1

PDE's, motion analysis and 3D reconstruction from movies

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Abstract: We study a non-linear second order PDE related to the image processing problem called "structure from motion". Its close relation to the monodimensional heat equation permits to define weak solutions and establish existence and uniqueness properties. We also point out a variational interpretation and present numerical simulations.

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1 Introduction

Throughout this paper, we study the partial differential equation

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + \left(\frac{u_\theta}{u_x}\right)^2 u_{xx}, \qquad (DCMA)$$

where $u(x, \theta, t)$ is a real-valued function depending on three scalar variables x (space), θ (time) and t (scale), and submitted to some boundary conditions. This is a degenerate parabolic equation which describes an anisotropic diffusion process : the left term is the second derivative of u evaluated in the direction $(-u_\theta/u_x, 1)$. Equation (DCMA) was introduced in [7] as the only multiscale analysis of movies compatible with the depth recovery (the so-called structure from motion problem). Evans also studied it as a way to extend the heat equation to multi-valued functions (see [6]).

The goal of structure from motion is to compute the tridimensional structure of objects that are seen by a camera from different points of view. In theory, two images are sufficient to recover the 3D structure of observed parts (this is the principle of stereovision), but in practive long sequences of images are used to guarantee robustness. Formally, this leads to consider *movies*, represented as functions u of two spatial coordinates x, y and the time coordinate θ. In the following, we shall only deal with *gray-level* movies, for which the scalar value $u(x, y, \theta)$ measures the intensity received by the camera at point (x, y) of the image plane and at time θ.

If the camera is moving along the x axis, pointing in a perpendicular direction Z towards a lambertian fixed surface, then it produces a movie u which ideally satisfies

$$-\frac{u_\theta}{u_x}(x, y, \theta) = -\frac{V(\theta)}{Z(M)}, \tag{1}$$

where $V(\theta)$ is the camera speed at time θ and $Z(M)$ the depth of the physical point M projected in (x, y) on the focal plane at time θ. In practice, the computation of the left term $-u_\theta/u_x$ (which corresponds to the apparent velocity v induced on the image plane by the camera motion) requires some smoothing process, first because the low sampling rates in the digitization of real movies make the use of finite difference estimations hazardous, second in order to take advantage of the redundancy of the depth information contained in the movie.

The most relevant smoothing process found so far consists precisely in the multi-scale representation $u(x, y, \theta, t)$ of a raw movie u_0, obtained as the solution u of the (DCMA) evolution satisfying the initial condition $u(x, y, \theta, 0) = u_0(x, y, \theta)$. Notice that the y variable is not involved in this equation and can be removed in its mathematical analysis. The following results justify in some way the use of the DCMA (see [8] or [9] for more precise formulations).

Theorem 1 *The DCMA is the only regular semigroup $T_t : u_0(\cdot) \mapsto u(\cdot, t)$ which is*

- *monotone : if $u_1 \leqslant u_2$, then $T_t u_1 \leqslant T_t u_2$ at any scale $t \geqslant 0$,*

- *contrast invariant : if $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $T_t\, g(u) = g(T_t u)$,*

- *Galilean invariant : T_t commutes with the change of Galilean referential $(x, \theta) \mapsto (x - \alpha\theta + x_0, \theta + \theta_0)$,*

- *zoom invariant : T_t commutes with spatial homotheties $(x, \theta) \mapsto (\lambda x, \theta)$.*

Theorem 2 *The Depth Compatible Multiscale Analysis (DCMA) is the only regular monotone semigroup preserving the depth map of ideal movies : if u_0 has interpretation $Z(X, Y)$ (depth map) and $V_0(\theta)$ (camera speed), then $T_t u_0$ has interpretation $Z(X, Y)$ (same depth map) and $V(\theta, t)$, where $V(\theta, 0) = V_0(\theta)$ and $V_t = V_{\theta\theta}$ (that is, the camera speed interpretation is smoothed by the monodimensional heat equation).*

2   Classical and weak solutions of the DCMA

Equation (DCMA) can be written under the form $u_t = F(D^2 u, Du)$, where Du and $D^2 u$ are the first and second derivatives of u and F is an elliptic operator (that is, nondecreasing with respect with $D^2 u$ for the natural order on symmetric 2x2 matrices). However, since F is not continuous, the theory of viscosity solutions of Crandall, Ishii and Lions (see [4]) does not apply. Even known generalizations to discontinuous

functions F (e.g. Evans-Spruck [5] and Chen-Giga-Goto [3] for the Mean Curvature Motion) do not work for the DCMA, because the singularity at points where $u_x = 0$ is too strong. However, it is possible to give a definition of weak solutions of (DCMA) ensuring uniqueness and existence for some class of initial conditions by noticing (like Evans in [6]) that the DCMA is the level set formulation of the linear heat equation. More precisely, if u solves (DCMA) and if we parameterize a level curve $u(x, \theta, t) = cst$ under the form $x = \varphi(\theta, t)$, then this level curve should evolve according to the one-dimensional heat equation $\varphi_t = \varphi_{\theta\theta}$.

2.1   Classical solutions of the DCMA

For the reason we explained before, we forget the y variable in the following, and a movie is defined on $\mathbb{R} \times \overline{I}$, with either $I =]\theta_1, \theta_2[$ or $I = S^1$. In the space variable, a periodization has no meaning in terms of scene interpretation, so that we shall rather suppose that u tends towards some constant when x grows to infinity (we shall say that u is "constant at infinity").

Definition 1 *For $c = (c^-, c^+) \in \mathbb{R}^2$ and $n \geqslant 0$, $\mathcal{C}_c^n$ is the space of movies $u \in C^n(\mathbb{R} \times I)$ such that*

$$\sup_{\theta \in \overline{I}} |u(-x, \theta) - c^-| + |u(x, \theta) - c^+| \to 0 \quad \text{as} \quad x \to +\infty. \tag{2}$$

From now on, we write $\Omega = \mathbb{R} \times I \times]0, +\infty[$, i.e. $\overline{\Omega}$ is the domain of movie analyses.

Definition 2 *For $c \in \mathbb{R}^2$ and $n, p \geqslant 0$, $\mathcal{C}_c^{n,p}$ is the space of movie analyses $u \in C^0(\overline{\Omega})$ such that*

1. *$\forall T > 0$, $\displaystyle\sup_{\theta \in \overline{I}, t \leqslant T} |u(-x, \theta, t) - c^-| + |u(x, \theta, t) - c^+| \to 0 \quad \text{as} \quad x \to +\infty$,*

2. *on Ω, $(x, \theta, t) \mapsto u(x, \theta, t)$ is of class C^n with respect to (x, θ) and C^p with respect to t.*

Definition 3 *Given $u_0 \in \mathcal{C}_c^0$, we say that u is a classical solution of the DCMA associated to the initial datum u_0 if*

(i) $u \in \mathcal{C}_c^{2,1}$,

(ii) on $\Omega = \mathbb{R} \times I \times]0, +\infty[$, $\quad \begin{cases} u_t = u_{\theta\theta} - 2\dfrac{u_\theta}{u_x} u_{\theta x} + \left(\dfrac{u_\theta}{u_x}\right)^2 u_{xx} & \text{when} \quad u_x \neq 0, \\[4mm] u_t = 0 \quad \text{when} \quad u_x = 0. \end{cases}$

(iii) $\forall (x, \theta, t) \in \partial\Omega, \quad u(x, \theta, t) = u_0(x, \theta)$.

This definition ensures the uniqueness result thanks to the following

Proposition 1 (comparison principle) *Let u and $\tilde{u}$ be classical solutions of the DCMA associated to initial data u_0 and $\tilde{u}_0$ respectively. If $u_0 \leqslant \tilde{u}_0$, then $u \leqslant \tilde{u}$ on Ω.*

The proof is rather classical : it consists to show that for any $\alpha, T, R > 0$, the map $(x, \theta, t) \mapsto u(x, \theta, t) - \tilde{u}(x, \theta, t) - \alpha t$ attains its max value on the boundary of $[-R, R] \times \overline{I} \times [0, T]$, and then to send α to zero and R to infinity.

Corollary 1 (uniqueness) *A classical solution of the DCMA associated to a given initial datum $u_0 \in \mathcal{C}_c^2$ is unique.*

In order to ensure the existence of classical solutions of the DCMA, we now restrain the space of initial data.

Definition 4 *For $n \geqslant 1$, we write $\mathcal{V}_c^n$ the space of movies $u \in \mathcal{C}_c^n$ for which there exists a movie $v \in \mathcal{C}_0^{n-1}$ such that*

$$u_\theta + v u_x = 0 \quad \text{on} \quad \mathbb{R} \times \overline{I}. \tag{3}$$

v is called a velocity map of u. In addition, the space $\mathcal{V}_c^{n,p}$ is defined as elements of $\mathcal{C}_c^{n,p}$ admitting a velocity map $v \in \mathcal{C}_0^{n-1,p}$.

Remark : Consider a movie $u \in \mathcal{V}_c^n$. When $u_x(x, \theta) = 0$, (3) implies $u_\theta(x, \theta) = 0$, and if $n \geqslant 2$, differentiating (3) with respect to θ and x shows that $u_{\theta\theta} + 2v u_{\theta x} + v^2 u_{xx} = 0$ as soon as $u_x = 0$. A consequence is that if $u \in \mathcal{V}_c^{2,1}$ is a classical solution of the DCMA, then any velocity map v of u satisfies on Ω

$$\begin{cases} u_\theta + v u_x = 0 \\ u_t = u_{\theta\theta} + v u_{\theta x} + v^2 u_{xx}. \end{cases} \tag{4}$$

We now build explicit solutions of the DCMA. As we said before, the main idea is to notice that the trajectories (i.e. the curves $x(\theta)$ along which u is constant) are smoothed by the monodimensional heat equation. For that purpose, we need to introduce the natural domain $I^\star$ for such trajectories. If $I =]\theta_1, \theta_2[$ then $I^\star = I$, and if $I = S^1$, then $I^\star = \mathbb{R}$ (the natural injection $S^1 \hookrightarrow [0, 2\pi[\subset \mathbb{R}$ being implicit). To simplify the notations, we suppose in the following that $0 \in \overline{I}$.

Definition 5 *A map $\varphi \in C^n(\mathbb{R} \times I^\star)$ ($n \geqslant 0$) is a θ-**graph** of $u \in \mathcal{C}_c^n$ if*

    1. for any $\theta \in \overline{I^\star}$, the map $x \mapsto \varphi(x, \theta)$ is increasing and bijective (and φ_x does not vanish if $n \geqslant 1$).

2. for any $(x,\theta) \in \mathbb{R} \times \overline{I^\star}$,

$$u(\varphi(x,\theta),\theta) = u(x,0),\tag{5}$$

3. for any $x \in \mathbb{R}$, $\varphi(x,0) = x$, and if $I = S^1$, then for any $(x,\theta) \in \mathbb{R} \times \overline{I^\star}$,

$$\varphi(x,\theta + 2\pi) = \varphi(\varphi(x,2\pi),\theta),\tag{6}$$

4. $\displaystyle\sup_{|x| \geqslant R, \theta \in \overline{I}} |\varphi_\theta(x,\theta)| \to 0$ as $R \to +\infty$ (in a generalized sense if $n = 0$).

Remark : Notice that in Condition 4, the sup is taken for $\theta \in \overline{I}$ and not for $\theta \in \overline{I^\star}$. If $n = 0$, the term $|\varphi_\theta(x,\theta)|$ must be replaced by

$$\limsup_{h \to 0} \left| \frac{\varphi(x,\theta + h) - \varphi(x,\theta)}{h} \right|.$$

Proposition 2 A movie $u \in \mathcal{C}_c^n$ $(n \geqslant 2)$ belongs to $\mathcal{V}_c^n$ if and only if it admits a θ-graph of class C^n.

Proposition 3 Let $u_0 \in \mathcal{V}_c^n$ $(n \geqslant 2)$, and φ_0 be a θ-graph of u_0 of class C^n. Define $(x,\theta,t) \mapsto \varphi(x,\theta,t)$ as the unique solution of the monodimensional heat equation

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial \theta^2}\tag{7}$$

on $\Omega^\star = \mathbb{R} \times I^\star \times]0,+\infty[$ submitted to the boundary condition

$$\forall (x,\theta,t) \in \partial\Omega^\star, \quad \varphi(x,\theta,t) = \varphi_0(x,\theta).\tag{8}$$

Then, the unique map $u : \overline{\Omega} \to \mathbb{R}$ defined by

$$\forall (x,\theta,t) \in \overline{\Omega}, \quad u(\varphi(x,\theta,t),\theta,t) = u_0(x,0)\tag{9}$$

belongs to $\mathcal{V}_c^{n,n}$ and is a classical solution of the DCMA associated to the initial datum u_0.

We think it is worth explaining here the link between the (DCMA) and the monodimensional heat equation stated in Proposition 3. Let us note $\mathbf{z}_1 = (\varphi(\mathbf{z}),\theta,t)$ for a given $\mathbf{z} \in \Omega$. If $u_x(\mathbf{z}_1) = 0$, differentiating (9) with respect to t yields

$$\varphi_t(\mathbf{z})u_x(\mathbf{z}_1) + u_t(\mathbf{z}_1) = u_t(\mathbf{z}_1) = 0$$

as expected. If $u_x(\mathbf{z}_1) \neq 0$, we obtain $u_t(\mathbf{z}_1) = -\varphi_t(\mathbf{z})u_x(\mathbf{z}_1)$,

$$\frac{d}{d\theta}\left(u_0(x,0)\right) = 0 = \varphi_\theta(\mathbf{z})u_x(\varphi(\mathbf{z}),\theta,t) + u_\theta(\varphi(\mathbf{z}),\theta,t), \qquad \text{and}$$

$$
\begin{aligned}
\frac{d^2}{d\theta^2}(u_0(x,0)) = 0 &= \frac{d}{d\theta}\Big(\varphi_\theta(\mathbf{z})u_x(\varphi(\mathbf{z}),\theta,t) + u_\theta(\varphi(\mathbf{z}),\theta,t)\Big) \\
&= \varphi_{\theta\theta}(\mathbf{z})u_x(\mathbf{z}_1) + \varphi_\theta^2(\mathbf{z})u_{xx}(\mathbf{z}_1) + 2\varphi_\theta(\mathbf{z})u_{x\theta}(\mathbf{z}_1) + u_{\theta\theta}(\mathbf{z}_1) \\
&= \Big(-u_t + u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}\Big)(\mathbf{z}_1).
\end{aligned}
$$

Hence, u is a classical solution of the DCMA associated to the initial datum u_0. We now have the

Proposition 4 (existence) *Given an initial datum $u_0 \in \mathcal{V}_c^n$ ($n \geqslant 2$), there exists a unique classical solution of the DCMA, and it belongs to $\mathcal{V}_c^{n,n}$.*

Proposition 3 proves that the DCMA Equation is a scalar formulation of the monodimensional heat equation (7), like two other important equations of image processing : the Mean Curvature Motion and the Affine Morphological Scale Space, which can be obtained by axiomatic formulations as well (see [1]). The difference between them only comes from the intrinsic parameter of the level lines : the Euclidean abscissa for the Mean Curvature Motion, the affine abscissa for the Affine Scale space. For the DCMA, the natural parameter is the time θ, which means that level lines are not considered as curves but as *graphs*. This remark allowed us to prove the existence of weak solutions for the DCMA, but in certain cases only : precisely, when the level lines of the initial datum can be described by graphs.

2.2   Weak solutions of the DCMA

We define weak (only continuous) solutions of the DCMA as uniform limits of classical solutions.

Definition 6 *Given a movie $u_0 \in \mathcal{C}_c^0$, we say that a map $u \in \mathcal{C}_c^{0,0}$ is a weak solution of the DCMA associated to the initial datum u_0 if*

$$
\forall (x,\theta,t) \in \partial\Omega, \ \ u(x,\theta,t) = u_0(x,\theta)
$$

and if there exists a sequence $(u^\varepsilon)_{\varepsilon>0}$ of classical solutions of the DCMA associated to the initial datum u_0 such that $u^\varepsilon \to u$ uniformly on $\overline{\Omega}$ when $\varepsilon \to 0$.

Proposition 5 (uniqueness) *A weak solution of the DCMA associated to a given initial datum is unique.*

Proposition 6 (existence) *Call $\overline{\mathcal{V}_c}$ the topological closure of $\mathcal{V}_c^2$ with respect to the $\|\cdot\|_\infty$ norm. Then, given $u_0 \in \overline{\mathcal{V}_c}$, there exists a unique weak solution u of the DCMA associated to the initial datum u_0.*

Once again, the uniqueness property results from a comparison principle. The existence can be shown using the approximation of the initial datum by elements of $\mathcal{V}_c^2$ and the existence property for regular solutions. One can also build an explicit weak solution, using the construction (the monodimensional heat equation) of Proposition 3.

Definition 7 *We write $\mathcal{V}_c^0$ the space of movies $u \in \mathcal{C}_c^0$ which admit a continuous θ-graph.*

Proposition 7 *Let $u_0 \in \mathcal{V}_c^0$, and φ_0 be a θ-graph of u_0. Define $(x, \theta, t) \mapsto \varphi(x, \theta, t)$ as the unique solution of the monodimensional heat equation (7) submitted to the boundary condition (8). Then, the unique map u defined from φ by (9) is a weak solution of the DCMA.*

A consequence of this characterization of weak solutions is that a weak solution of the DCMA associated to an initial datum $u_0 \in \mathcal{V}_c^n$ admits a kind of velocity movie as soon as $t > 0$, as stated by

Corollary 2 *Let u be the weak solution of the DCMA associated to an initial datum $u_0 \in \mathcal{V}_c^0$. If u is locally Lipschitz in the x variable, then there exists a continuous map v defined on $\Omega = \mathbb{R} \times I \times]0, +\infty[$ such that on Ω,*

$$u(x + \tau v(x, \theta, t), \theta + \tau, t) = u(x, \theta, t) + o(\tau)$$
$$\text{and} \quad u(x + \tau v(x, \theta, t), \theta + \tau, t - \frac{\tau^2}{2}) = u(x, \theta, t) + o(\tau^2).$$

Notice that this property is a generalization of (4).

2.3   Further existence properties

In the previous sections, we did not prove the existence of (weak or classical) solutions of the DCMA in the general case, that is to say when the initial datum admits no θ-graph. In fact, we do not believe that the DCMA admits a solution in general, at least a solution in the sense we defined. When the initial datum u_0 admits a θ-graph, the DCMA is obtained by applying the linear monodimensional heat equation to the level lines of u_0. For an ordinary continuous map u_0, the level lines have no reason to be graphs in the θ variable, since to a given value of θ, several values of x will correspond in general. Hence, defining general solutions of the DCMA is somewhat equivalent to defining solutions of the heat equation for multi-valued data. It is in that spirit that in [6] Evans studied (DCMA) as the limit when $\varepsilon \to 0$ of the more regular equation

$$u_t = \frac{u_x^2 u_{\theta\theta} - 2u_x u_\theta u_{x\theta} + u_\theta^2 u_{xx}}{u_x^2 + \varepsilon^2 u_\theta^2}. \tag{10}$$

Equation (10) admits viscosity solutions because it is more or less the Mean Curvature Motion (actually, the case $\varepsilon = 1$ is exactly the Mean Curvature Motion). He noticed that in the general case (that is, when the level lines of the initial datum are not graphs), the regularizing effects of the heat equation are so strong that the limit of approximate solutions is not continuous at scale $t = 0$, because the level lines are constrained to become graphs instantaneously.

3   Variational interpretation of the DCMA

Proposition 8 *The DCMA induces on v a flow associated to the minimization of*

$$\mathcal{E}(v) = \frac{1}{2} \iint (v_\theta + vv_x)^2 \, dxd\theta. \tag{11}$$

Let us consider the functional $\mathcal{E}(v)$ defined by (11) on compactly supported movies of class C^2. Differentiating $\mathcal{E}$ yields, after integrations by parts,

$$D_v\mathcal{E}(h) = -\iint \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} h \, dxd\theta,$$

where $\frac{\mathrm{D}}{\mathrm{D}\theta} = \frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}$ represents the total derivative operator. Then, for a classical solution of the DCMA $u \in \mathcal{V}_0^{4,1}$ associated to a compactly supported initial datum and admitting a velocity map v, one has

$$\frac{d}{dt}\left(E(v) \right) = -\iint \left(\frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} \right)^2 \, dxd\theta,$$

which means that the flow induced on v by the DCMA is associated to the minimization of $\mathcal{E}$. This mimimization property proves that the DCMA "idealizes" movies and tend to give them a coherent depth interpretation as scale increases, since the apparent acceleration $Dv/D\theta = v_\theta + vv_x$ is globally decreasing.

4   Numerical scheme

In order to apply the DCMA evolution to real movies, we need to devise a numerical scheme. A "naive" discretization of the partial derivatives of u cannot be used, because in practice it is well known that the time discretization is not thin enough. Moreover, such a discretization is not likely to satisfy the axioms that we imposed to the DCMA. This is the reason why we focus our attention on an inf-sup scheme. To this end, given a movie $u : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$, we define

$$
\begin{aligned}
IS_h u(x_0, y_0, \theta_0) &= \inf_{v \in \mathbb{R}} \sup_{-h \leqslant \theta \leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\
SI_h u(x_0, y_0, \theta_0) &= \sup_{v \in \mathbb{R}} \inf_{-h \leqslant \theta \leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\
\text{and} \quad T_h u &= \frac{1}{2}\left(IS_h u + SI_h u \right).
\end{aligned}
$$

We have a consistency result (see [9] for a proof) at points where u_x does not vanish.

Theorem 3 *If u is a bounded movie locally C^3 near $\mathbf{z}_0$, with $u_x(\mathbf{z}_0) \neq 0$, then*

$$T_h u(\mathbf{z}_0) = u(\mathbf{z}_0) + \frac{1}{2} h^2 u_{\xi\xi}(\mathbf{z}_0) + O(h^3),$$

and the $O(h^3)$ is uniform in a neighborhood of $\mathbf{z}_0$.

Theorem 3 proves the consistency of the numerical scheme given by the iteration of T_h with respect to the DCMA evolution. Due to the h^2 coefficient in the expansion of T_h, it is natural to consider the numerical scheme which associates, to a given movie u_0 and a scale $t \geqslant 0$, the sequence of movies $(u_{n,t})_{n \geqslant 1}$ given by

$$u_n = (T_{h_n})^n u_0, \quad \text{with} \quad h_n = \sqrt{2t/n},$$

and satisfying the boundary constraint $u_n(x, y, \theta) = u_0(x, y, \theta)$ on $\partial(\mathbb{R}^2 \times I)$. Thanks to Theorem 3, we know that such a scheme is consistent, and one could prove that u_n converges towards the DCMA of u_0 when the partial derivative of u_0 with respect to x never vanishes. In the general case, the existence of a solution is not guaranteed, even if numerically the monotonicity of the scheme ensures the convergence of the algorithm. In fact, at singular points where no velocity can be defined, the scheme should produce an instantaneous evolution, as stated by the following

Proposition 9 *Let $P(x, \theta)$ be a polynomial with degree at most two and such that $P_x(x_0, \theta_0) = 0$. Then, in (x_0, θ_0) we have, as $h \to 0$,*

$$T_h P = P + \frac{h}{2} |P_\theta| \operatorname{sgn}(P_{xx}) + O(h^2).$$

Proposition 9 suggests that the numerical scheme we proposed may induce a projection of the initial datum from $\mathcal{C}_c^0$ to $\mathcal{V}_c^0$, defined by the asymptotic state of

$$u_t = \begin{cases} |u_\theta| \operatorname{sgn}(u_{xx}) & \text{if} \quad u_x = 0, \\ 0 & \text{else.} \end{cases}$$

Notice that if we follow Evans (see [6]) and consider the DCMA as the limit of (10), we obtain a different projection operator in general.

One may notice the extreme simplicity of the algorithm we presented : in particular, it can be implemented very easily on a massive parallel machine. Our optimized code in C language for one iteration consists of only 23 instructions. Numerical simulations realized with this algorithm are presented in Figure 1.

5   Conclusion

We presented a study of the DCMA equation, based on its interpretation as a level set formulation of the monodimensional heat equation. When the initial condition admits trajectories, we prove the existence and uniqueness of weak solutions. For general initial conditions, difficulties appear because the time variable imposes to consider level curves as graphs. Defining solutions in that case would probably require a weaker formulation of the DCMA allowing occlusion fronts to arise an propagate. The variational interpretation we pointed out might then be helpful to build the proper definition of solution in the case of occlusions. It is not sure, however, that an inf-sup-like scheme would still exist then and allow to estimate *indirectly* the velocity field.

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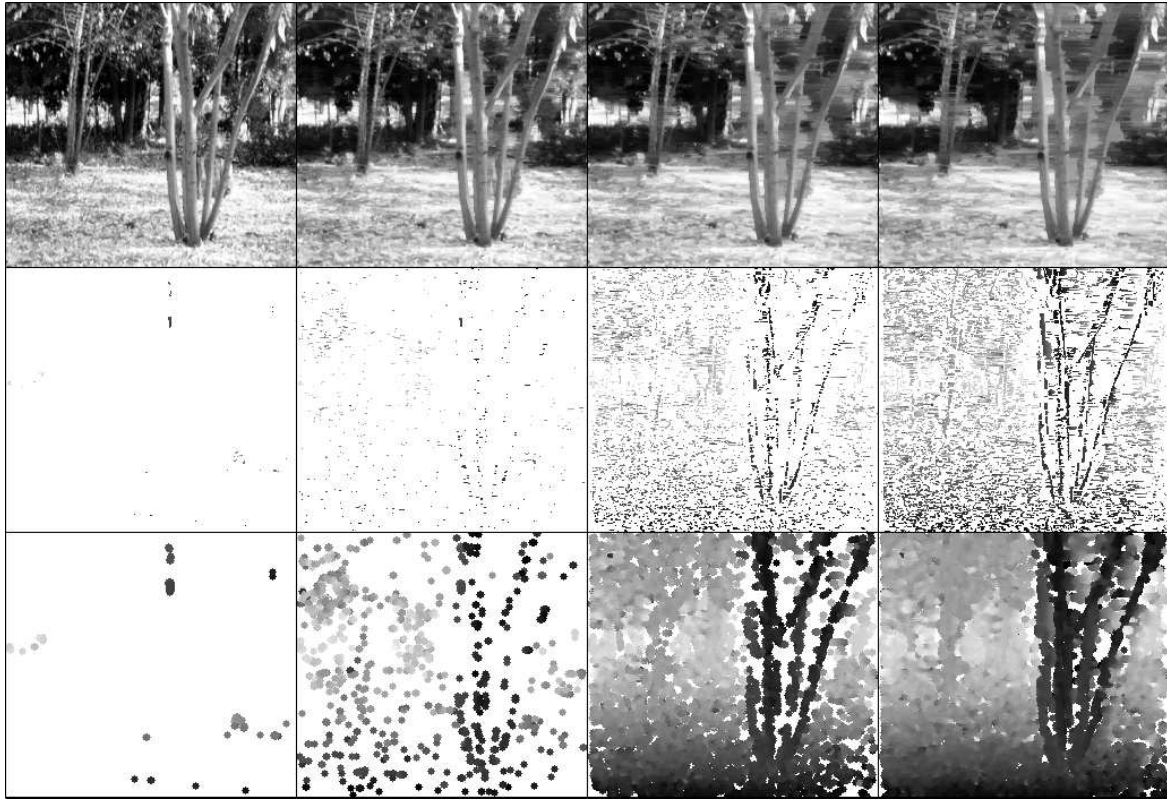
Figure 1: Computation of the velocity field (minimum of 15 matchings).

The following results were produced from a real movie produced by the SRI center (see [2]) and available with anonymous ftp at periscope.cs.umass.edu. *The four images on the first row are taken from four different movies : each image is the 20th image (over 64) of the movie it belongs to : column 1: original "TREES" movie, column 2: movie processed with DCMA (5 iterations), column 3: processed movie (15 iterations), column 4: processed movie (31 iterations). Then, the velocity field of each movie was computed on the 20th image simply by looking for trajectories with a matching constraint of 15 images. These velocities are represented on row 2 : the white color means points where no matching was found with respect to the constraint, and the grey scale (from light grey to black) measures the velocity from 0.0 to 2.0 pixels per image. On the third row, the velocity images of row 2 were simply "dilated" to produce more readable results. Notice how the velocity information, which is almost inexistant on the original movie (for the matching constraint we imposed), progressively appears on the DCMA as the scale increases. Since the distance of objects to the image plane is inversely proportional to their velocity, closest points appear in black and farthest ones in light grey. On the last image of row 3, we distinguish the foreground tree in black, the ground from black to middle grey, the background tree in middle grey, and the far background in light grey.*