenation of convex curves (a C-set), and to process separately the convex pieces. The major advantage is that the affine erosion of convex curves does not involve intersections in general, unless non-regular chords arise, which is rare in practice. Hence, this simplified algorithm is even faster than the exact one we just described (it allows to process a complicated curve in less than one second, see [12]). However, some theoretical problems still are to be investigated, and it is not the aim of this paper to discuss them.

C. Iterating the affine erosion

So far, we know how to compute exactly the affine erosion of a polygon. To iterate this process, we require to quantize the resulting curve (which is, as we shown, the concatenation of hyperbola pieces and segments) in order to get a new polygon. Fortunately, there is a simple way to sample a piece of hyperbola in an affine-invariant way. Considering the parameterization

$$H \ : \ M(t) = \lambda(e^t \boldsymbol{v}_1 + e^{-t}\boldsymbol{v}_2), \qquad t_1 \leqslant t \leqslant t_2,$$

then one can prove easily that $(t, t+x)$ is an ε-chord set of H if and only if $\varepsilon = \lambda^2(\operatorname{sh} x - x)$, sh meaning the hyperbolic sine. Hence, the polygon $P_0P_1...P_n$ defined by

$$P_k = M\left((1-\frac{k}{n})t_1 + \frac{k}{n}t_2\right)$$

is a discrete affine invariant quantization of H of "area step"

$$\varepsilon(n) = \lambda^2(\operatorname{sh}\frac{1}{n} - \frac{1}{n}).$$

Given $\varepsilon > 0$, we can quantize the affine erosion of a polygon up to the area step ε by choosing, for each piece of hyperbola, the minimum entire value of n such that $\varepsilon(n) \leqslant \varepsilon$.

Not surprisingly, this quantization step is a kind of discrete affine erosion of scale ε. Thus, as we want to minimize its influence on the computation, we must choose $\varepsilon \ll \sigma$, where σ is the scale of the computed affine erosion. This condition will force the second iteration of E_σ to be *non-local* in the sense that the σ-chord sets of the resulting approximate polygon will contain many edges (i.e. $k - i \gg 1$ for the valid $H_{i,k}$). In that sense, our algorithm is quite different from a point evolution scheme, for which the scale quantization step is supposed to be small compared to the space quantization step. Here, the inverse phenomenon happens : the scale quantization step (σ) is much larger than the space quantization step (ε). The important consequence is that we can effectively iterate only a few times (i.e. with large scale steps) the affine erosion to compute the affine scale space. Indeed, we do not loose accuracy since ε can remain small and the affine erosion is a good approximation of the affine scale space even for rather large scales, as we noticed previously in §2.4.

V. Experiments

On Figure 14 is computed the affine scale space of a non-convex polygon. Each curve corresponds to one iteration of the affine erosion plus dilation, computed using the exact algorithm described in the previous section. As predicted by the theory, the curve collapse in a "elliptically shaped" point (see [14]). Computing the 29 iterations displayed on Fig. 14 takes 6 minutes (CPU time) on a HP 735/125 station. The number of sampled points reaches 700 for some iterations and the number of computed curves (hyperbolae and segments) attains 1600.

Conclusion

We presented in this paper the first purely geometrical and fully consistent scheme for the affine scale space of curves, based on the iteration of a non-local operator called *affine erosion*. This operator appears to be fully consistent in the sense that it satisfies most the properties of the affine scale space, in particular the monotonicity and the affine invariance. It permits to define an algorithm which computes accurately and rather quickly the affine scale space of a polygonal curve, as illustrated by the experiments.

This scheme should be useful, for instance, to shape recognition tasks (e.g. [6] in the case of partially occluded shapes), since in any method based on the computation of characteristic points, the regularization process must be accurate, reliable and strongly invariant in order to create no artifacts.

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Lionel Moisan was born in Bègles, France, in 1970. From 1990 to 1995, he studied Mathematics and Computer Science in the Ecole Normale Supérieure de Paris. In 1997, he received the PhD degree in applied mathematics from Ceremade, Paris-Dauphine University, where he has been working on applications of Partial Differential Equation to Image and Movie Processing. He has been teaching Mathematics in the Ecole Normale Supérieure de Cachan since 1995.

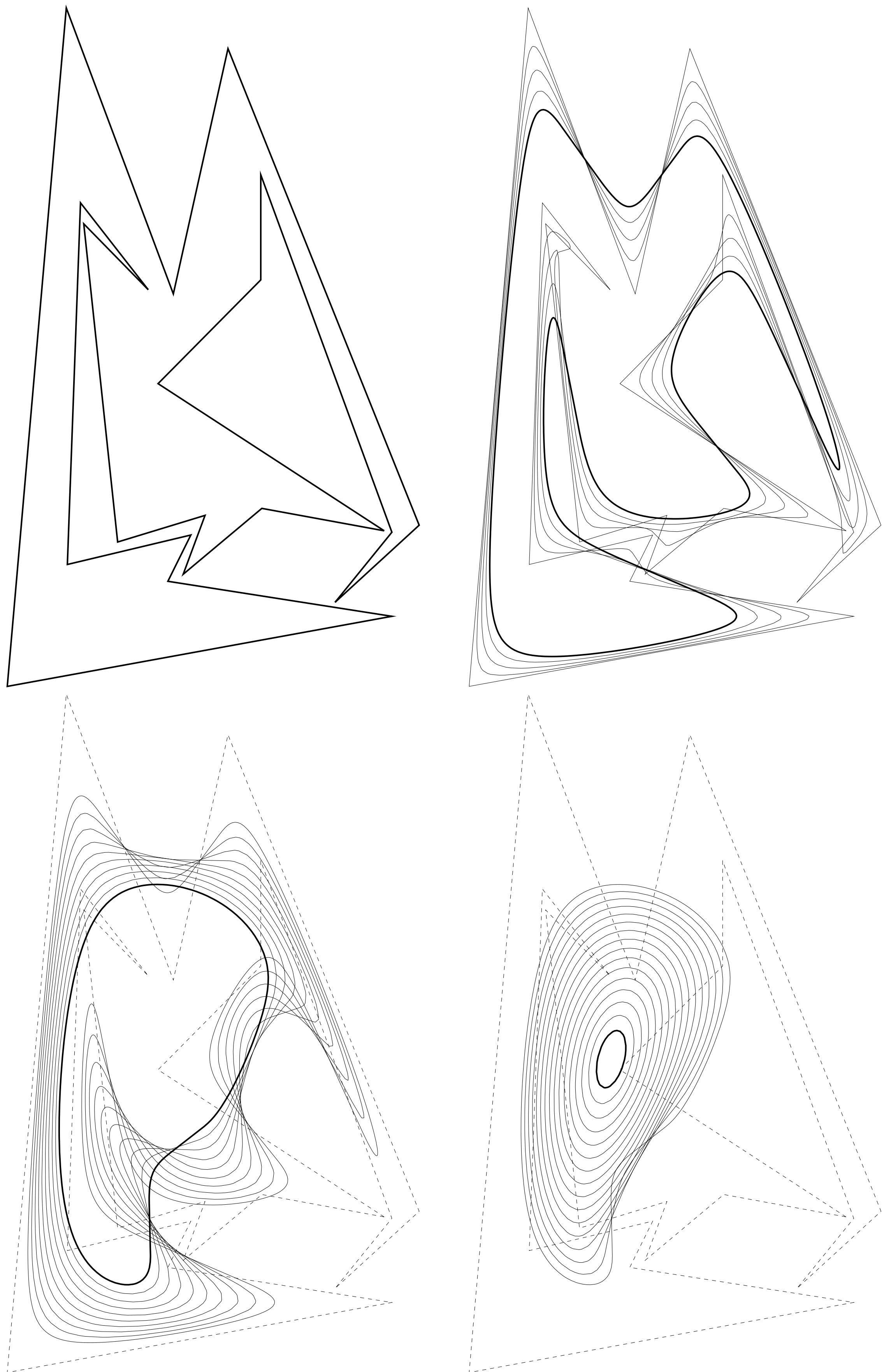

Fig. 14. Affine scale space of a weird polygon