Total Variation Restoration of Images Corrupted by Poisson Noise with Iterated Conditional Expectations

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The TV-ICE model

Observation model

Let $u : \Omega \to \mathbb{R}_+$ denote an (unobserved) intensity image defined on a discrete domain Ω, and v a random photon-count observation of the ideal image u, following the **Poisson probability density function**

$$p(v|u) = \prod_{x\in\Omega} \frac{u(x)^{v(x)}}{v(x)!} e^{-u(x)} \propto e^{-\langle u - v\log u, \mathbb{1}_\Omega\rangle} ,$$

where $\mathbb{1}_\Omega$ denotes the constant image equal to 1 on Ω and $\langle \cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^\Omega$.

Maximum a posteriori (MAP)

The equivalent of the Rudin-Osher-Fatemi (ROF) model [1] in the case of a Poisson noise model corresponds to the unique minimizer of the convex energy $E(u) = \langle u - v\log u, \mathbb{1}_\Omega\rangle + \lambda\,\mathrm{TV}(u)$ which is usually reinterpreted from a Bayesian point of view as the unique **maximizer of the posterior density**

$$\pi(u) = p(u|v) \propto p(v|u)p(u)$$

taking $p(u) \propto e^{-\lambda\,\mathrm{TV}(u)}$. The main drawback of this approach is that the restored image generally suffers from the **staircasing effect** due to the non-differentiability of the Total Variation (TV) term.

Least square error (TV-LSE)

In the case of Gaussian noise, this undesirable staircasing effect can be avoided by considering the **posterior mean** instead of the maximizer of $p(u|v)$, that is,

$$\widehat{u}_{\mathrm{LSE}} = \mathbb{E}_{u\sim\pi}(u) = \int_{\mathbb{R}^\Omega} u\,\pi(u)\,du ,$$

which is the image that **reaches the Least Square Error** under π (see [2]). As in the Gaussian case, the numerical computation of $\widehat{u}_{\mathrm{LSE}}$ could be done using a Markov Chain Monte Carlo Metropolis-Hasting algorithm, but would exhibit a **slow convergence rate**.

Iterated Conditional Means (TV-ICE)

To overcome this computational limitation, in [3] a new variant was proposed based on the iteration of **conditional marginal posterior means** (in the case of a Gaussian noise). The $\widehat{u}_{\mathrm{ICE}}$ estimate is defined as the limit of the iterative scheme

$$\forall x \in \Omega,\ u^{n+1}(x) = \mathbb{E}_{u\sim\pi}\left(u(x)\middle|u(x^c) = u^n(x^c)\right) ,$$

noting $x^c = \Omega\setminus\{x\}$. Like in the Gaussian case, the iterates are relatively **easy to compute** as they only involve the computation of integrals over $\mathbb{R}$.

In order to obtain explicit formulae we need to consider the **anisotropic** version of the discrete total variation

$$\mathrm{TV}(u) = \frac{1}{2}\sum_{x\in\Omega}\sum_{y\in\mathcal{N}_x} |u(x) - u(y)| ,$$

where $\mathcal{N}_x$ represents the 4-neighborhood of the pixel x. Noting $a = (a_1, a_2, a_3, a_4)$ the values of $u^n(\mathcal{N}_x)$ sorted in nondecreasing order (set $a_0 = 0 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5 = +\infty$). We obtain

$$u^{n+1}(x) = \frac{\sum_{1\le k\le 5} c_k\, I^{\mu_k, v(x)+1}_{a_{k-1},a_k}}{\sum_{1\le k\le 5} c_k\, I^{\mu_k, v(x)}_{a_{k-1},a_k}} , \qquad (1)$$

where c_k, μ_k are simples constants that explicitly depend on k, a and λ, while $I^{\mu,p}_{x,y}$ denotes the integral

$$I^{\mu,p}_{x,y} = \int_x^y s^p e^{-\mu s} ds ,$$

whose numerical evaluation is closely related to that of lower and upper **incomplete gamma functions**.

The Poisson TV-ICE algorithm

Numerical computation of the Poisson TV-ICE

The practical computation of the TV-ICE recursion raises several **numerical issues**:

1. For some values of the parameters, the numerator and the denominator in (1) **cannot be represented by the machine arithmetic** although the **actual value of the ratio is representable**. To solve that issue we represent each integral $I^{\mu,p}_{x,y}$ under the form $\rho \times e^\sigma$.
2. An integral $I^{\mu,p}_{x,y}$ can be computed as the difference between generalized lower (γ_μ) or/and upper (Γ_μ) incomplete gamma functions:
   - $I^{\mu,p}_{x,y} = \gamma_\mu(p+1, y) - \gamma_\mu(p+1, x)$,
   - $I^{\mu,p}_{x,y} = \Gamma_\mu(p+1, x) - \Gamma_\mu(p+1, y)$,
   - $I^{\mu,p}_{x,y} = \frac{p!}{\mu^{p+1}} - \gamma_\mu(p+1, x) - \Gamma_\mu(p+1, y)$.

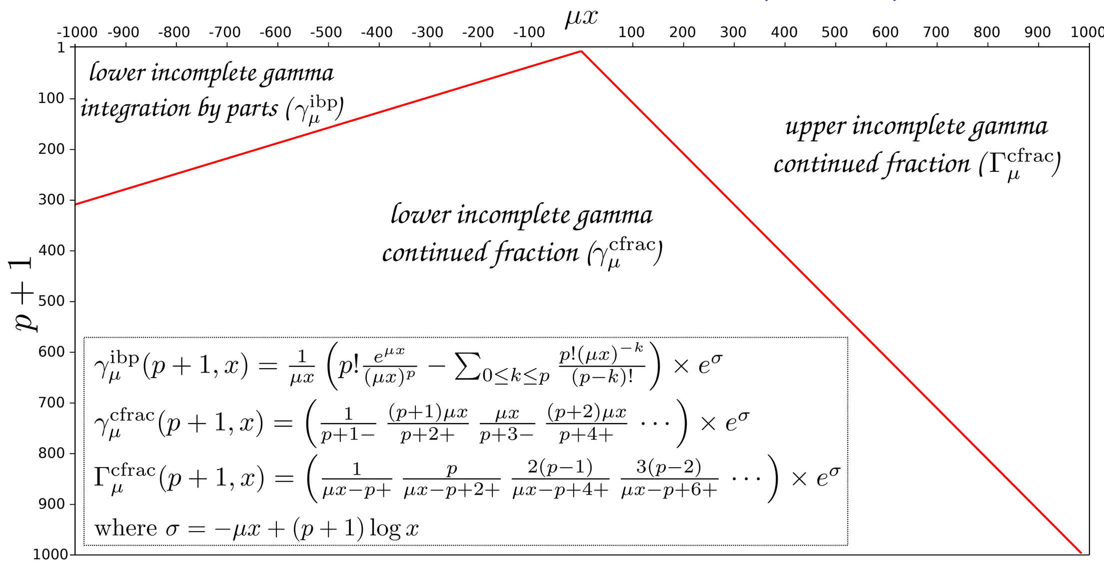
   We derived a partition of the plane $(\mu x, p)$ that allows an **efficient computation** of γ_μ or Γ_μ.

Generalized incomplete gamma functions

$$\gamma_\mu(p,x) = \int_0^x s^{p-1}e^{-\mu s}ds \quad (\forall \mu)$$

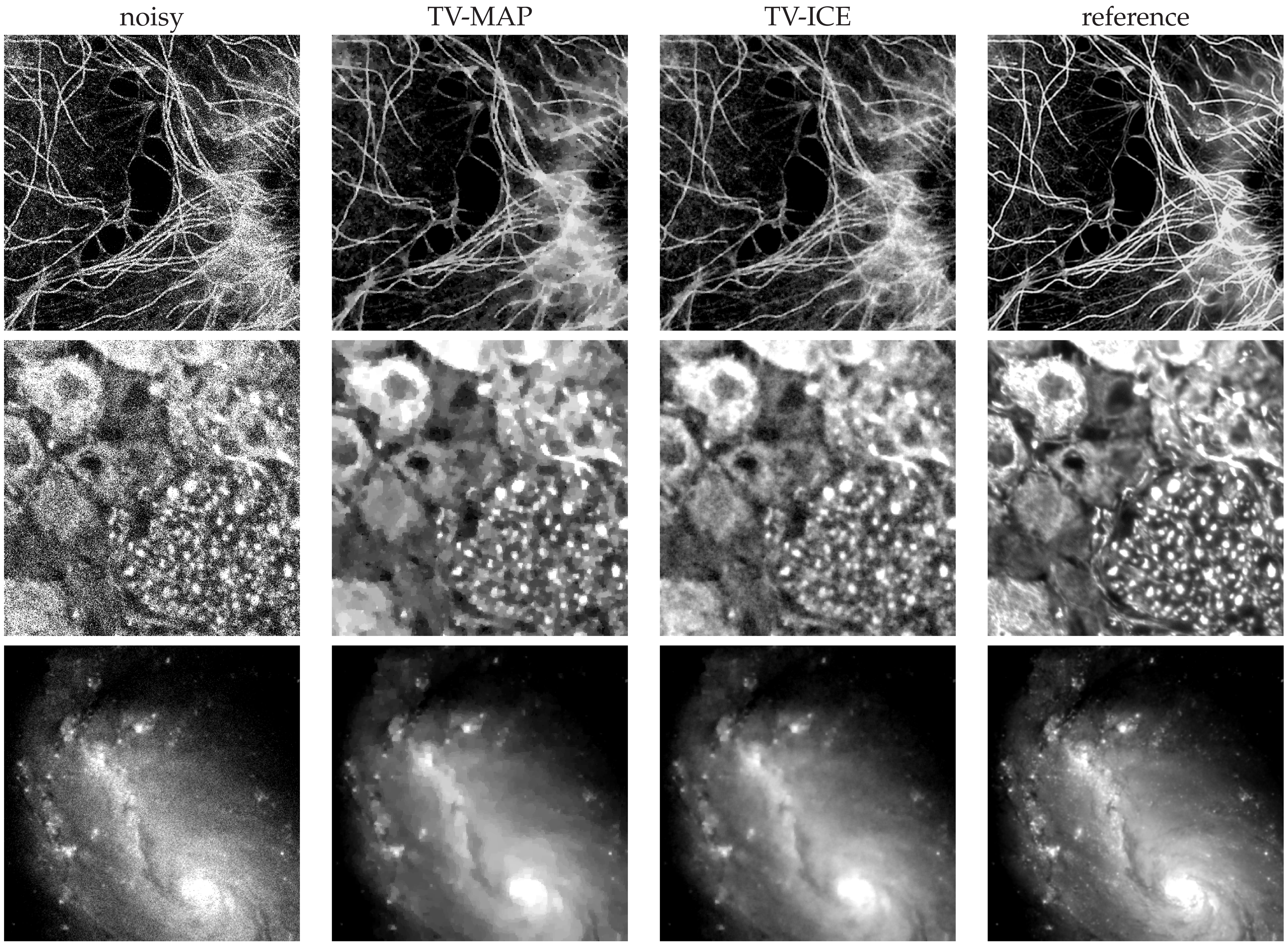
$$\Gamma_\mu(p,x) = \int_x^{+\infty} s^{p-1}e^{-\mu s}ds \quad (\mu > 0)$$

Numerical computation: γ_μ or Γ_μ?



Experiments

noisy | TV-MAP | TV-ICE | reference



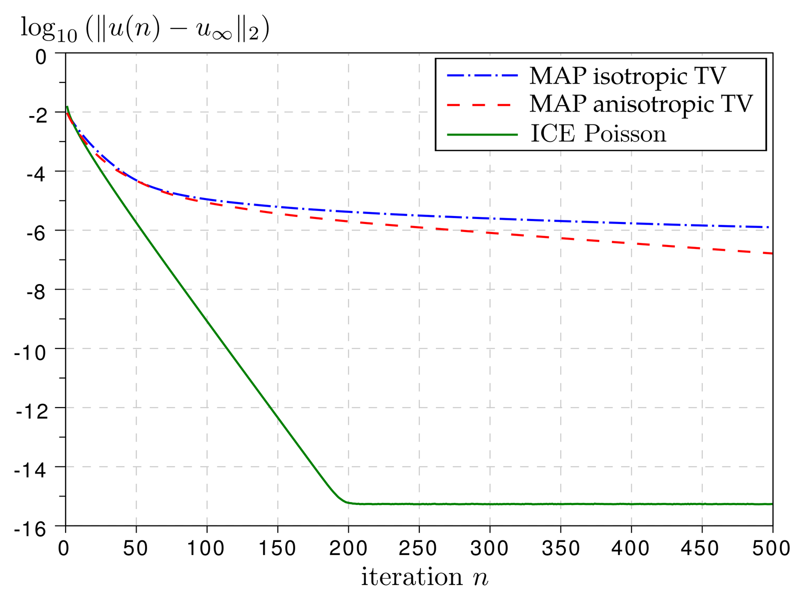
Mathematical properties

Theorem 1 (linear convergence). Given an image $v \in \mathbb{R}^\Omega$, the sequence of images $(u^n)_{n\ge0}$ defined by $u^0 = 0$ and the TV-ICE recursion **converges linearly** to an image $\widehat{u}_{\mathrm{ICE}}$.

Theorem 2 (no staircasing). Let $v : \Omega \to \mathbb{N}$ be a noisy image, and $\widehat{u}_{\mathrm{ICE}}$ its denoised version. Let x and y be two pixels of Ω, then if $\widehat{u}_{\mathrm{ICE}}$ is constant over $\mathcal{N}_x \cup \mathcal{N}_y \cup \{x, y\}$, necessarily $v(x) = v(y)$.

More intuitively, Theorem 2 proves that Poisson **TV-ICE cannot produce large constant regions** that were not at least partially present in the initial data.

Convergence rate



Main perspective: use the Poisson TV-ICE model in **more complex inverse problems** involving TV terms.

Main references

[1] Rudin, L. I., Osher, S., Fatemi, E.: “Nonlinear total variation based noise removal algorithms”. (1992)

[2] Louchet, C., Moisan, L.: “Posterior Expectation of the Total Variation model: Properties and Experiments”. SIAM J. Imaging Sci. (2013)

[3] Louchet, C., Moisan, L.: “Total Variation Denoising Using Iterated Conditional Expectation”. Proc. European Signal Processing Conf. (2014)

[4] Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W.: “NIST Handbook of Mathematical Functions”. Cambridge University Press, N-Y. (2010)