Random matrices, determinantal point processes and hyperuniformity

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Chapter 1

Introduction

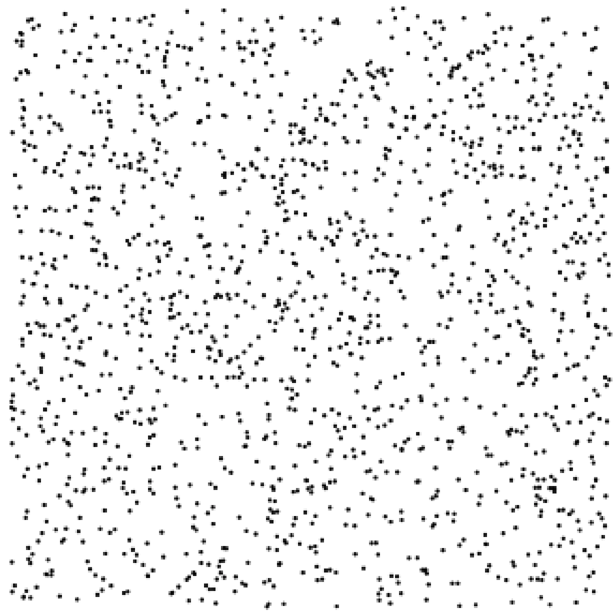


Figure 1.1: N Independent points

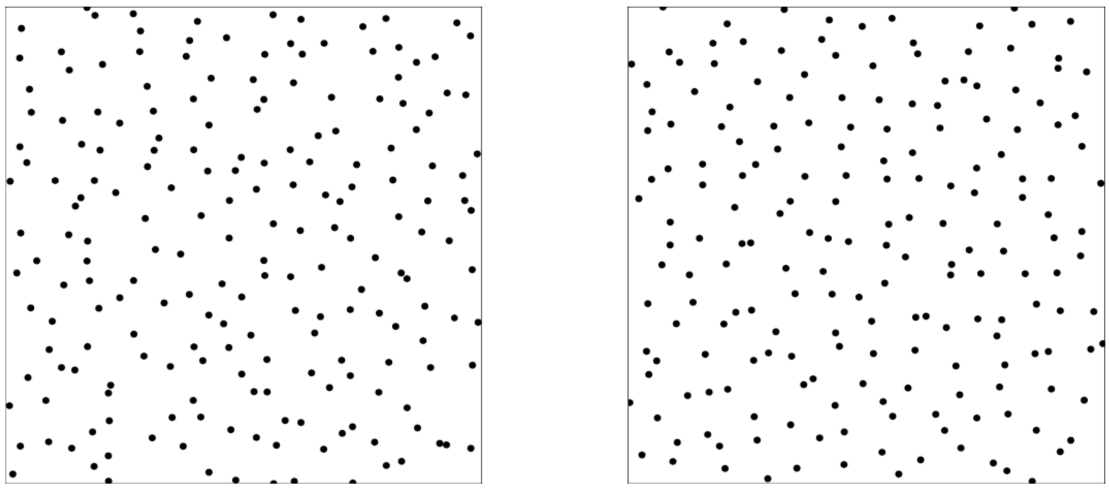


Figure 1.2: *Left:* Eigenvalues of $N \times N$ Ginibre random matrices / Coulomb particles. *Right:* Zeros of random polynomial $\sum_{n=0}^{N} X_n \frac{z^n}{\sqrt{n}}$

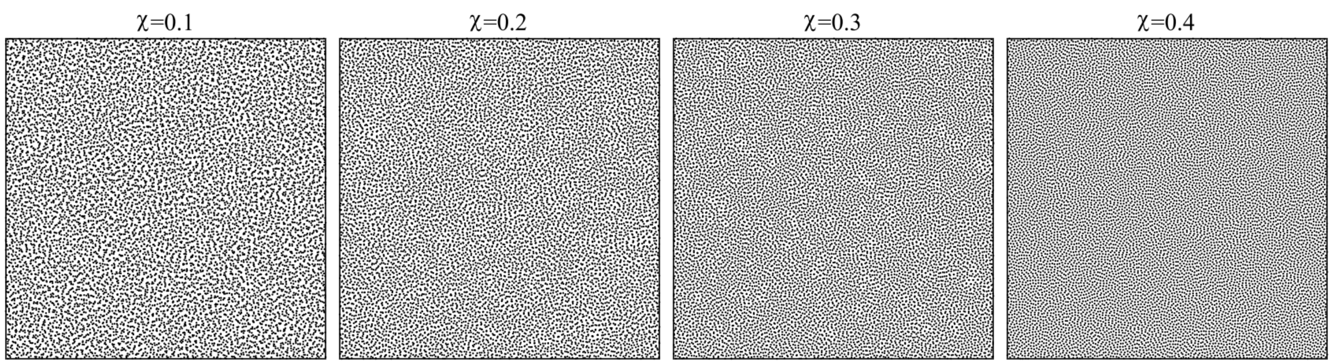


FIG. 1. Stealthy hyperuniform point patterns of $N = 2 \times 10^6$ particles with $\chi = 0.1, 0.2, 0.3$, and 0.4. Each image only shows 1/16th of all the data for better visualizations. Full configuration data are deposited through Princeton Data Commons [35].

Figure 1.3:

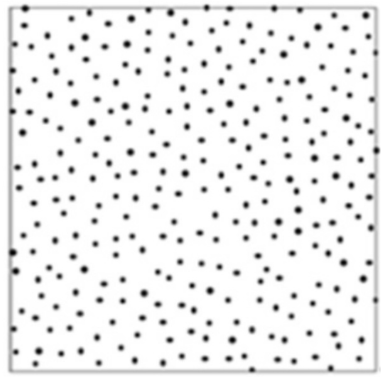


Figure 1.4: Photoreceptors in a chicken' eye

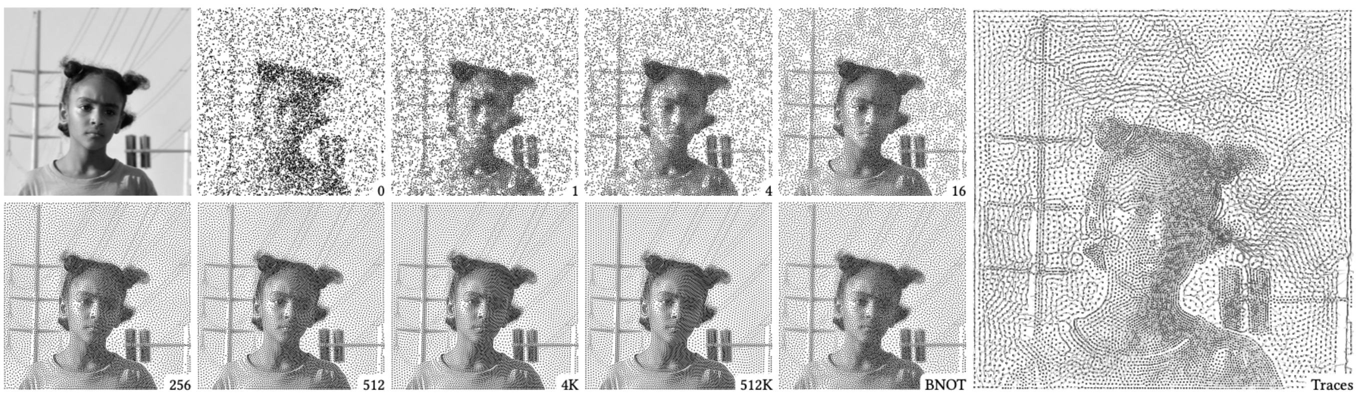


Figure 1.5:

- Theory of point processes
- Poisson point processes
- Zeros of random complex polynomials
- Random matrices / Coulomb systems
- Determinantal point processes

Survey: **Hyperuniform random measures, transport and rigidity**, arXiv preprint, `https://arxiv.org/abs/2510.18392` [?]

Other important source: [?]

Chapter 2

Some finite models

Definition 1. *For $A \subset \mathbb{R}^d$, call $\mathscr{N}(A)$ the class of subsets P of A that are "locally finite", i.e. such that for all compact set $K \subset \mathbb{R}^d$,*

$$\#P \cap K < \infty.$$

$\mathscr{N}(A)$ is called the class of (point) configurations over A. If $A = \mathbb{R}^d$, we write $\mathscr{N} = \mathscr{N}(\mathbb{R}^d)$.

A *point process* (on A), denoted by P in these notes, is a random element of $\mathscr{N}(A)$. Before giving a formal framework, let us examine some examples. They are all measurable in the sense that they result from a measurable construction based on (measurable) random variables on appropriate spaces.

2.1 n IID points

Denote by $\mathscr{L}^d$ the Lebesgue measure on $\mathbb{R}^d$.

Definition 2. *Let A a Borel set with $\mathscr{L}^d(A) \in (0, \infty)$. The most basic example is to draw n iid points $X_1, \dots, X_n$ uniform on A, i.e. with law $\mathscr{L}^d(\cdot \cap A)/\mathscr{L}^d(A)$, and put $\mathsf{P} = \{X_1, \dots, X_n\}$. It is called the binomial process (of n points) on A, its law is denoted* $\mathrm{Bin}(n, A)$.

2.2 Random zeros

Hereafter, $\mathbb{K}$ represents either $\mathbb{R}$ or $\mathbb{C}$.

Let $A_k, k = 1, \dots, n$ random variables in $\mathbb{C}$ with $A_n \neq 0$ a.s., and the degree n random polynomial

$$\mathsf{F}(X) = \sum_{k=0}^{n} A_k X^k.$$

Call $\mathscr{Z}_{\mathsf{F}} \subset \mathbb{C}$ the (random) set of roots, or zeros, of F. The cardinality of F is random, but in many situations we have $\#\mathscr{Z}_{\mathsf{F}} = n$. In some setups, F only has real roots, and we consider $\mathscr{Z}_{\mathsf{F}}$ as a point process of $\mathbb{R}$. We sometimes consider a unitary polynomial, i.e. set $A_n = 1$ a.s..

Definition 3 (Complex Gaussian variable)**.** *Let $G = A + \imath A'$ where A, A' are i.i.d $\mathscr{N}(0, 1/2)$ Gaussian. Equivalently, G has the elegant density in $\mathbb{C}$*

$$\frac{1}{\pi} \exp(-|z|^2).$$

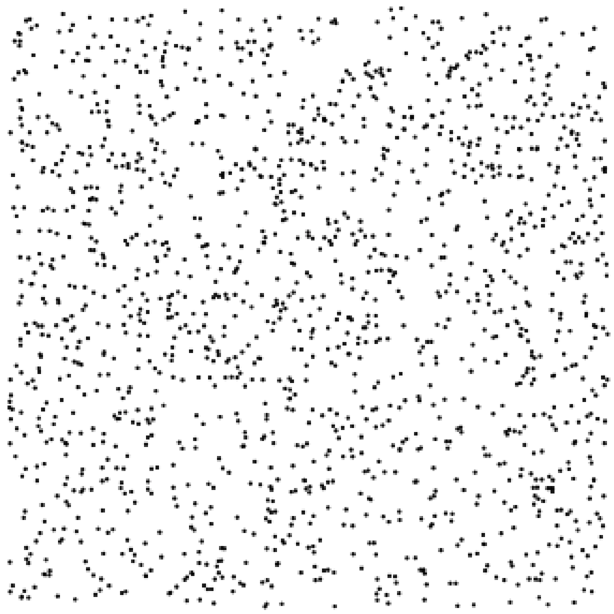


Figure 2.1: $\text{Bin}(n, [0,1]^2)$

For $\sigma > 0$, the law of σG is $\mathscr{N}_{\mathbb{C}}(0, \sigma^2)$. Note that

$$\mathbf{E}|\sigma G|^2 = \sigma^2.$$

The law of G is invariant under rotations of $\mathbb{C}$ (i.e. multiplication by $e^{i\theta}, \theta \in \mathbb{R}$).

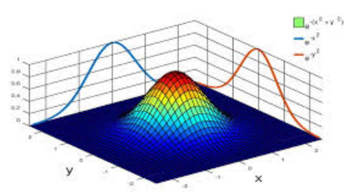


Figure 2.2: Complex Gaussian variable

Example 1 (Weyl polynomial / truncated Planar GAF)**.** *Let G_k i.i.d $\mathscr{N}_{\mathbb{C}}(0,1)$) and*

$$F(z) = \sum_{k=0}^{n} G_k \frac{z^k}{\sqrt{k}}$$

It seems like

- *Uniform on the disk*
- *Better distribution than i.i.d*

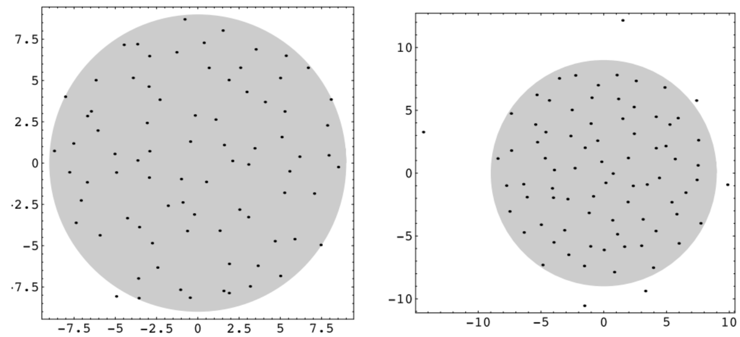


Figure 1. A typical realization of the eigenvalues of a 81 × 81 complex Gaussian random matrix (leftmost plot) and the zeros of a complex Gaussian random polynomial of degree 81 with variances given by (1.6). The shaded region represents the disc $|z| < 9$ which is the leading order support of the density in both cases. Outside the disc the density has a $1/r^2$ tail in the case of the zeros, whereas it falls off as a Gaussian for the eigenvalues, in keeping with the realizations in the figure.

Figure 2.3: [?]

2.3 Random matrices

A random matrix is a matrix with random entries in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We consider in this course two classes:

- (Symmetric/Hermitian matrices) Let $A_{i,j}, i \leqslant j$ random variables in $\mathbb{K}$, symmetric: $A_{j,i} := A_{i,j}$ ($\mathbb{K} = \mathbb{R}$), or Hermitian: $A_{i,j} = \bar{A}_{j,i}$ ($\mathbb{K} = \mathbb{C}$), and

$$\mathsf{M}_n = (A_{i,j})_{1 \leqslant i,j \leqslant n}$$

  To obtain asymptotic tractable results, we assume $(A_{i,j}, i \leqslant j)$ has a density with respect to $\mathscr{L}^d_{\mathbb{K}}[n(n-1)/2] \times \mathscr{L}^d\mathbb{R}[n]$. Later on, we will only consider independent $A'_{i,j}s$.

- Let $A_{i,j}, 1 \leqslant i, j \leqslant n$,

$$\mathsf{M}_n = (A_{i,j}, 1 \leqslant i, j \leqslant n),$$

  with $(A_{i,j}, 1 \leqslant i, j \leqslant n)$ having a density with respect to $\mathscr{L}^d_{\mathbb{K}}[n^2]$ (hence a.s. M_n is neither symmetric nor Hermitian).

In both cases, let P_n the point process formed by its eigenvalues. If M is symmetric or Hermitian, $\mathsf{P}_n \subset \mathbb{R}$. Otherwise, $\mathsf{P}_n \subset \mathbb{C}$.

2.3.1 GOE and GUE ensembles

Definition 4. *Let $A = (A_{i,j})_{1 \leqslant i,j \leqslant n}$ with the $A_{i,j} \sim \mathscr{N}(0, 1/2)$. Then $\mathsf{M}_n^{GOE} := A + A^T$ is a random real symmetric matrix, called Gaussian Orthogonal Ensemble (GOE).*

Proposition 1. *Let $M_{i,j}, 1 \leqslant i \leqslant j \leqslant n$ the entries of M_n^{GOE}. Then they are independent real Gaussian variables with*

$$Var(M_{i,j}) = \begin{cases} 1 \text{ if } i < j \\ 2 \text{ if } i = j \end{cases}$$

and $M_{j,i} = M_{i,j}$ for $j > i$.

Proof. For $i < j$, $M_{i,j} = A_{i,j} + A_{j,i} \sim \mathscr{N}(0, 1)$. For $i = j, M_{i,i} = 2A_{i,i} \sim \mathscr{N}(0, 2)$. □

Let us explain the GOE name and the different variance on the diagonal. Let $\mathscr{S}_n(\mathbb{R})$ the space of real symmetric matrices. For $m = (m_{i,j}, i \leqslant j) \in \mathbb{C}^{n(n+1)/2}$ and $M = (m_{i,j}) \in \mathscr{S}_n(\mathbb{R})$ the corresponding symmetric matrix, the density of the vector $(m_{i,j})_{i\leqslant j}$ in m is

$$\begin{aligned}\prod_{i<j}(2\pi)^{-1/2}\exp(-m_{i,j}^2/2)\prod_{i=1}^{n}(\pi)^{-1/2}\exp(-m_{i,i}^2/4) &\propto \prod_{i\neq j}\exp(-m_{i,j}^2/4)\prod_{i=1}^{n}\exp(-m_{i,i}^2/4)\\ &= \exp(-\sum_{i,j} m_{i,j}^2/4)\\ &= \exp(-\|M\|_2^2/4)\end{aligned}$$

with the Frobenius/Hilbert-Schmidt/2−norm

$$\|M\|_2^2 := \sum_{i,j}|a_{i,j}|^2 = \mathrm{Tr}(MM^*)$$

This is also the reason for the invariance under conjugation. Denote by $\mathscr{O}_n$ the orthogonal group

$$\mathscr{O}_n = \{O \in \mathscr{M}_n(\mathbb{R}) : OO^T = I_n\}.$$

Proposition 2. *For $O \in \mathscr{O}_n$, $O\mathsf{M}_nO^T \overset{(d)}{=} \mathsf{M}_n$.*

Proof. The density in $M \in \mathscr{S}_n(\mathbb{R})$ is the same as in OMO^T :

$$\exp(-\mathrm{Tr}(MM^T)/4) = \exp(-\mathrm{Tr}(OMO^TOM^TO^T)/4) = \exp(-\mathrm{Tr}((OMO^T)(OMO^T)^T)/4)$$

□

Direct proof: $X \overset{(d)}{=} OXO^T$ hence $\psi(X) := X + X^T = \mathsf{M}_n^{GOE}$ has the same law as $\psi(OXO^T) = O\mathsf{M}_n^{GOE}O^T$.

Let us now treat the GUE case. Denote

$$M^* = \overline{M^T}.$$

Let $\mathscr{H}_n(\mathbb{C})$ the space of Hermitian matrices, i.e. such that $M^* = M$.

Definition 5. *Let $G_{i,j}$ i.i.d $\mathscr{N}_{\mathbb{C}}(0, 1/2)$ variables. Then $\mathsf{M}_n^{GUE} = G + G^*$ is a random Hermitian matrix, called the GUE Hermitian model.*

Proposition 3. *Let $M_{i,j}$ the entries of M_n^{GUE}. Then*

$$M_{i,j} \overset{(d)}{=} \begin{cases}\mathscr{N}_{\mathbb{C}}(0,1) \text{ if } i < j\\ \mathscr{N}(0,1) \text{ if } i = j\end{cases}$$

and $M_{j,i} = \bar{M}_{i,j}, i < j$.

Proposition 4. *M_n^{GUE}'s law has density $\propto \exp(-\mathrm{Tr}(MM^*)/2)$ in $M \in \mathscr{H}_n(\mathbb{C})$. It is invariant under the action of the unitary group*

$$\mathscr{U}_n := \{U \in \mathscr{M}_n(\mathbb{C}) : UU^* = I_n\},$$

i.e.

$$\forall U \in \mathscr{U}_n, U\mathsf{M}_nU^* \overset{(d)}{=} \mathsf{M}_n.$$

2.3.2 Non-Hermitian Ginibre model

Let $G_{i,j}, 1 \leqslant i,j \leqslant n$ i.i.d $\mathscr{N}_{\mathbb{C}}(0,1)$ variables, and

$$\mathsf{M}_n^{\mathrm{Gin}} = (G_{i,j})_{1\leqslant i,j\leqslant n}.$$

Exercise 1. 1. $\mathsf{M}_n^{\mathrm{Gin}}$ is invariant under the action $M \mapsto UMV$ for U, V unitary matrices.

2. Under what similar transformation is the real Ginibre invariant? (same definition as for complex Ginibre but with $\mathscr{N}(0,1)$ i.i.d. entries)

2.4 Number of roots

2.4.1 Random polynomials

Recall

$$\mathsf{F}(z) = \sum_{k=0}^{n} A_k z^k$$

with $A_n \neq 0$ a.s.

Proposition 5. *If $(A_0/A_n, \ldots, A_{n-1}/A_n)$ has a density with respect to $\mathbb{K}^n$, $\#\mathscr{Z}_{\mathsf{F}} = n$ a.s.*

Note that if $(A_0, \ldots, A_n)$ has a density in $\mathbb{K}^{n+1}$, it works as well.

Proof. Call $\Lambda_1, \ldots, \Lambda_n$ the roots of F, repeated according to multiplicity, and sorted by lexicographic order. Define the discriminant of F as

$$\Delta(\mathsf{F}) = \prod_{i\neq j}(\Lambda_i - \Lambda_j).$$

The important features is that it vanishes iff F has multiple roots, and it is a symmetric polynomial in the $\Lambda_i's$. We aim to show $\Delta(\mathsf{F}) = Q_n(A_0/A_n, \ldots, A_{n-1}/A_n)$ for some deterministic polynomial Q_n. Since Q_n is not the null polynomial, we will prove that its zero set has zero Lebesgue measure.

For instance if $n = 2$, $\mathsf{F}(X) = A_0 + A_1X + A_2X^2 = A_2(X - \Lambda_1)(X - \Lambda_2)$,

$$\Delta(\mathsf{F}) = (\Lambda_1 - \Lambda_2)(\Lambda_2 - \Lambda_1) = -(\Lambda_1 - \Lambda_2)^2 = -\left[(\Lambda_1 + \Lambda_2)^2 - 4\Lambda_1\Lambda_2\right] = \frac{4A_0}{A_2} - \left(\frac{A_1}{A_2}\right)^2$$

and this vanishes almost never because for all a, $\{(b,c) : a^2 = 4bc\}$ is $\mathscr{L}^2$-negligible.

We have

$$\mathsf{F}(z) = A_n(z - \Lambda_1)\ldots(z - \Lambda_n).$$

Hence

$$\begin{aligned} A_0 =& A_n(-1)^n \prod_i \Lambda_i \\ &\ldots \\ A_n =& A_n. \end{aligned}$$

We write it as $A_k = A_n s_k(\Lambda_1, \ldots, \Lambda_n)$ where s_k is symmetric, i.e. invariant under permutation of the Λ_i. The s_k are called *elementary symmetric polynomials.* We then exploit the

Theorem 1 (Fundamental theorem of symmetric polynomials)**.** *Any polynomial $P(Z_1, \ldots, Z_n)$ symmetric, i.e. $= P(Z_{\sigma(1)}, \ldots, Z_{\sigma(n)})$ for any permutation σ, can be expressed as a polynomial expression evaluated in these elementary symmetric polynomials, i.e.*

$$P(Z_1, \ldots, Z_n) = Q_n(s_0(Z_1, \ldots, Z_n), \ldots, s_n(Z_1, \ldots, Z_n))$$

for some polynomial Q_n.

Ideas of proof.

- **Leading monomial argument.** Order monomials lexicographically. For a symmetric polynomial P, subtract a polynomial in $s_1, \dots, s_n$ whose symmetrization matches the leading monomial of P. The remainder is symmetric with strictly smaller leading term, and an induction concludes.

- **Roots–coefficients viewpoint.** Consider

$$\prod_{i=1}^{n}(X - Z_i) = X^n - s_1 X^{n-1} + \cdots + (-1)^n s_n.$$

  The elementary symmetric polynomials are the coefficients of the polynomial with roots $Z_1, \dots, Z_n$. Any symmetric polynomial in the roots can therefore be written as a polynomial in the coefficients.

- **Algebraic structure.** The invariant ring $\mathbb{K}[Z_1, \dots, Z_n]^{S_n}$ is generated by the elementary symmetric polynomials $s_1, \dots, s_n$. One shows that every symmetrized monomial can be reduced to a polynomial in these generators.

Reference. I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, 1995.

Back to proof:
Consider the discriminant, which is indeed symmetric

$$\Delta(\mathsf{F}) = P(\Lambda_1, \dots, \Lambda_n) = Q_n(s_0(\Lambda_1, \dots, \Lambda_n), \dots, s_n(\Lambda_1, \dots, \Lambda_n)) = Q_n(A_0/A_n, \dots, A_{n-1}/A_n, 1),$$

where Q does not depend on F. This is indeed polynomial in the A_k/A_n. This concludes the proof with:

Theorem 2. *For $Q \neq 0$ polynomial in n variables, its zero set is Lebesgue negligible in $\mathbb{R}^n$ or $\mathbb{C}^n$.*

Proof.

- $n = 1$: OK since Q is not the null polynomial.

- Then if it is true for $n-1$, induction on n: Then fix $z_2, \dots, z_n$. Then the set of z_1 such that it vanishes is Let us write for $z' \in \mathbb{C}^{n-1}$

$$Q(z_1, z') = \sum_{k=0}^{n} z_1^k q_k(z')$$

  with not all polynomials q_k identically 0. Let E the set of z' such that all $q_k(z')$ vanish. By the induction hypothesis, E is $\mathscr{L}^d_{\mathbb{C}^{n-1}}$-negligible. For $z' \notin E$, $\{z_1 : Q(z_1, z') = 0\}$ is negligible. Then Fubini gives the result.

There is an alternate proof with the implicit function theorem. □

□

2.4.2 Random matrices

Theorem 3. *Let $\mathsf{M}_n = (M_{i,j})_{1 \leqslant i,j \leqslant n}$*

- *(Symmetric / Hermitian) Assume M_n is Hermitian. Assume $(M_{i,j})_{i \leqslant j}$ has a density in $\mathbb{K}^{n(n+1)/2}$. Then $\#\mathsf{P}_n = n$ a.s. in $\mathbb{R}$.*

- *(Non-Hermitian) Assume that $(M_{i,j})_{1 \leqslant i,j \leqslant n}$ has a density in $\mathbb{K}^{n^2}$. Then $\#\mathsf{P}_n = n$ a.s. in $\mathbb{C}$.*

Proof. Consider

$$\mathsf{F}(\lambda) = \det(\lambda I_n - \mathsf{M}_n) = \sum_{i=0}^{n} A_i \lambda^i$$

polynomial in the Gaussian entries forming real and imaginary parts of the matrix. It hence has coefficients which are polynomial in the entries $A_0 = P_0((A_{i,j})), ..., A_{n-1} = P_n((A_{i,j})), A_n = 1$. Call $\Lambda_1, \ldots, \Lambda_n$ the eigenvalues in LG order with possible repetitions. With the previous result, we hence have

$$\Delta(\mathsf{F}) = \prod_{i \neq j} (\Lambda_i - \Lambda_j) = Q_n(A_0, \ldots, A_{n-1}) = Q_n(P_0((A_{i,j})), \ldots, P_{n-1}((A_{i,j})))$$

is a polynomial in the $A_{i,j}$, hence its zero set is $\mathscr{L}^N$-negligible, with $N = n(n+1)/2$ or $N = n^2$. Hence a.s. it is $\neq 0$ and the roots are simple.

□

A large part of random matrix theory is devoted to deriving results under weak assumptions on the distributions of the $A_{i,j}$, and on their dependence. In this introductory course, we only consider Gaussian ensembles with independent coordinates up to the symmetry constraints.

Exercise 2. Let a random symmetric matrix M with centred Gaussian entries in the upper diagonal forming a non-degenerate Gaussian vector. Prove that $\det(M)$ has a bounded density in 0

$$\mathbf{P}(|\det(M)| < a) \leqslant Ca.$$

Chapter 3

Semi-circular and circular laws

3.1 Topology

Recall the configurations space $\mathscr{N}(\mathbb{R}^d)$. We consider a random element P_n with n points, and wish to study its convergence.

Before introducing convergence for random measures, let us introduce it for deterministic ones:

Definition 6. *For measures m_n, m with finite mass, $m_n \to m$ weakly if $m_n(f) \to m(f)$ for f bounded continuous. Denote it by $m_n \xrightarrow[n\to\infty]{w} m$.*

This actually corresponds to the convergence in law for random variables : let $X_n, n \geqslant 1$ and X whose laws are m_n, m resp. Then

$$X_n \xrightarrow[n\to\infty]{\text{Law}} X \Leftrightarrow \mathbf{E}f(X_n) = \int f dm_n \to \mathbf{E}f(X) = \int f dm, \text{ for } f \text{ continuous bounded} \Leftrightarrow m_n \xrightarrow[n\to\infty]{\text{w}} m.$$

In particular, the method of moments can be used in $\mathbb{R}$, recalling the following.

Lemma 1. *For X a bounded variable in $\mathbb{R}$ (or equivalently if m has a bounded support),*

$$X_n \xrightarrow[n\to\infty]{\text{Law}} X \Leftrightarrow \forall k \in \mathbb{N}, \mathbf{E}X_n^k \to \mathbf{E}X^k.$$

Proof. Convergence in law is equivalent to convergence of the characteristic function, i.e. for all $t \in \mathbb{R}$

$$\mathbf{E}\exp(itX_n) \to \mathbf{E}\exp(itX).$$

Then expand exp in Taylor series and apply Lebesgue's theorem. □

3.1.1 Results

The limit laws for GUE and Ginibre ensemble are the following:

- Let the semi-circular law on $\mathbb{R}$

$$\mu_{\mathsf{sc}}(dt) = \mathsf{sc}(t)dt \text{ with } \mathsf{sc}(t) = \frac{1}{2\pi}\sqrt{4-t^2}1_{|t|\leqslant 2}, t \in \mathbb{R}$$

- Let the circular law on $\mathbb{C}$

$$\mu_{\text{circ}}(dz) = \frac{1}{\pi}\mathbf{1}\{z \in B(0,1)\}dz, z \in \mathbb{C}.$$

The constants ensure that these two laws are probability laws: note that $\sqrt{4-t^2}$ is the upper height of the half circle with radius 2 at abscissae t, hence the integral is half its surface

$$\int_{-2}^{2} \sqrt{4-t^2}dt = \frac{1}{2}\pi 2^2 = 2\pi.$$

Theorem 4. *Let* $\mathsf{P}_n = \{\Lambda_1, \ldots, \Lambda_n\}$ *either the GUE or GOE or complex Ginibre eigenvalues, and the rescaled renormalisation*

$$\tilde{\mathsf{P}}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\Lambda_i/\sqrt{n}}.$$

Then,

- *For the GOE and GUE models,* $\tilde{\mathsf{P}}_n \xrightarrow[n\to\infty]{\mathrm{w}} \mu_{SC}$ *a.s. on* $\mathbb{R}$.
- *For the Ginibre ensemble, on* $\mathbb{C}$, $\tilde{\mathsf{P}}_n^{\mathrm{Gin}} \xrightarrow[n\to\infty]{w} \mu_{\mathrm{circ}}$*.a.s..*

Method of proof for GOE/GUE:

- **Prove convergence of moments:** a.s., for all $\mathbf{k} \in \mathbb{N}^d$,

$$\tilde{\mathsf{P}}_n(x^{\mathbf{k}}) \to \mu(x^{\mathbf{k}}).$$

To that end:

1. First prove convergence of expectations:

$$\mathbf{E}\tilde{\mathsf{P}}_n(x^{\mathbf{k}}) \to \mu(x^{\mathbf{k}})$$

2. Then control the variance

$$\mathrm{Var}\left(\tilde{\mathsf{P}}_n(x^{\mathbf{k}})\right) \to 0$$

sufficiently fast.

3. Apply Borel-Cantelli lemma.

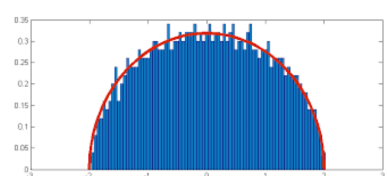


Figure 3.1: Empirical histogram for GOE eigenvalues

For Ginibre, the strategy is the same but we will show point 1 with the theory of DPPs.
End of first session

3.2 Symmetry considerations

Proposition 6. *The sets of eigenvalues* $\mathsf{P}_n^{GUE}, \mathsf{P}_n^{GOE}$ *are invariant in law under the operation* $x \mapsto -x$, *i.e.* $\mathsf{P}_n \stackrel{(d)}{=} -\mathsf{P}_n$.
For $\mathsf{P}_n^{\mathrm{Gin}} \subset \mathbb{C}$, *the invariance in law is under each rotation* $z \mapsto e^{i\theta}z$ *for* $\theta \in \mathbb{R}$.

Proof. GOE/GUE: Based on $\mathsf{M}_n \overset{(d)}{=} -\mathsf{M}_n$, hence

$$\mathsf{P}_n = \{\lambda : \det(\mathsf{M}_n - \lambda I) = 0\} \overset{(d)}{=} \{\lambda : \det(-\mathsf{M}_n - \lambda I) = 0\} = -\mathsf{P}_n.$$

For Ginibre, we want to have $\mathsf{M}_n \overset{(d)}{=} e^{i\theta}\mathsf{M}_n$, which is the direct consequence of the fact that for $G \sim \mathscr{N}_{\mathbb{C}}(0,1), e^{i\theta}G \overset{(d)}{=} G$.

Then

$$\mathsf{P}_n^{\mathrm{Gin}} = \{\lambda : \det(\lambda - \mathsf{M}I) = 0\} \overset{(d)}{=} \{\lambda : \det(\lambda - e^{i\theta}\mathsf{M}I)\} = e^{-i\theta}\mathsf{P}_n^{\mathrm{Gin}}.$$

□

Consequences on moments: For the GOE, GUE, $k \in \mathbb{N}$,

$$\mathbf{E}\mathsf{P}_n(x^k) = \mathbf{E}\sum_i \Lambda_i^k = \mathbf{E}\sum_i (-\Lambda_i)^k = (-1)^k\mathbf{E}\mathsf{P}_n(x^k).$$

Hence the expected moments vanish for k odd. It means that the limit distribution should be even (and indeed $\mathsf{sc}(t)$ is an even function).

Expected moments. **3.2.1 GOE Proof**

For $k = 2p$, treat first $p = 1$ to get the idea.

$$\mathbf{E}\mathsf{P}_n(t^2) = \mathbf{E}\sum_i \Lambda_i^2 = \mathbf{E}\|\mathsf{M}_n\|_2^2 = \mathbf{E}\sum_{i,j}|A_{i,j}|^2 = \sum_{i,j}\mathrm{Var}\,(A_{i,j}) = n^2 + 2n \sim n^2.$$

Hence the renormalisation yields for GUE and GOE

$$\mathbf{E}\tilde{\mathsf{P}}_n(t^2) = \frac{1}{n}\mathbf{E}\sum_{i=1}^n (\Lambda_i/\sqrt{n})^2 \to 1.$$

For general $p \geqslant 1$,

$$\mathbf{E}\mathsf{P}_n(t^{2p}) = \mathbf{E}(\sum_i \Lambda_i^{2p}) = \mathbf{E}\sum_{i_1}(M^p\bar{M}^p)_{i_1,i_1} = \sum_{i_1,\dots,i_{2p-1}} \mathbf{E}A_{i_1,i_2}\bar{A}_{i_2,i_3}A_{i_3,i_4}\dots\bar{A}_{i_{2p},i_1}$$

Let us start with the real GOE case $A_{i,j} = \bar{A}_{i,j}$. Let us try to see when the variance cancels:

- If a term $A_{i,j}$ is not "compensated", it gives a factor $\mathbf{E}A_{i,j} = 0$, and all the product vanishes.

Therefore, since $\mathbf{E}A_{i,j} = 0$ for all i, j,

$$\mathbf{E}A_{i_1,i_2}\dots A_{i_{2p},i_1} \neq 0$$

if and only if each pair (i, j) appears an even number of times, example with $2p = 8$

$$\mathbf{E}A_{3,2}A_{2,3}A_{3,4}A_{4,6}A_{6,6}A_{6,6}A_{6,4}A_{4,3} = (\mathbf{E}A_{1,2}^2)^6(\mathbf{E}A_{1,1}^2)^2 = 1^6 2^2 = 4.$$

(This provides another proof of why odd powers vanish $\mathbf{E}\mu_n(t^{2p+1}) = 0$).

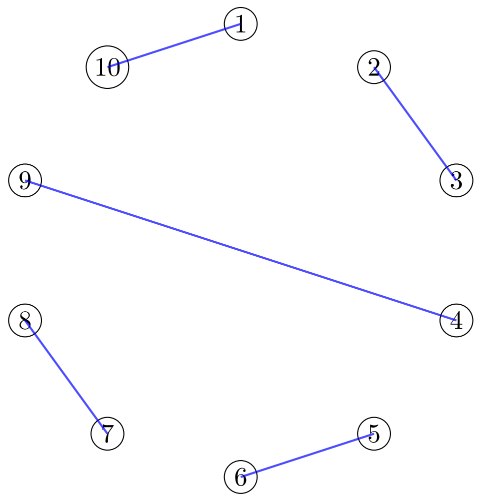
Each term of the product can be seen as an opening bracket or a closing bracket, for instance

$$\underbrace{A_{12}}_{(_1}\ \underbrace{A_{23}}_{(_2}\ \underbrace{A_{32}}_{)_2}\ \underbrace{A_{24}}_{(_3}\ \underbrace{A_{45}}_{(_4}\ \underbrace{A_{54}}_{)_4}\ \underbrace{A_{46}}_{(_5}\ \underbrace{A_{64}}_{)_5}\ \underbrace{A_{42}}_{)_3}\ \underbrace{A_{21}}_{)_1}$$

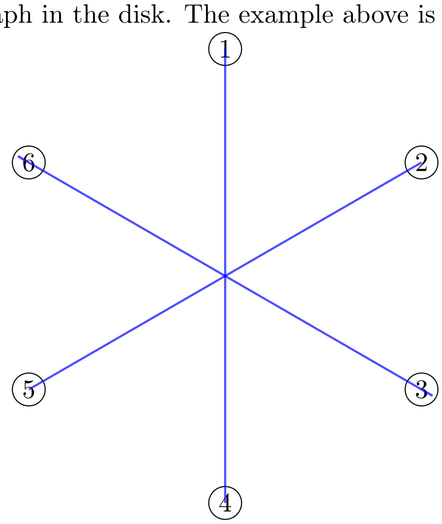
We represent such terms by a graph on the locations of terms $\{1,\dots,2p\}$ (not on indexes between 1 and n). We first focus on the case where each $A_{i,j}$ appears at most two times to simplify the problem. The number of possibilities is the number of ways to pair points.

For instance above the pairings are

$$\{1,10\},\{2,3\},\{4,9\},\{5,6\},\{7,8\}.$$



We say that the pairing is not crossing when there is no crossing if we represent the pairings by a graph in the disk. The example above is non crossing with $\{1,4\},\{2,5\},\{3,6\}$



Here is a crossing graph associated to $\{1,4\},\{2,5\},\{3,6\}$ (i.e. to for instance $\mathbf{E}A_{12}A_{23}A_{31}A_{12}A_{23}A_{31}$)

To each term is associated a graph G on $\{1,2,\dots,2p\}$. Conversely there are several terms $\mathbf{E}A_I$ associated with one graph, which we denote $I\to G$. Hence the expectation is

$$\mathbf{E}\mathsf{P}_n(t^{2p})=\sum_G\sum_{I:G\to I}\mathbf{E}A_I.$$

Lemma 2. *The non crossing graphs dominate the sum:*

$$\sum_{G\text{ not crossing}}\sum_{I:G\to I}\mathbf{E}A_I\sim n^{p+1}$$

$$\sum_{G\text{ crossing}}\sum_{I:G\to I}\mathbf{E}A_I=o(n^{p+1})$$

This is the subtle point of the proof.

Proof. We ignore for now indexes I having diagonal terms $A_{i,i}$, or with powers larger than 2, e.g. $A^4_{i,j}$, as they remove degrees of freedom.

To count the number of degrees of freedom, we actually have to consider the graph on $[[2p]] := \{1, \ldots, 2p\}$ corresponding to constraint on the $2p$ bullet points

$$A \underbrace{\bullet, \bullet}_{1,2} \ldots A \underbrace{\bullet, \bullet}_{2p-1,2p}.$$

For $I = (i_1, i_2, \ldots, i_{2p})$, call H_I the corresonding graph on $[[2p]]$ depending on equality between successive indexes, and repeating terms $A^2_{i,j}$ or $A^2_{j,i}$. For instance for $I = (1, 2, 2, 3, 3, 2, 2, 1)$, corresponding to $\mathbf{E}A_{12}A_{2,3}A_{3,2}A_{2,1}$, the graph connects $2 \to 3, 4 \to 5, 6 \to 7, 8 \to 1$ and $1 \to 8, 2 \to 7$ because of terms $A_{1,2}, A_{2,1}$, $3 \to 6, 4 \to 5$ because of terms $A_{2,3}, A_{3,2}$. It has $3 = p + 1$ degrees of freedom (here $p = 2$). This is typically the kind of graphs that will contribute to the dominating term. We observe that graph H_I gives a degree two to each index, but this will not matter so much (but it is a way to ensure your pictures are right).

We see that for every I, we already have in H_I the constraint that 2 is connected to 3 because of $A_{i_1,\mathbf{i_2}}A_{\mathbf{i_2},i_3}\ldots$, and similarly $4 \to 5, 6 \to 7, \ldots, 2p \to 1$. The positions of repeating terms $A^2_{i,j}$ give the rest of the connections.

We will discuss the other constraints below, but the key point is that we will count indexes I depending on the number of connected of components of the graph H_I, denoted d_I, because this is what gives the number of degrees of freedom. Indeed, all the points in the same connected component must be equal, hence when one is chosen, all the other ones are determined, which gives a magnitude in n^{d_I}: for $d \in \mathbb{N}$,

$$\sum_{I:d_I=d} \mathbf{E}A_I = O(n^d).$$

We consider also the graph on G on $[[p]]$ induced by H_I, but remark that several possible H might corresponding to a given G, depending on wether one has $A^2_{i,j}$ or $A_{i,j}A_{j,i}$. We will see that terms $A^2_{i,j}$ induce less degrees of freedom, hence they are negligible.

Of particular importance are the I for which $d_I = p + 1$. We want to prove that these are the dominating terms, and that their contribution is

$$\sum_{I:d_I=p+1} \mathbf{E}A_I = \#\{G \text{ without crossings}\}n^{p+1}.$$

There are several steps:

- Start from a given graph G not crossing on $[[p]]$. There is a unique graph H_G on $[[2p]]$ such that I yielding graph G has $d_I = p + 1$ implies that I has graph $H_I = H_G$ (said differently, there are indexes I giving graph G but not giving a graph H_I with $p + 1$ degrees of freedom). The graph H_G is the unique graph that you can draw that corresponds to G where there is no crossing for H (the best way to see that is to draw pictures of examples).
- For such a graph H without crossings corresponding to a graph G (also necessarily without crossings), you have $p + 1$ connected components. You can prove it by induction.
  - Ok for $p = 1$.
  - For any p, take such a graph H and the corresponding graph G. There are necessarily at least two neighbours $i, i + 1$ connected in G, hence $2i + 1, 2i + 2$ are connected in H (make a picture). They are also connected by default, hence these two indexes form a closed loop in H. Denote by $\tilde{H}$ the graph obtained after removing this loop, and $\tilde{G}$ the corresponding graph on $[[p]]$. $\tilde{G}$ is a not crossing graph on $[[p - 1]]$, and $\tilde{H}$ on $[[2p - 2]]$, hence by induction it has $p - 1 + 1 = p$ connected components. Since H is obtained from $\tilde{H}$ by adding a component, H has $p + 1$ components.

- Let us finally give the idea why other type of graph has less connected components. If one starts from a non-crossing graph H on $[[2p]]$ and crosses two edges, you merge two connected components, hence the number of CCs strictly decreases. Any graph with crossings can be obtained from a graph without crossing (hence with $p+1$ CCs) by crossing successively some of its CCs, hence its number of CCs is $< p+1$. We do not give a formal prove, and encourage you to try to convince yourself first with examples.

This finally yields the formula

$$\begin{aligned}\sum_I \mathbf{E}A_I &= \sum_G \sum_{H:G_H=G} \sum_{I=H_I=H} \mathbf{E}A_I \\ &= \sum_{G \text{ not crossing}} \sum_{H \text{ not crossing}:G_H=G} \sum_{I=H_I=H} \mathbf{E}A_I + o(n^{p+1}) \\ &= \sum_{G \text{ not crossing}} \underbrace{1}_{\text{only one possible } H} \times \underbrace{1}_{(\mathbf{E}A_{i,j}A_{j,i})^{p+1}}\end{aligned}$$

$$= \#\{G \text{ not crossing}\}$$

□

Theorem 5. *The number of noncrossing pairings over $k = 2p$ is the Catalan number*

$$C_p = \frac{1}{p+1}\binom{p}{2p}.$$

Hence after renormalisation we arrive at

$$\mathbf{E}\tilde{\mathsf{P}}_n(t^{2p}) = \frac{1}{n}\left(\frac{1}{\sqrt{n}}\right)^{2p} \mathbf{E}\mathsf{P}_n(t^{2p}) \sim \frac{1}{n^{p+1}} n^{p+1} C_p \to C_p.$$

Proof. There is also a purely computational proof, more intricate. Proof by generating functions Let

$$F(x) = \sum_{p\geqslant 0} C_p x^p.$$

Using the closed formula for C_p,

$$F(x) = \sum_{p\geqslant 0} \frac{1}{p+1}\binom{2p}{p} x^p.$$

Now use the standard series

$$\sum_{p\geqslant 0} \binom{2p}{p} x^p = \frac{1}{\sqrt{1-4x}}.$$

So

$$F(x) = \frac{1}{x}\sum_{p\geqslant 0}\binom{2p}{p}\frac{x^{p+1}}{p+1} = \frac{1}{x}\int_0^x \frac{dt}{\sqrt{1-4t}}.$$

Evaluating the integral gives

$$F(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Now rearrange:

$$2xF(x) = 1 - \sqrt{1-4x}.$$

Square both sides:

$$(2xF(x) - 1)^2 = 1 - 4x.$$

Expanding:

$$4x^2F(x)^2 - 4xF(x) + 1 = 1 - 4x.$$

Cancel 1 and divide by $4x$:

$$xF(x)^2 - F(x) + 1 = 0.$$

Hence

$$F(x) = 1 + xF(x)^2.$$

Now compare coefficients of x^p on both sides. For $p \geqslant 1$,

$$C_p = [x^p]F(x) = [x^{p-1}]F(x)^2.$$

But

$$F(x)^2 = \left(\sum_{i\geqslant 0} C_i x^i\right)\left(\sum_{j\geqslant 0} C_j x^j\right),$$

so the coefficient of x^{p-1} is

$$\sum_{i=0}^{p-1} C_i C_{p-1-i}.$$

Therefore

$$C_p = \sum_{i=0}^{p-1} C_i C_{p-1-i}.$$

If we set $l = i + 1$, this becomes

$$C_p = \sum_{l=1}^{p} C_{l-1} C_{p-l},$$

which is exactly the desired recursion. □

Let us show that the semi-circular law has the same moments using classical results. Simplify to

$$\begin{aligned} m_{2p} :=& \int_{-2}^{2} t^{2p}\sqrt{4-t^2}dt = 2\int_0^2 t^{2p}\sqrt{4-t^2}dt = 2^{3+2p}\int_0^1 s^{2p}\sqrt{1-s^2}ds \\ =& 2^{3+2p}\int_0^1 u^{p-1/2}(1-u)^{1/2}\frac{du}{2} = 2^{2p+3}\frac{1}{2}\mathsf{B}(\frac{2p+1}{2}, \frac{3}{2}) \end{aligned}$$

and $\mathsf{B}(a,b) = \frac{\Gamma(a)}{\Gamma(b)}\Gamma(a+b)$.

Method: $x = ut, y = t(1-u)$

$$\Gamma(a)\Gamma(b) = \int x^{a-1}e^{-x}dx \int y^{b-1}e^{-y}dy = \int\int (ut)^{a-1}(t(1-u))^{b-1}e^{-t}tdtdu = \Gamma(a+b)B(a,b)$$

TOC

We finally have

$$m_{2p} = \frac{1}{2\pi} 2^{2p+2} \frac{\Gamma(3/2)\Gamma((2p+1)/2)}{\Gamma(p+2)}$$

These are actually known values:

- $\Gamma(p+1/2) = \frac{(2p)}{4^p p}\sqrt{\pi}$
- $\Gamma(3/2) = \frac{2}{4}\sqrt{\pi}$
- $\Gamma(p+2) = (p+1)$

Finally

$$m_{2p} = \frac{1}{2\pi} 2^{2p+2} \frac{\frac{1}{2}(2p)/(4^p p)\pi}{(p+1)} = \frac{1}{p+1}\binom{p}{2p}$$

□

3.2.2 Higher moments

Proof. Let $X_{n,k} = \tilde{\mathsf{P}}_n(t^k)$. We proved $\mathbf{E}X_{n,k} \to \mathbf{E}\mu_{SC}(t^k)$. Now that the expectation converges to the right value, let us bound the variance.

$$\mathbf{E}\left(X_{n,k}\right) = \left(\frac{1}{n^{p+1}}\right)^2 \sum_{I^1=(i_1^1,\dots,i_{2p}^1), I^2=(\dots)} \mathbf{E}\, \underbrace{A_{i_1^1,i_2^1}\dots A_{i_{2p}^1 i_1^1}}_{A_{I^1}} \underbrace{A_{i_1^2 i_2^2}\dots A_{i_{2p}^2 i_1^2}}_{A_{I^2}}$$

$$\mathrm{Var}\left(X_{n,k}\right)^2 = \frac{1}{n^{2p+2}} \sum_{I^1,I^2} \mathbf{E}[A_{I^1}A_{I^2}] - \mathbf{E}[A_{I^1}]\mathbf{E}[A_{I^2}].$$

As before, the first term is over pairings of $I^1 \cup I^2$. Say a pairing is disconnected if it is the union of a pairing over I^1 and a pairing over I^2. Such pairings give a 0 contribution to the variance. It only remains connected pairings, that we have to count.

To be more explicit, a connected pairing is such that one (i_m^2, j_m^2) is the closing bracket of some (i_l^1, i_{l+1}^1), i.e. $(i_m^2, j_m^2) = ((i_{l+1}^1, i_l^1))$. Denote by N_p the number of connected pairings over $4p$ indexes.

Once the way to pair indexes has been chosen, it remains to "fill the locations" with indexes from $\{1,\dots,n\}$. The major difference with before is that we cannot choose (i_m^2, j_m^2), which removes two degrees of freedom. In total there will be $2p$ "set of brackets". The final number of degrees of freedom is

$$\underbrace{\text{choice of } (i_1^1, i_2^1)}_{2} + \underbrace{\text{choice of second index in "opening bracket"}}_{2p-1}.$$

Therefore it yields

$$\mathrm{Var}\left(X_{n,k}\right) \sim \frac{1}{n^{2p+2}} N_p n^{2p+1} = O(n^{-1}).$$

It yields convergence in probability with Byenaimé-Tchebyshev inequality:

$$\mathbf{P}(|X_{n,k} - \mathbf{E}X_{n,k}| > \varepsilon) \leqslant \frac{\mathrm{Var}\left(X_{n,k}\right)}{\varepsilon^2} \to 0.$$

Unfortunately n^{-1} is not summable, we cannot deduce a.s. convergence. We have still proved a.s. convergence for some subsequences: $\sum_n \frac{1}{2^n} < \infty$ hence

$$X_{2^n,k} \to X_k \text{ a.s.}$$

To prove a.s. convergence $X_{n,k} \to X_k$, we must look at the 4th moment:

$$\mathbf{E}\left(\sum_I A_I - \mathbf{E}A_I\right)^4 = \mathbf{E}A^4 - 4\mathbf{E}A^3\mathbf{E}A + 6\mathbf{E}A^2\mathbf{E}A^2 - 4\mathbf{E}A\mathbf{E}A^3 + \mathbf{E}A^4$$
$$= \sum_{I^1, I^2, I^3, I^4} \mathbf{E}A_{I^1 \dots I^4} - 3\mathbf{E}A_{I^1}\mathbf{E}A_{I^2, I^3, I^4} + \dots$$

We can show that if one I^l, say I^3, is disconnected from the others (i.e. it has its own opening and closing brackets), or if two of them are disconnected from the two others, then separating the expectations the corresponding term directly vanishes, as for the variance computation. It remains the tuples $I^1, \dots, I^4$ completely connected. Reasoning similarly, we can show that constraining to such tuples removes 2 degrees of freedom. Finally

$$\mathbf{E}|\tilde{\mathsf{P}}_n(t^{2p}) - \mathbf{E}\tilde{\mathsf{P}}_n(t^{2p})|^4 \leqslant \frac{1}{n^{4(p+1)}} n^{4(p+1)-2} = O(n^{-2}).$$

We can then use Borel Cantelli lemma with $\varepsilon = \varepsilon_n = 1/\ln(n)$,

$$\sum_n \mathbf{P}(|X_{n,k} - X_k| > \varepsilon_n) \leqslant \sum_n \frac{\mathbf{E}|X_{n,k} - X_k|^4}{\varepsilon_n^4} < \infty$$

hence a.s., for n sufficiently large, $|X_{n,k} - X_k| < \varepsilon_n$, and $X_{n,k} \to X_k$ a.s.. Formally, to talk about a.s. convergence, one must first make sure that variables are built on the same probability space, but it changes nothing to the computation.

□

Exercise 3. Make the same proof for the GUE model.

Let $P_n = \sum_i \delta_{\Lambda_i}$ the empirical measure of the GUE, and $\tilde{\mathsf{P}}_n = n^{-1}\sum_i \delta_{\Lambda_i/\sqrt{n}}$. Let

$$Y_{n,k} = \tilde{\mathsf{P}}_n(t^k) = n^{-1}\sum_i \left(\frac{\Lambda_i}{\sqrt{n}}\right)^k = \frac{1}{n^{k+1}}\mathsf{P}_n(t^k).$$

The method is the same as for the GOE. The only difference is the following: for $G \sim \mathscr{N}_{\mathbb{C}}(0,1)$,

$$\mathbf{E}G^2 = \int z^2 e^{-|z|^2} dz = 0$$

with the change of variables $w = iz$. Hence with $\mathsf{M} = \mathsf{M}_n^{GUE}$, in the sum

$$\mathbf{E}\mathsf{P}_n(t^{2p}) = \mathbf{E}\sum_i \Lambda_i^{2p} = \mathbf{E}\mathrm{Tr}((P\mathsf{M}\bar{\mathsf{M}}P^*)^p) = \mathrm{Tr}((\mathsf{M}\bar{\mathsf{M}})^p)$$
$$= \sum_i \mathbf{E}(\mathsf{M}\bar{\mathsf{M}}\mathsf{M}\bar{\mathsf{M}}\dots\mathsf{M}\bar{M})_{i,i}$$
$$= \sum_{i_1, \dots, i_{2p-1}} \mathbf{E}A_{i_1,i_2}\bar{A}_{i_2,i_3}A_{i_3,i_4}\dots\bar{A}_{i_{2p},i_1}$$

We will treat the Ginibre case later with DPPs and hyperuniformity.
End of 2d course

Chapter 4

Gibbs measures

4.1 Systems of particles

The GOE, GUE, and Ginibre models, descend from a more general class of models in statistical physics defined through a Hamiltonian. Consider a pairwise potential, i.e. a function $\varphi : \mathbb{R}^d \to \mathbb{R}$, and the energy function

$$H^\circ(\{x_1, \dots, x_n\}) := \sum_{i<j} \varphi(x_i - x_j).$$

A *Gibbs measure* with energy H° is roughly speaking a system of particles that tend to arrange themselves randomly while keeping a low-energy configuration. Traditionnally, $\varphi(x)$ is a decreasing function of $\|x\|$, hence the more the particles shift apart at infinity the lowest is, in general, the energy, one must add a confinement term to ensure the particles have an antagonising force to localise them. This could be a hard confinment in a ball (or another shape) so that each particle has approximately a constant volume for itself. There are also smoother ways to confine the particles, we will typically consider an energy term of the form

$$H(x_1, \dots, x_n) = H^\circ(x_1, \dots, x_n) + \sum_i V(x_i)$$

and V_n is called a confinment potential, it is supposed to have compact sublevel sets. A hard confinment penalisation consists formally in choosing $V(x_i) = \infty \times \mathbf{1}\{\|x_i\| > 1\}$ (with $\infty \times 0 = 0$), but this is less usual in the mathematics literature. A frequent and convenient choice is $V(x_i) \propto \|x_i\|^2$, and it emerges naturally in the theory of random matrices as we saw with previous models. We indicate [**?**, **?**, **?**] as an introduction to the mathematical aspects of such topics.

The deterministic configurations minimising this energy are called *ground states*, or *optimal configurations*, and are highly ordered. To add some randomness among low energy configurations and reflect disordered states of matter such as gases and liquids, one balances the energy by an entropy term, favoring more disordered random states, parametrized by some temperature $T > 0$, or the inverse temperature $\beta = 1/T$. The minimisers of this combined quantity called *free energy* are random point configurations called Gibbs measure at inverse temperature $\beta > 0$, see [**?**, **?**] for details. The Gibbs measure P_n^H with n particles and energy H is defined by the density

$$\rho_H^n(x_1, \dots, x_n) \propto \exp(-\beta H(x_1, \dots, x_n)). \tag{4.1}$$

The particles of P_n^H tend to approach global minimisers of the energy in the low temperature regime $\beta \to \infty$, and at the opposite converge towards independent "totally disorded" processes with density $\propto \prod_i e^{-\sum_i V(z_i)}$ when $\beta \to 0$ in the "high temperature regime".

A popular example is s-Riesz gases, $s \in \mathbb{R}$, where $\varphi(x-y) = \|x-y\|^{-s}$. When $s > d$, φ is integrable at ∞ and the model is said to be *short range*.

A particular case is obtained with *Coulomb gases*, also known under the terminology *jellium*, or *One Component Plasmas* (OCPs). Call φ_d the Coulomb potential in dimension d, i.e.

$$\varphi_d(x) = \begin{cases} -\ln(\|x\|) \text{ if } d = 2 \\ \frac{1}{\|x\|^{d-2}} \text{ if } d \neq 2, \end{cases}$$

easily seen to satisfy in the distributional sense with some κ_d

$$\Delta\varphi_d(x) = -\kappa_d \delta_0 \tag{4.2}$$

and $\kappa_d > 0$ in dimension $d \geqslant 2$. The Coulomb gas in dimension 1 also exists with $\varphi_1(x) = |x|$ (clearly the Laplacian vanishes on $\mathbb{R}\backslash\{0\}$).

Exercise 4. Prove that $\Delta\varphi_d(x) = -\kappa_d\delta_0$.

Many recent studies instead focus on the so-called log-gas, or β-ensemble (more in the context of random matrices), with potential that we denote $\varphi_{\log}(x) = -\ln(\|x\|)$, in particular for its importance in random matrices (but this is not the Coulomb potential).

Call $\mathsf{P}_n^{d,\beta} \in \mathscr{N}(\mathbb{R}^d)$ the simple point process with exactly n points and inverse temperature $\beta \geqslant 0$ whose density is

$$\rho_{d,\beta}^n(x_1, \ldots, x_n) \propto \exp(-\beta \sum_{i<j} \varphi_d(x_i - x_j)) \exp(-\beta \sum_i \|x_i\|^2 \times (d/4)) \tag{4.3}$$

and $\mathsf{P}^{\log,\beta}$ with the logarithmic potential in dimension 1. Note that $\mathsf{P}^{\log,\beta} = \mathsf{P}^{2,\beta}$. It indeed corresponds to Boltzmann density (4.1) with potential φ_d (the reason for the $(4/d)$-renormalisation simplifies the connection with random matrices).

Remark 1. *We study the potential φ_d here because of its relations with random matrices, but it is a "complicated Gibbs measures". If φ_d is replaced with an integrable potential, i.e. in the short-range case, a probabilistic study is much easier.*

4.2 Coulomb systems and random matrices

A striking fact of random matrix theory is that many ensembles of eigenvalues can be represented as a Coulomb gas.

Theorem 6. *We have*

$$\mathsf{P}_n^{GOE} \overset{(d)}{=} \mathsf{P}_n^{\log,1}$$

$$\mathsf{P}_n^{GUE} \overset{(d)}{=} \mathsf{P}_n^{\log,2}$$

$$\mathsf{P}_n^{\text{Gin}} \overset{(d)}{=} \mathsf{P}_n^{2,2} \overset{(d)}{=} \mathsf{P}^{\log,2}.$$

The proof of this result is pretty involved through a matrix decomposition and a change of variables and we only present here the GOE and GUE cases below.

We also study the possible existence of a limit point process $\mathsf{P}^{d,\beta}$ such that for some rescaling $a_n > 0$,

$$a_n \mathsf{P}_n^{d,\beta} \xrightarrow[n\to\infty]{w} \mathsf{P}^{d,\beta}.$$

Several (difficult) questions arise, that we will not study here:

- Such a limit exists? Is it stationary?
- Such a limit is unique? The answer is in general yes for $d = 1$ and it is believed to be no for $d \geqslant 2$, and there might be limit processes P that are stationary and others that are not stationary. An exception is the Ginibre case, i.e. $d = 2, \beta = 2$, where there is a unique limit $\mathsf{P}^{\mathrm{Gin}} = \lim_n \mathsf{P}_n^{\mathrm{Gin}}$.
- Does the set of limit points change if we change the confining potential $V(x) = \|x\|^2$?
- It is believed in dimension 2 from physical experiments that the structure of the set of limit points (unique or not unique) changes around $\beta \approx 140$, but proving is not currently possible as one cannot even prove that there exists β with several limit points.

There are other examples not covered in this course. For any $\beta > 0$, the β-ensemble $\mathsf{P}_n^{1,\beta}$ has been shown by [?] to constitute the eigenvalues of an explicit tridiagonal matrix model $\mathsf{M}^{\beta,(n)}$. The case $\beta = 4$ involves matrices of quaternions and is called the Gaussian Symplectic Ensemble (GSE), but we will not explicit further cases $\beta \notin \{1, 2\}$. Furthermore, Valko and Virag [?] recently shown the following: there exists a stationary point process $\mathsf{P}^{1,\beta}$ on $\mathbb{R}$, called Sine_β process such that $\sqrt{n}\mathsf{P}_n^{\log,\beta} \xrightarrow[n\to\infty]{w} \mathsf{P}^{1,\beta}$.

4.3 Change of variables

We refer to the monographs [?, ?, ?] for other models.

Proof that the eigenvalues of M_n^{GOE} *have density* (4.3) *for* $\beta = 1$. Denote by $\mathsf{M}_n = \mathsf{M}_n^{GOE}$ the GOE random matrix. Since M is a symmetric matrix, there is a.s. a random matrix Q_n in the orthogonal group $\mathscr{O}_n$ and a random matrix $\mathsf{D}_n = \mathrm{diag}(\Lambda_1, \dots, \Lambda_n)$ in the space $\mathscr{D}_n$ of diagonal matrices such that $\mathsf{M} = \mathsf{Q}_n\mathsf{D}_n\mathsf{Q}_n^T$. $\mathscr{D}_n$ is assimilated to $\mathbb{R}^n$ with the Euclidean metric and endowed with $\mathscr{L}^n$.

The fact that the eigenvalues of M are a.s. distinct is proved at Theorem 3. Hence M is .a.s in $\mathcal{S}_n^*$ the set of real symmetric matrices with distinct eigenvalues, and D_n is a.s. in $\mathcal{D}_n^*$ the class of diagonal matrices with distinct entries.

Our interest is the exact density of $\mathsf{D}_n = (\Lambda_1, \dots, \Lambda_n)$, i.e. to identify the law ν on $\mathbb{R}$' such that for a test function ψ on $\mathbb{R}^n$

$$\mathbf{E}\psi(\Lambda_1, \dots, \Lambda_n) = c_n \int_{\mathbb{R}^n} \psi(\lambda_1, \dots, \lambda_n) d\nu(\lambda_1, \dots, \lambda_n).$$

Recall the density of M_n in $M \in \mathscr{S}_n$: for φ a test function,

$$\mathbf{E}\varphi(\mathsf{M}_n) = c_n' \int_{\mathscr{S}_n^*} \varphi(M) \exp(-\frac{1}{4}\mathrm{Tr}(MM^T))dM. \tag{4.4}$$

We drop the "1/4" for notational simplification in the rest of the proof (it amounts to a global rescaling).

There is no unicity for the law of Q_n.

Lemma 3. *There is a Haar measure on* $\mathscr{O}_n(\mathbb{R})$*, i.e. a probability measure* σ *such that for* $\mathsf{Q} \sim \sigma$*, for all* $Q \in \mathscr{O}_n$*,* $Q\mathsf{Q} \overset{(d)}{=} \mathsf{Q}Q \overset{(d)}{=} \mathsf{Q}$.

Proof. It is a consequence of Haar theorem on the compact topological group $\mathscr{O}_n$. We can also build it explicitly:

- Let V_1 random uniform in $\mathbb{S}^{n-1}$ (in particular V_1's law is invariant under multiplication by $Q \in \mathscr{O}_n$).
- Let V_2 uniform in $V_1^\perp \cap \mathbb{S}^{n-1}$
- Let V_3 uniform in $(\mathbb{R}V_1 + \mathbb{R}V_n)^\perp \cap \mathbb{S}^{n-1}$

- ...
- Let V_n a coin flip to choose between the two elements of $(\oplus_{i=1}^{n-1}\mathbb{R}V_i) \cap \mathbb{S}^{n-1}$.

Then the matrix Q with columns $(V_1, \ldots, V_n)$ is invariant under rotations. □

Lemma 4. *Let Q_n' with law σ independent of M_n. Then*

$$\mathsf{M}_n \overset{(d)}{=} \mathsf{Q}_n' D_n (\mathsf{Q}_n')^T$$

Proof. Let $\mathsf{Q} \sim \sigma$ independent. Let $\mathsf{Q}_n' = \mathsf{Q}\mathsf{Q}_n$. Let μ the joint law of $(\mathsf{Q}_n, \mathsf{M}_n)$. We characterise the joint law $(\mathsf{Q}_n', \mathsf{M}_n)$ with bounded continuous test functions φ, ψ

$$\begin{aligned}
\mathbf{E}\left[\varphi(\mathsf{Q}_n')\psi(\mathsf{M}_n)\right] &= \mathbf{E}\varphi(\mathsf{Q}\mathsf{Q}_n)\psi(\mathsf{M}_n) \\
&= \int\int \varphi(QQ')\sigma(dQ)\varphi(M)d\mu(Q', M) \\
&= \int \left(\underbrace{\int \varphi(QQ')\sigma(dQ)}_{=\int \varphi(Q)\sigma(dQ)} \right) \psi(M)d\mu(Q', M) \\
&= \int \varphi(Q)\sigma(dQ) \int \psi(M)\mu(dQ', dM)
\end{aligned}$$

which proves at the same time that $\mathsf{Q}_n' \sim \sigma$ and is independent of $(\mathsf{Q}_n, \mathsf{M}_n)$.

Remember that for fixed $Q \in \mathscr{O}_n, Q\mathsf{M}_n Q^T \overset{(d)}{=} \mathsf{M}_n$ (this is a property of the GOE). Hence

$$\mathbf{E}\left[\varphi(\mathsf{Q}_n' \mathsf{M}_n (\mathsf{Q}_n')^T)\right] = \mathbf{E}\mathbf{E}\left[\varphi(\mathsf{M}_n)|\mathsf{Q}_n'\right] = \mathbf{E}[\varphi(\mathsf{M}_n)]$$

therefore

$$\mathsf{M}_n \overset{(d)}{=} \mathsf{Q}\mathsf{Q}_n\mathsf{D}_n\mathsf{Q}_n^T\mathsf{Q}^T = \mathsf{Q}_n'\mathsf{D}_n(\mathsf{Q}_n')^T$$

which concludes the proof. □

Hence without loss of generality (up to changing Q_n) we write $\mathsf{M}_n = \mathsf{Q}_n\mathsf{D}_n\mathsf{Q}_n^T$ with the law of $(\mathsf{Q}_n, \mathsf{D}_n)$ is $\sigma \times \nu$ for some measure ν on $\mathcal{D}_n^*$ that we seek to explicit.

The idea of the proof is to compute the Jacobian of the mapping $(Q, D) \mapsto M = QDQ^T$ for $(Q, D) \in \mathscr{O}_n \times \mathscr{D}_n^*$. It is tricky to directly perform a change of variables on the non-Euclidean manifold $\mathscr{O}_n$, we therefore locally linearise it first with the space of skew-symmetric matrices

$$\mathscr{A}_n(\mathbb{R}) = \{A \in \mathscr{M}_n(\mathbb{R}) : A^T = -A\}$$

, assimilated to $\mathbb{R}^{n(n-1)/2}$ and endowed with $\mathscr{L}^{n(n-1)/2}$.

Lemma 5. *The exponential function induces a bijective mapping from some neighbourhood U of 0 in $\mathscr{A}_n$ to $\exp(U)$ in $\mathscr{O}_n$.*

Proof. $\exp(0 + H) = I + H + o(H)$, clearly implies that the derivative of exp at 0 is bijective. The inverse function theorem gives the conclusion. □

We introduce the mapping $\Gamma : \mathscr{A}_n \times \mathscr{D}_n^* \to \mathscr{S}_n^*$

$$\Gamma(A, D) := \exp(A)D\exp(A)^T.$$

Let some diagonal matrix D_0. We wish to compute the density of ν at the point D_0. To that end, let us compute the absolute Jacobian determinant $J_\Gamma(\cdot, \cdot)$ in $(0, D_0)$.

Lemma 6. *For $D_0 = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathscr{D}_n^*$, the Jacobian matrix of Γ has absolute determinant in the VanderMonde form*

$$J_\Gamma(0, D_0) = c_n \prod_{i<j} |\lambda_i - \lambda_j| \text{ for some } c_n > 0.$$

This lemma is proved later. In particular, the determinant does not vanish. By the inverse function theorem, Γ is a $\mathcal{C}^1$-diffeomorphism on a neighbourhood $V \times U_{D_0}$ of $(0, D_0)$ in $\mathscr{A}_n \times \mathscr{D}_n^*$, and for $M \in \Gamma(V \times U_{D_0})$, there is a unique $(A_M, D_M) \in V \times U_{D_0}$ satisfying $M = \Gamma(A_M, D_M)$ (there is no such unicity on $\mathscr{S}_n^*$). By independence, the law of $(\exp^{-1}(\mathsf{Q}_n), \mathsf{D}_n)$ is of the form $\tilde{\sigma}(dA)\nu(dD)$ on this neighbourhood.

For ψ supported on U_{D_0} and another "mute" test function ξ supported by V, and $\varphi(M) := \psi(D_M)\xi(Q_M)$,

$$\begin{aligned}\mathbf{E}\psi(\mathsf{D}_n)\xi(\mathsf{Q}_n) = \mathbf{E}\varphi(\mathsf{M}_n) &= \int_W \psi(D_M)\xi(Q_M)e^{-\text{Tr}\xi(\mathsf{Q}_n)(MM^T)}dM \\ &= \int_{V \times U_{D_0}} \psi(D)\xi(\exp(A))e^{-\text{Tr}(DD^T)}J_\Gamma(A, D)dAdD\end{aligned}$$

On the other hand

$$\mathbf{E}\varphi(\mathsf{M}_n) = \int \varphi(QDQ^T)\sigma(dQ)\nu(dD) = \int \varphi(\exp(A)D\exp(-A))\xi(\exp(A))\tilde{\sigma}(dA)\nu(dD).$$

We can therefore identify on this neighbourhood

$$e^{-\text{Tr}(DD^T)/4}J_\Gamma(A, D) \propto \tilde{\sigma}(dA)d\nu(D).$$

It yields that ν has a density $f(D_0) \propto \prod_{i<j} |\lambda_i - \lambda_j|$, and this Jacobian form is valid for any $D \in U_{D_0}$, which concludes the proof because $\exp(-\text{Tr}(DD^T)) = \exp(-\sum_i \lambda_i^2)$.

Let us finally prove Lemma 6. We see Γ as a function from $\mathbb{R}^{n(n-1)/2} \times \mathbb{R}^n$ to $\mathbb{R}^{n(n+1)/2}$, and denote its components $\Gamma_{i,j}, i \leqslant j$. Let $\delta = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathscr{D}_n$, $H = (H_{i,j})_{1 \leqslant i,j \leqslant n} \in U$.

$$\begin{aligned}\Gamma(H, D_0 + \delta) =& (I + H + o(H))(D_0 + \delta)(I - H + o(H)) \\ =& \Gamma(0, D_0) + \delta + HD_0 - D_0H + o((H, \delta)).\end{aligned}$$

Note that $HD_0 - D_0H = (H_{i,j}(\lambda_j - \lambda_i))_{1 \leqslant i,j \leqslant n}$ vanishes on the diagonal, and at the opposite δ is supported by the diagonal. For $i < j$, we can read the partial derivatives on the lower diagonal

$$\begin{aligned}\frac{\partial \Gamma_{i,j}}{\partial H_{k,l}}(0, D_0) =& \lambda_i - \lambda_j \text{ iff } (i,j) = (k,l), \text{ for } k < l \\ \frac{\partial \Gamma_{i,j}}{\partial \lambda_k}(0, D_0) =& 0, 1 \leqslant k \leqslant n,\end{aligned}$$

and for $1 \leqslant i \leqslant n$

$$\begin{aligned}\frac{\partial \Gamma_{i,i}}{\partial \lambda_k}(0, D_0) =& \delta_{k,i} \\ \frac{\partial \Gamma_{i,i}}{\partial H_{k,l}}(0, D_0) =& 0, k < l.\end{aligned}$$

Seeing the Jacobian matrix $\nabla\Gamma$ as blocks of dimension $n(n-1)/2$ or n, the $n \times n$ block gives determinant 1, the $n(n-1)/2$ block is diagonal and gives $\prod_{i<j}(\lambda_i - \lambda_j)$, and the other blocks vanish; this gives the desired expression.

□

4.3.1 GUE Case

End of 4th course

The proof is essentially the same with $\mathsf{M}_n = \mathsf{M}_n^{(2)}$. The main difference is that the decomposition is

$$\mathsf{M}_n = \mathsf{U}\mathsf{D}_n\mathsf{U}_n^* \in \mathscr{H}_n^*$$

where $\mathsf{D}_n \in \mathscr{D}_n^*, \mathsf{U}_n \in \mathscr{U}_n$ and $\mathscr{H}_n^*$ is the class of Hermitian matrices with n simple eigenvalues. We still use the exponential mapping with the space of skew-hermitian matrices

$$\mathscr{A}_n(\mathbb{C}) := \{A \in \mathscr{M}_n(\mathbb{C}) : \bar{A}^T = -A\},$$

and we have the decomposition

$$\mathsf{M}_n = \exp(A) D_n \exp(\bar{A}^T)$$

recalling $\exp(-A) = \exp(\bar{A}^T) = \overline{\exp(A)^T}$. Such matrices A decompose in

$$A = A^{\mathbb{R}} + iA^{\mathbb{C}}$$

with $A^{\mathbb{R}} \in \mathscr{A}_n(\mathbb{R}), A^{\mathbb{C}} \in \mathscr{S}_n(\mathbb{R})$. The main difference occurs in the Jacobian determinant for the mapping

$$\Gamma^{\mathbb{C}}(A, D) = \exp(A) D \exp(-A).$$

We have as before

$$\Gamma^{\mathbb{C}}(H, D_0 + \delta) = \delta + AD_0 - D_0 A + o(\delta) + o(H)$$

and $\Gamma^{\mathbb{C}}$ has more components than Γ :

- n real diagonal $\Gamma_{i,i}$
- $n(n-1)/2$ real anti-diagonal $\Gamma_{i,j,\mathbb{R}}, i < j$
- $n(n+1)/2$ real diagonal $\Gamma_{i,j,\mathbb{C}}, i < j$

And the antidiagonal expression is for $i < j$

$$\Gamma_{i,j} = \Gamma_{i,j,\mathbb{R}} + i\Gamma_{i,j\mathbb{C}} = H_{i,j}(\lambda_i - \lambda_j) + o(\delta) + o(H) = H_{i,j}^{\mathbb{R}}(\lambda_i - \lambda_j) + iH_{i,j}^{\mathbb{C}}(\lambda_i - \lambda_j)$$

which yields more non-zero diagonal Jacobian components

$$\begin{aligned}\frac{\partial \Gamma_{i,j,\mathbb{R}}}{\partial H_{i,j}^{\mathbb{R}}} =& \lambda_i - \lambda_j \\ \frac{\partial \Gamma_{i,j,\mathbb{C}}}{\partial H_{i,j}^{\mathbb{C}}} =& \lambda_i - \lambda_j\end{aligned}$$

it ultimately gives products of $(\lambda_i - \lambda_j)^2$ instead of $(\lambda_i - \lambda_j)$ for the GOE.

4.3.2 Ginibre case:

Based on Schur decomposition $M = QTQ^{-1}$. See the proof of Dyson in [?], or [?].

Chapter 5

Point processes theory

For $n \to \infty$, in many cases, the P_n seem to converge to an infinite object P. It raises several questions:

- Weak topology is supposed to apply to finite measures. What topology and σ-algebra?
- How to describe its law?
- How to prove a convergence $\mathsf{P}_n \to \mathsf{P}$?

Let Λ_n be a cube with volume n

$$\Lambda_n = [-n^{1/d}/2, n^{1/d}/2]^d.$$

5.1 Vague topology

As before a point configuration $P \in \mathscr{N}$ can be seen as a measure $\tilde{P} := \sum_{x \in P} \delta_x$, i.e.

$$\tilde{P}(B) = \# P \cap B, B \subset \mathbb{R}^d.$$

We often use the abuse of notation

$$P(A) = \tilde{P}(A), P(f) = \int f d\tilde{P} = \sum_{x \in \mathsf{P}} f(x).$$

A locally finite set P is necessarily countable:

$$P = \cup_{n=0}^{\infty} \underbrace{(P \cap \Lambda_n)}_{\text{finite}}$$

hence we can label its points $P = \{x_1, x_2, \dots\}$. We sometimes write without specification "let $P = \{x_i; i \geqslant 1\}$" a point process.

Let us endow $\mathscr{N}$ with a classical topology on the space of measures. Let $P, P_n \in \mathscr{N}, n \geqslant 1$. Say that $P_n \to P$ in the vague topology if for any $f \in \mathsf{C}_c^0(\mathbb{R}^d)$ (continuous function with compact support),

$$P_n(f) \to P(f).$$

Compact support ensures that it is well defined. It is a *local convergence*, it means that $P_n \to P$ in a reasonable sense on each compact.

We consider convergence in distributions of random point processes weakly in the vague topology, denoted

$$\mathsf{P}_n \xrightarrow[n \to \infty]{w} \mathsf{P}$$

it means by definition $\mathbf{E}\Phi(\mathsf{P}_n) \to \mathbf{E}\Phi(\mathsf{P})$ for Φ bounded continuous.

Theorem 7 (Th. 14.16-(ii) of Kallenberg [**?**]). $\mathsf{P}_n \xrightarrow[n\to\infty]{w} \mathsf{P}$ *if and only if*

$$\mathsf{P}_n(f) \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}(f), f \in \mathsf{C}^0_c(\mathbb{R}^d).$$

if and only if for all $A \subset \mathbb{R}^d$ *bounded such that* $\mathsf{P}(\partial A) = 0$ *a.s.,*

$$\mathsf{P}_n(A) \xrightarrow{Law} \mathsf{P}(A).$$

Sets A such that $P(\partial A) = 0$ for some configuration $P \in \mathscr{N}$ are called *continuity sets* for P because $Q \mapsto Q(A)$ is continuous in P.

The general idea of this result is to approximate any continuous bounded functional Φ by a function over finitely many "coordinates" formed by linear statistics

$$\Phi(\mathsf{P}) \approx F(\mathsf{P}(f_1), \dots, \mathsf{P}(f_m))$$

for F bounded continuous function on $\mathbb{R}^n$. The joint convergence $(\mathsf{P}(f_1), \dots, \mathsf{P}(f_m))$ based on individual convergences $\mathsf{P}(f)$ is based on Laplace transforms of linear combinations $\sum_i t_i \mathsf{P}(f_i) = \mathsf{P}(f)$ with $f = \sum_i t_i f_i$.

Exercise 5. $\mathsf{P}_n := \{1 + 1/n\}$ converges weakly to $\mathsf{P} := \{1\}$, but $\mathsf{P}_n([-1, 1]) = 0$ does not converge to $\mathsf{P}([-1, 1]) = 1$.

We can then apply the following general theorem:

Theorem 8 (Continuous mapping theorem). *Let* $\mathsf{P}_n, n \geqslant 1$ *point processes converging weakly to some point process* P*. Let* $\Phi : \mathscr{N} \to \mathbb{R}^{(m)}$ *a mapping continuous a.s. in* P*. Then*

$$\Phi(\mathsf{P}_n) \xrightarrow{Law} \Phi(\mathsf{P}).$$

Equality in law and tightness are dealt with the following result:

Theorem 9 ([**?**, **?**]). *1. Any point processes* P, P' *have the same law iff* $\mathsf{P}(A) \overset{(d)}{=} \mathsf{P}'(A)$ *for every compact* $A \subset \mathbb{R}^d$.

2. A sequence of random measures $\mathsf{P}_n, n \geqslant 1$ *is tight if for all* A *compact,*

$$\sup_n \mathbf{E}\mathsf{P}_n(A) < \infty.$$

Since $\mathscr{N}$ *is metrisable (closed subspace of a metrisable Polish separated space), tight sequences have converging subsequences.*

[Admitted] Some insights:

1. The first point means that the σ-algebra is generated by functionals $\Phi_{1_A} : P \to P(A)$.
2. The second point can be decomposed in two facts: P_n is tight if for all bounded sets B, the restriction $\mathsf{P}_{n,B} = \mathsf{P}_n \cap B$ is a compact sequence of point processes (i.e. of laws on $\mathscr{N}(B)$), this is a diagonal extraction argument. Secondly, on $\mathscr{N}(B)$, $\mathsf{P}_n \xrightarrow[n\to\infty]{w} \mathsf{P}$ if $\mathsf{P}_n(A) \xrightarrow[n\to\infty]{w} \mathsf{P}(A)$ for $A \subset B$ closed.

To better understand these results, we can draw a parellel with random variables $X_n, n \geqslant 1, X, Y$ on the real line $\mathbb{R}$:

- Two real random variables X, Y have the same law if and only if $\mathbf{E}f(X) = \mathbf{E}f(Y)$ for f bounded continuous, if and only if $\mathbf{P}(X \in A) = \mathbf{P}(Y \in A)$ for all A with $\mathbf{P}(X \in \partial A) = 0$

- $X_n \xrightarrow[n\to\infty]{w} X$ if and only if

$$\mathbf{E}f(X_n) \to \mathbf{E}f(X)$$

for f bounded continuous, if and only if $\mathbf{P}(X_n \in A) \to \mathbf{P}(X \in A)$ for A with $\mathbf{P}(X \in \partial A) = 0$.

- The sequence $\{X_n; n \geqslant 1\}$ is tight if

$$\sup_n \mathbf{E}|X_n| < \infty$$

(but this is not necessary).

Real random variables can be seen as point processes with a.s. one point.

5.1.1 The homogeneous Poisson process

Proposition 7. *$\text{Bin}(n, \Lambda_n)$ converges to some point process P on $\mathbb{R}^d$ called homogeneous Poisson point process with intensity 1. It is characterised by*

$$\mathsf{P}(A) \stackrel{(d)}{=} \text{Poiss}(\mathscr{L}^d(A))$$

for bounded measurable $A \subset \mathbb{R}^d$.

Proof. Let A fixed bounded. For n sufficiently large (such that $A \subset \Lambda_n$), each point has a probability $n^{-1}\mathscr{L}^d(A)$ to fall in A, hence the number of points falling into A has law $\text{Binomial}(n, n^{-1}\mathscr{L}^d(A))$. In particular, the expectation is $\leqslant \mathscr{L}^d(A) < \infty$, hence the compactness result applies and P_n has at least one limit point. We have the classical exercise: convergence in law $\text{Binomial}(n, n^{-1}\mathscr{L}^d(A)) \to \text{Poiss}(\mathscr{L}^d(A))$, i.e. for all $k \in \mathbb{N}$

$$\begin{aligned}\mathbf{P}(\text{Binomial}(n, n^{-1}\mathscr{L}^d(A)) = k) =& \binom{k}{n}(n^{-1}\mathscr{L}^d(A))^k(1 - n^{-1}\mathscr{L}^d(A))^{n-k} \\ =& \dots \\ \to& e^{-\mathscr{L}^d(A)}\frac{(\mathscr{L}^d(A))^k}{k!} \\ =& \mathbf{P}(\text{Poiss}(\mathscr{L}^d(A)) = k).\end{aligned}$$

Therefore $\mathsf{P}(A) \stackrel{(d)}{=} \text{Poiss}(\mathscr{L}^d(A))$, which gives uniqueness of the limit. □

Henceforth, we will by default assume intensity 1.

Theorem 10 (Complete randomness)**.** *For P a Poisson process, for $A_1, \dots, A_m$ disjoint with $\mathscr{L}^d(\partial A_j) = 0$, the $\mathsf{P}(A_j), j = 1, \dots, m$ are independent Poisson variables.*

Proof. Let $\mathsf{P}_n = \{X_1, \dots, X_n\}$ the binomial process on Λ_n.

The basic point is actually pretty intuitive: for $A \subset \Lambda_n, 0 \leqslant k \leqslant n$, conditionally on the event that exactly k points fall into A, i.e. $\mathsf{P}_n(A) = k$, then $\mathsf{P}_n \backslash A$ is binomial with $n - k$ points: for $B \subset \Lambda_n \backslash A$,

$$(\mathsf{P}_n(B)|\mathsf{P}_n(A) = k) \sim \text{Binomial}(n - k, \mathscr{L}^d(B)/(n - k))$$

A formal proof can be derived with probabilities of the binomial law by summing over all k-tuple of indexes of the points falling into A.

Let N_j independent Poisson variables with resp. parameters $\mathscr{L}^d(A_j)$. Let $m = 2$. For $k_1, k_2 \in \mathbb{N}$, let P'_n a binomial point process on $\Lambda_n \backslash A$

$$P(\mathsf{P}_n(A_1) = k_1 | \mathsf{P}_n(A_2) = k_2) = \mathbf{P}(\mathsf{P}'_{n-k_2}(A_1)) \to \mathbf{P}(N_1 = k_1)$$
$$P(\mathsf{P}_n(A_1) = k_1, \mathsf{P}_n(A_2) = k_2) = \mathbf{P}(\mathsf{P}_n(A_1) = k_1 | \mathsf{P}_n(A_2) = k_2)\mathbf{P}(\mathsf{P}_n(A_2) = k_2) \to \mathbf{P}(N_1 = k_1)\mathbf{P}(N_2 = k_2)$$

For an arbitrary number m, integers $k_1, \dots, k_m \in \mathbb{N}$, we have the same facts that if one conditions on k_2 points falling in A_2, k_3 points falling in A_3, etc..., until k_m points in A_m, then the remaining points have law $\text{Bin}(n - (\sum_{j=0}^{m-1} k_j), \Lambda_n \backslash (A_1 \cup \dots \cup A_{m-1}))$:

$$\mathbf{P}(\mathsf{P}_n(A_1) = k_1 | \mathsf{P}_n(A_j) = k_j; 2 \leqslant j \leqslant m) \to \mathbf{P}(N_1 = k_1)$$

successive conditionings (or an induction) gives

$$\begin{aligned}
\mathbf{P}(\mathsf{P}_n(A_j) = k_j; 1 \leqslant j \leqslant m) =& \mathbf{P}(\mathsf{P}_n(A_1) = k_1 \mid \mathsf{P}_n(A_j) = k_j, j = 2, \dots, m)\mathbf{P}(\mathsf{P}_n(A_j) = k_j, j = 2, \dots, m) \\
\sim& \mathbf{P}(N_1 = k_1)\mathbf{P}(\mathsf{P}_n(A_j) = k_j, j = 2, \dots, m) \\
\sim& \mathbf{P}(N_1 = k_1)\mathbf{P}(N_2 = k_2)\mathbf{P}(\mathsf{P}_n(A_j) = k_j, j = 3, \dots, m) \\
\sim& \dots \\
&\to \mathbf{P}(N_1 = k_1) \dots \mathbf{P}(N_m = k_m).
\end{aligned}$$

Hence $(\mathsf{P}_n(A_1), \dots, \mathsf{P}_n(A_m))$ converge in law towards $(N_1, \dots, N_m)$ where the N_j are independent (and Poisson).

Does it mean that $(N_1, \dots, N_m) \stackrel{(d)}{=} (\mathsf{P}(A_1), \dots, \mathsf{P}(A_m))$?

We saw that $\mathsf{P}(\partial A_j) = 0$ a.s., hence the A_j are continuity sets, and the functions

$$\Phi_j : P \mapsto P(A_j)$$

are a.s. continuous in P. Hence the function

$$\tilde{\Phi}(P) = (\mathbf{1}\{\mathsf{P}(A_j) = k_j\})_{1 \leqslant j \leqslant m}$$

is bounded and is continuous in P. The continuous mapping theorem gives

$$\mathbf{E}\tilde{\Phi}(\mathsf{P}_n) \to \mathbf{E}\tilde{\Phi}(P)$$

by uniqueness of the limit, we have independence

$$\mathbf{P}(\mathsf{P}(A_j) = k_j; 1 \leqslant j \leqslant m) = \mathbf{E}\tilde{\Phi}(P) = \lim_n \mathbf{E}\tilde{\Phi}(\mathsf{P}_n) = \prod_j \mathbf{P}(N_j = k_j) = \prod_j \mathbf{P}(\mathsf{P}(A_j) = k_j).$$

□

Remark 2. *There is a much simpler characterisation of law equality and convergence for point processes by [?, 14.17] but it is not intuitive:* $\mathsf{P}_n \xrightarrow[n\to\infty]{w} \mathsf{P}$ *if and only if for A a continuity set of* P,

$$\mathbf{P}(\mathsf{P}_n(A) = 0) = \mathbf{P}(\mathsf{P}(A) = 0).$$

For P' *another point process,* $\mathsf{P} \stackrel{(d)}{=} \mathsf{P}'$ *if* $\mathbf{P}(\mathsf{P}(A) = 0) = \mathbf{P}(\mathsf{P}'(A) = 0)$ *for all continuity set A.*

5.2 Stationarity and intensity

Definition 7. *For a point process* P*, denote by* $\mu_{\mathsf{P}}(A) = \mathbf{E}\mathsf{P}(A)$*. It is a measure called* **intensity measure** *of* P*.*

Denote by τ_x the translation operator by $x \in \mathbb{R}^d$.

Definition 8 (Stationarity)**.** *A (random) point process* P *is invariant, or stationary, if for all* $x \in \mathbb{R}^d$*,* $\tau_x\mathsf{P} := \{x + y; y \in \mathsf{P}\} \overset{(d)}{=} \mathsf{P}$.

Proposition 8. *Let* P *a stationary point process. Then its intensity measure is* $\lambda\mathscr{L}^d$*, a multiple of Lebesgue measure for some* $\lambda \geqslant 0$*.* λ *is also called the intensity of* P*.*

Any measure invariant under translations is a multiple of Lebesgue measure by construction.

Proposition 9. *Let* $\lambda > 0$ *and* P *a homogeneous Poisson point process. Let* $\mathsf{P}_\lambda := \lambda^{-1/d}\mathsf{P} = \{\lambda^{-1/d}x; x \in \mathsf{P}\}$*. Then* P_λ *is a stationary point process with intensity* λ*, called Poisson process with intensity* λ*.*

Proof.

- **intensity** For $A \subset \mathbb{R}^d$,

$$\mathsf{P}_\lambda(A) = \mathsf{P}(\lambda^{1/d}A) \overset{(d)}{=} \mathrm{Poiss}(\mathscr{L}^d(\lambda^{1/d}A))$$

  hence the intensity measure in indeed $\lambda\mathscr{L}^d(A)$.

- **Stationary** (Proof for $\lambda = 1$)

$$\tau_x\mathsf{P}(A) = \mathsf{P}(\tau_{-x}A) \overset{(d)}{=} \mathrm{Poiss}(\mathscr{L}^d(\tau_{-x}A)) = \mathrm{Poiss}(\mathscr{L}^d(A)) \overset{(d)}{=} \mathsf{P}(A).$$

□

Remark 3. *For any topological measured space* (X, μ) *with* μ *locally finite, a point process* P *on* X *such that for* $A \subset \mathsf{X}$, $\mathsf{P}(A) \overset{(d)}{=} \mathrm{Poiss}(\mu(A))$ *is called a Poisson point process with intensity* μ *(the* σ*-algebra is also that induced by continuous functions with compact support).*

Proposition 10. *For a stationary point process* P*, the intensity measure is of the form* $\mu_{\mathsf{P}} = \lambda\mathscr{L}^d$ *for some* $\lambda \geqslant 0$*. The "intensity"* λ *can be computed on any bounded set* A *that is not Lebesgue-negligible:*

$$\lambda = \frac{\mathbf{E}\mathsf{P}(A)}{\mathscr{L}^d(A)}.$$

Proof. First of all, the intensity measure is invariant under shifts: since $\mathsf{P} \overset{(d)}{=} \tau_{-x}\mathsf{P}$,

$$\mu_{\mathsf{P}}(\tau_xA) = \mathbf{E}\mathsf{P}(\tau_xA) = \mathbf{E}\tau_{-x}\mathsf{P}(A) = \mathbf{E}\mathsf{P}(A) = \mu_{\mathsf{P}}(A)$$

hence μ_{P} is a multiple of Lebesgue measure.

We know $\mathbf{E}\mathsf{P}(A) = \lambda\mathscr{L}^d(A)$, hence the result follows immediately. □

Exercise 6.

1. Let U uniform in $[0,1]^d$, and

$$\mathsf{P} = \{k + U; k \in \mathbb{Z}^d\}.$$

   Show that P is stationary with intensity 1.

2. Let $X_k, k \in \mathbb{Z}^d$ i.i.d variables with law $\mathcal{U}_{[0,1]^d}$. Let $\mathsf{P} = \{k + X_k; k \in \mathbb{Z}\}$. Show that P has Lebesgue intensity but is not stationary.

3. Assume U and the X_k are independent. Show that $\{k + U + X_k\}$ is stationary with unit intensity.

End of 3d course

TOC

5.3 Factorial moment measures

Factorial moment measures give a more analytic way to decompose P's law in projections of orders $1, 2, ...$ and characterise it in the same way that the law of a reasonable real random variable is characterised by its moments of every order.

Say that P has finite moments of order m (or finite m-moments) if for A bounded, $\mathbf{E}[\mathsf{P}(A)^m] < \infty$.

Denote the diagonal of $(\mathbb{R}^d)^m$ by $\Delta_m = \{(x_1, \dots, x_m) : \exists i \neq j, x_i = x_j\}$

Definition 9. *Let $\mathsf{P} = \{x_i; i \geqslant 1\}$ with finite m-moments. Define $\mu_{\mathsf{P}}^{(m)}$ the m-th factorial moment measure of P on $(\mathbb{R}^d)^m$ by*

- $\mu_{\mathsf{P}}^{(m)}(\Delta_m) = 0$
- *for disjoint sets $A_1, \dots, A_m$*

$$\mu_{\mathsf{P}}^{(m)}(\underbrace{A_1 \times \cdots \times A_m}_{\subset \mathbb{R}^d \setminus \Delta_m}) = \mathbf{E}\mathsf{P}(A_1) \dots \mathsf{P}(A_m)$$

Also characterised on test functions $f \in \mathsf{C}_c^0(\mathbb{R}^d)$ by

$$\mu_{\mathsf{P}}^{(m)}(f) = \mathbf{E}\left[\sum_{i_1, \dots, i_m \text{ distinct}} f(x_{i_1}, \dots, x_{i_m})\right]$$

Proof. The formula with f is true with $f = \mathbf{1}\{A_1 \times \cdots \times A_m\}$.

For f supported by $(\mathbb{R}^d)^m \setminus \Delta_m$, it is a general fact from measure theory that f can be approximated by linear combinations of such products with $\mathbf{1}\{A_1^n \times \cdots \times A_m^n\}$ where the diameter of the A_j^n is bounded by $\frac{1}{n}$. Use MCT to approximate $f \geqslant 0$ and prove equality, then decompose $f = f_+ - f_-$.

For general f, it can be approximated on $\mathbb{R}^d \setminus \Delta_m$ by f_n supported by $\mathbb{R}^d \setminus \Delta_m$. $\square$

For instance for $m = 2$,

$$\mu_{\mathsf{P}}^{(2)}(f) = \mathbf{E} \sum_{x \neq y \in \mathsf{P}} f(x, y).$$

It is a measure because

- $= 0$ if $f \equiv 0$
- Additive in f
- σ-additivity: monotone convergence theorem on the right hand side.

The number of terms in the sum is determined by the number of point in $\mathsf{P} = \{x_i; i \geqslant 1\}$. When $\mu_{\mathsf{P}}^{(m)}$ has a density with respect to $(\mathscr{L}^d)^m$, it is denoted by $\rho_{\mathsf{P}}^{(m)}$.

It is furthermore **symmetric**: if one applies a permutation to the coordinates, the result does not change

$$\mu_{\mathsf{P}}^{(m)}(A_1 \times \cdots \times A_m) = \mu_{\mathsf{P}}^{(m)}(A_{\sigma(1)} \times \cdots \times A_{\sigma(m)}).$$

Theorem 11. *For the Poisson process P_λ, we have $\rho_{\mathsf{P}_\lambda}^{(m)} = \lambda^m$.*

Proof. By independence,

$$\mu_{\mathsf{P}_\lambda}^{(m)}(A_1 \times \cdots \times A_m) = \mathbf{E}\left[\prod_i \mathsf{P}_\lambda(A_i)\right] = \prod_i \mathbf{E}\left[\mathsf{P}_\lambda(A_i)\right] = \prod_i (\lambda \mathscr{L}^d(A_i)) = \lambda^m (\mathscr{L}^d)^m (A_1 \times \cdots \times A_m)$$

$\square$

- For $k = 1$, one retrieves the intensity $\mu_{\mathsf{P}}^{(1)}(A) = \mathbf{E}\mathsf{P}(A)$ for $A \subset \mathbb{R}^d$. If P is stationary, $\mu_{\mathsf{P}}^{(1)}$ is invariant under translations, hence $\rho_{\mathsf{P}}^{(1)} \equiv \lambda$, with the **intensity** $\lambda \geqslant 0$.

  The variance of a linear statistic $\mathsf{P}(f) = \sum_i f(x_i)$ is directly expressible in terms of $\mu_{\mathsf{P}}^{(1)}, \mu_{\mathsf{P}}^{(2)}$:

  Proposition 11. *Let* P *with finite second moments. Then for* $f \in \mathsf{C}_c^b(\mathbb{R}^d)$,

$$Var(\mathsf{P}(f)) = \mu_{\mathsf{P}}(f^2) + \mu_{\mathsf{P}}^{(2)}(f \otimes f) - \mu_{\mathsf{P}}(f)^2.$$

  For P_λ *a Poisson process with intensity* $\lambda \geqslant 0$,

$$Var(\mathsf{P}_\lambda(f)) = \lambda \int_{\mathbb{R}^d} f(x)^2 dx.$$

$$\begin{aligned}\mathrm{Var}(\mathsf{P}(f)) =& \mathbf{E}\left[\sum_{x,y \in \mathsf{P}} f(x)f(y)\right] - \left(\mathbf{E}\left[\sum_{x\in\mathsf{P}} f(x)\right]\right)^2 \\ =& \mathbf{E}\sum_{x\in\mathsf{P}} f(x)^2 + \mathbf{E}\sum_{x\neq y} f(x)f(y) - \mu_{\mathsf{P}}(f)^2 \\ =& \mu_{\mathsf{P}}(f^2) + \int f(x)f(y)d\mu_{\mathsf{P}}^{(2)}(x,y) - \mu_{\mathsf{P}}(f)^2.\end{aligned}$$

Exercise 7. Prove the inclusion exclusion formula:

$$\mathbf{P}(\mathsf{P}(B) = 0) = \sum_{m=0}^{\infty} \frac{(-1)^{(m)}}{m!} \mu_{\mathsf{P}}^{(m)}(B^m). \tag{5.1}$$

5.3.1 Law characterisation and convergence

See [?, Chap. 9] for a more complete treatment. Recall the result on non-negative real variables:

Proposition 12. *Let* X *a variable with some exponential moment, i.e.* $\exists t > 0 : \mathbf{E}\exp(tX) < \infty$.

If a variable Y *has the same moments as* $X, Y \stackrel{(d)}{=} X$.

If some variables X_n *satisfy for all* $k \in \mathbb{N}, \mathbf{E}X_n^k \to \mathbf{E}X^k$, *then* $X_n \xrightarrow{Law} X$.

Proof. Use characteristic function and Lebesgue theorem.

$$\mathbf{E}\exp(iuY) = \sum \frac{(iu)^k}{k!}\mathbf{E}Y^k = \mathbf{E}\exp(iuX).$$

$(X_n)_n$ is tight by Markov's inequality.

We will prove that any subsequential limit has the same moments as X under the assumption $X \geqslant 0, X_n \geqslant 0$. Take a subsequence. By tightness, there is a further subsequence, denoted X_{n_j}, such that

$$X_{n_j} \xrightarrow{d} Y$$

for some $Y \geqslant 0$.

Fix $k \geqslant 1$. Choose $r > k$. Since $m_r^{(n)} \to m_r < \infty$,

$$\sup_j \mathbf{E}[X_{n_j}^r] < \infty.$$

For $M > 0$, define

$$\varphi_M(x) := x^k \wedge M^k.$$

This is bounded and continuous, so from $X_{n_j} \Rightarrow Y$,

$$\mathbf{E}[\varphi_M(X_{n_j})] \to \mathbf{E}[\varphi_M(Y)].$$

Also,

$$0 \leqslant X_{n_j}^k - \varphi_M(X_{n_j}) \leqslant X_{n_j}^k \mathbf{1}_{\{X_{n_j} > M\}} \leqslant M^{k-r} X_{n_j}^r.$$

Therefore

$$0 \leqslant \mathbf{E}[X_{n_j}^k] - \mathbf{E}[\varphi_M(X_{n_j})] \leqslant M^{k-r} \sup_j \mathbf{E}[X_{n_j}^r].$$

Letting $j \to \infty$,

$$0 \leqslant m_k - \mathbf{E}[\varphi_M(Y)] \leqslant C_r M^{k-r},$$

where $C_r := \sup_j \mathbf{E}[X_{n_j}^r] < \infty$.

Now let $M \to \infty$. Since $\varphi_M(Y) \uparrow Y^k$, monotone convergence gives

$$\mathbf{E}[Y^k] = m_k = \mathbf{E}[X^k].$$

So Y and X have the same moments of every order. Hence $Y \overset{(d)}{=} X$.

□

Definition 10. *Say a point process* P *has exponential moments if for all* A *bounded,* $\exists t_A > 0$ *such that*

$$\mathbf{E}\exp(t\mathsf{P}(A)) < \infty.$$

Proposition 13. *The law of a point process* P *having some exponential moments is characterised by the* $\mu_{\mathsf{P}}^{(m)}(A), m \geqslant 1$ *for* A *a bounded* P*-continuity set.*

Example 2. *A Poisson point process has exponential moments because* $N \overset{(d)}{=} \text{Poiss}(\lambda\mathscr{L}^d(A))$ *has exponential moments:*

$$\mathbf{E}\exp(tN) = \sum_k \exp(tk) \frac{(\lambda\mathscr{L}^d(A))^k}{k!} e^{-\lambda\mathscr{L}^d(A)} = e^{-\lambda\mathscr{L}^d(A)} \exp(\exp(t)\lambda\mathscr{L}^d(A)).$$

This is actually the MGF of Poisson's law.

Proof. It is classical that the law of a real random variable X is characterised by its moments $\mathbf{E}X^{(m)}, m \geqslant 1$ if $\mathbf{E}\exp(t|X|) < \infty$ for some $t > 0$. Similarly, if some point process P satisfies $\mathbf{E}\exp(t_A\mathsf{P}(A)) < \infty$ for some $t_A > 0$ for all A bounded, in which case we say that P *has some exponential moments*, then the law of the $\mathsf{P}(A), A$ bounded, and hence the law of P ([**?**, Th. 9.2.XII]), is characterised by the moments $\mathbf{E}\mathsf{P}(A)^m, m \geqslant 1, A$ bounded. In turn, the $\mathbf{E}\mathsf{P}(A)^m, m \geqslant 1$ can be recovered from the $\mu_{\mathsf{P}}^{(m)}, m \geqslant 1$ with (**??**). □

Proposition 14. *If the factorial moment measures are known to satisfy* $\mu_{\mathsf{P}}^{(m)}(B^m) \leqslant c_B^{(m)}$ *for some* $c_B < \infty$*, it implies finite exponential moments on* B*:*

$$\mathbf{E}\exp(t\mathsf{P}(B)) < \infty, \tag{5.2}$$

Proof. Let $X = \mathsf{P}(B) \in \mathbb{N}$. Denote by

$$X^{(m)} := X(X-1)\cdots(X-m+1),$$

hence by assumption

$$\forall m \geqslant 0, \qquad \mu_{\mathsf{P}}^{(m)}(B^m) = \mathbf{E}[X^{(m)}] \leqslant c^m$$

(with $(X)_0 = 1$), then for all $u \geqslant 0$,

$$(1+u)^X = \sum_{m=0}^{X} \binom{X}{m} u^m = \sum_{m=0}^{\infty} \frac{X^{(m)}}{m!} u^m.$$

Since all terms are non-negative, Tonelli gives

$$\mathbf{E}[(1+u)^X] = \sum_{m=0}^{\infty} \frac{u^m}{m!} \mathbf{E}[X^{(m)}] \leqslant \sum_{m=0}^{\infty} \frac{(cu)^m}{m!} = e^{cu}.$$

Now, for $t \geqslant 0$, let $u = e^t - 1$. Then

$$e^{tX} = (e^t)^X = (1+u)^X,$$

hence

$$\mathbf{E}[e^{tX}] = \mathbf{E}[(1+u)^X] \leqslant e^{c(e^t-1)} < \infty.$$

$$\mathbf{E}[(X)_m] \leqslant c^m \ \forall m \quad \Longrightarrow \quad \mathbf{E}[e^{tX}] < \infty \ \forall t \geqslant 0.$$

□

hence under such an assumption for all B, the $\mu_{\mathsf{P}}^{(m)}$ uniquely define a distribution. This will in particular allow to define properly the class of DPPs in Section 6 through their factorial moment measures.

Similarly, the convergence between random variables $X_n \to X$ for X with some exponential moment is implied by the convergence of the m-th moment $\mathbf{E}X_n^{(m)} \to \mathbf{E}X^{(m)}$ for each $m \geqslant 1$. Recall that the convergence between point processes $\mathsf{P}_n \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}$ is implied by $\mathsf{P}_n(A) \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}(A)$ for each bounded A ([**?**, 11.1.VII]). Hence if for all A bounded, for all $m \geqslant 1, \mathbf{E}\mathsf{P}_n(A)^m \to \mathbf{E}\mathsf{P}(A)^m$, we have indeed the weak convergence $\mathsf{P}_n \to \mathsf{P}$. Finally, since $\mathbf{E}\mathsf{P}(A)^m$ is a linear combination of the $\mu_{\mathsf{P}}^{(k)}(A), 1 \leqslant k \leqslant m$, we have:

Proposition 15. *If for some random measures P_n, P with P having some exponential moments, we have $\mu_{\mathsf{P}_n}^{(m)}(A^m) \to \mu_{\mathsf{P}}^{(m)}(A^m)$, for each P-continuity bounded A, then $\mathsf{P}_n \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}$.*

End of 5th course

Chapter 6

Determinantal point processes

In order to prove results for the GUE and Ginibre process, we must derive some basic definitions and concepts related to the theory of determinantal processes. Even though instances of determinantal processes occur throughout probability theory and statistical physics, the general concept in the continuous space was introduced by Macchi [**?**], see references in [**?**].

6.1 DPPs and Ginibre

Definition 11. *Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ a Hermitian function, i.e.*

$$K(x, y) = \overline{K(y, x)}, x, y \in \mathbb{R}^d.$$

A point process P on $\mathbb{R}^d$ is a DPP with kernel K if it admits factorial moment densities of the form, for $m \geqslant 1$,

$$\rho_K^{(m)}(x_1, \dots, x_m) := \det((K(x_i, x_j))_{1 \leqslant i,j \leqslant m}), x_1, \dots, x_m \in \mathbb{R}^d. \tag{6.1}$$

In this case P is denoted by P_K.

The first example, sometimes considered degenerate, is the homogeneous Poisson process with intensity $\lambda > 0$, for which $K(x, y) = \lambda \mathbf{1}\{x = y\}$. More generally, $\rho_K^{(1)}(x) = K(x, x) \in \mathbb{R}_+$ on $\mathbb{R}^d$. Denote by P_K the DPP with kernel K, if it exists.

We make the running assumption in this course that $(x, y) \mapsto K(x, y)$ is locally bounded: for $A \subset \mathbb{R}^d$ bounded,

$$M_A := \sup_{x,y \in A} |K(x, y)| < \infty.$$

It implies that P has finite moments of all orders, in particular

$$\mu_{\mathsf{P}}(A) = \mathbf{E}\mathsf{P}(A) = \int_A K(x, x)dx < \infty. \tag{L^1_{loc}}$$

We actually have much more:

Lemma 7. *for B bounded,*

$$\mu_{\mathsf{P}}^{(m)}(B^m) \leqslant M_B^{(m)} \mathscr{L}^d(B)^m$$

whence P has exponential moments.

Proof. Remark that by (6.1), each principal minor of K has a non-negative determinant, hence each submatrix $(K(x_i, x_j))$ is necessarily semi-definite positive. Therefore, Hadamard's inequality

$$\det(M) \leqslant \prod_i \|(\text{column i})\|_2$$

yields for a SDP matrix $S = M^2$ (for some Hermitian M)

$$|\det(S)| = |\det(M)|^2 \leqslant \prod_i \underbrace{\|(M)_i\|^2}_{\sum_k M_{i,k}^2} = \prod_i \underbrace{S_{i,i}}_{\sum_k M_{i,k}M_{k,i}}$$

hence for compact measurable $B \subset \mathbb{R}^d$

$$|\mu_{\mathsf{P}}^{(m)}(B^n)| \leqslant \int_{B^n} \prod_i K(x_i, x_i)dx_1 \ldots dx_m = \left(\int_B K(x,x)dx\right)^{(m)} \leqslant \mathscr{L}^d(B)^{(m)} M_B^{(m)}.$$

□

From Propositions 14, 13 we have the following uniqueness result: There is at most one DPP with a given kernel K that is locally bounded. The convergence to a DPP is also simplified with Proposition 15: a sequence of point processes $\mathsf{P}_n, n \geqslant 1$ converges towards DPP P_K if for all $m \geqslant 1, A$ bounded continuity set

$$\rho_{\mathsf{P}_n}^{(m)}(A^m) \to \rho_K^{(m)}(A^m).$$

A particularly important class is that of *projection canonical kernels*, defined to be of the form

$$K(x,y) = \sum_{k=1}^n \varphi_k(x)\bar{\varphi}_k(y) \tag{6.2}$$

where the φ_k form an orthonormal family over $\mathbb{R}^d$.

Let us illustrate this concept with the Ginibre process. Denote $\mathsf{P}_n = \mathsf{P}_n^{\text{Gin}}$ the n-th order Ginibre process.

Theorem 12. *The Ginibre P_n is a DPP with canonical projector kernel*

$$K_n(z,w) = \sum_{k=0}^{n-1} \varphi_k(z)\bar{\varphi}_k(w).$$

with $\varphi_k(z) := \frac{1}{\sqrt{\pi k!}} z^k e^{-|z|^2/2}$.

Let us first check that the φ_k form an orthonormal family. Let $\alpha_k := (\pi k!)^{-1/2}$:

$$\int_{\mathbb{C}} \varphi_k(z)\bar{\varphi}_j(z)dz = \alpha_k\bar{\alpha}_j \int z^k \bar{z}^j e^{-|z|^2} dz = \alpha_k\bar{\alpha}_j \int_0^\infty \int_0^{2\pi} \rho^{k+j} e^{i\theta(k-j)} e^{-\rho^2} \rho d\rho d\theta$$

in polar coordinates. We see in particular that it vanishes for $k \neq j$ due to the angular integral, and for $k = j$, it gives with the change of variables $u = \rho^2$

$$2\pi|\alpha_k|^2 \int_0^\infty \rho^{2k+1} d\rho e^{-\rho^2} = \pi|\alpha_k|^2\Gamma(k+1) = 1.$$

The proof is in 3 steps:

1. Let us first treat the $\rho^{(m)}_{\mathsf{P}_n}, m > n$.

   For P_n, we have

$$\mu^{(m)}_{\mathsf{P}_n}(f) = \mathbf{E}\left[\sum_{(x_1,\dots,x_m)\text{ distinct}} f(x_1,\dots,x_m)\right] = 0$$

   because there are no such m distinct points in P_n because P_n has a.s. n points. For a DPP with canonical kernel with n terms, the matrix $((K(x_i,x_j))_{1\leqslant i,j\leqslant m})$ has 0 determinant: indeed, each matrix $(\varphi_k(x_i)\bar{\varphi}_k(x_j))_{1\leqslant i,j\leqslant m}$ has rank 1, and a sum of n rank-1 matrices has rank at most n : Consider the column vector $V_k = (\varphi_k(x_i))_{1\leqslant i\leqslant n+1}$. Column j is $\sum_k \bar{\varphi}_k(x_j)V_k$ the linear combination of column vectors $(\bar{\varphi}_k(x_i)) = V_k$. Therefore it is rank n.

   We therefore proved $\rho^{(m)}_{\mathsf{P}_n} = \rho^{(m)}_{K_n} \equiv 0$ for $m > n$.

2. Let us now prove it for $m = n$. The starting point is the density of the Ginibre process as a Coulomb gas

$$\rho^n(z_1,\dots,z_n) = \frac{1}{Z_n}\prod_{i<j}|z_i - z_j|^2\prod_{i=1}^n \exp(-|z_i|^2)$$

   in the sense that calling $Z_1,\dots,Z_n$ the points of P_n, for a symmetric test function f

$$\mathbf{E}[f(P_1,\dots,P_n)] = \int_{(\mathbb{R}^d)^n} f\rho^n,$$

   see Chapter 4.

   Remark 4. *If the factorial densities of a point process* P *are of the form*

$$\rho^{(k)}_{\mathsf{P}}(x_1,\dots,x_k) = \det(\tilde{K}(x_i,x_j))\varphi(x_1)\dots\varphi(x_n)$$

   then they correspond to the DPP with kernel

$$K(x,y) = \tilde{K}(x,y)\sqrt{\varphi(x)\bar{\varphi}(y)}$$

   by multilinearity of the determinant. More explicitly

$$\det((K(x_i,x_j))) = \det((\tilde{K}(x_i,x_j)\sqrt{\varphi(x_i)\varphi(x_j)})) = \sum_\sigma \varepsilon(\sigma)\prod_i \dots\sqrt{\varphi(x_i)\varphi(x_{\sigma(i)})}$$
$$= \det((\tilde{K}(x_i,x_j)))\prod_i \varphi(x_i)$$

   because each $\varphi(x_i)$ *appears twice, once as a line and once as a column. This corresponds to a change of measure* $dx \mapsto \sqrt{\varphi(x)}dx$.

   So in our case it is enough to prove

$$\prod_{i<j}|z_i - z_j|^2 = \det(\tilde{K}(z_i,z_j))$$

   for the kernel

$$\tilde{K}(z,w) = \sum_k \frac{z^k\bar{w}^k}{\pi k!}.$$

   In general, Ginibre is rather considered a DPP on $\mathbb{C}$ *endowed with the measure* $e^{-|z|^2}dz$.

We recognize for the first term the Vandermonde determinant. For $\mathsf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$\prod_{i<j}(z_i - z_j) = \det((z_i^{k-1})_{1\leqslant i,k\leqslant n}).$$

Let us multiply each column by the scalar α_{k-1} and take the square modulus: let $M(\mathsf{z}) = (\alpha_{k-1}z_i^{k-1})_{1\leqslant i,k\leqslant n}$, then

$$\prod_{i<j}|z_i - z_j|^2 \propto |\det(M(\mathsf{z}))|^2 = \det(M(\mathsf{z})M(\mathsf{z})^*) = \det(\tilde{K}_n(z_i, z_j))$$

with

$$\tilde{K}_n(z, w) := \sum_{k=1}^{n} |\alpha_{k-1}|^2 z^{k-1}\bar{w}^{k-1}.$$

It proves that $\rho^{(n)}_{\mathsf{P}_n}$ has the DPP form (6.1) with the kernel $K_n(z, w) = \tilde{K}_n(z, w)e^{-\frac{|z|^2+|w|^2}{2}}$, up to a constant. We write $\rho^{(n)}_{\mathsf{P}_n} = \lambda_n \rho^{(n)}_K$ and we show later $\lambda_n = 1$ (because we know that P_n exists...).

Note that it could imply that P_n and the K_n-DPP have the same density ρ_n, and therefore they have the same law. Unfortunately we do not know (yet) if a DPP with kernel K_n exists at all, so we really have to prove that all factorial moment measures of P_n correspond. Our strategy in particular proves that any DPP with CPK exists.

3. We are missing other factorial measures $\rho^{(k)}_{\mathsf{P}_n}, k < n$. We will discuss that now.

Those are typically the form of kernels for finite point processes P_n coming from finite matrix models. These kernels enjoy the *reproducing* property:

Definition 12. *Say that K is reproducing if for $x, y \in \mathbb{R}^d$,*

$$K(x, y) = \int_{\mathbb{R}^d} K(x, z)K(z, y)dz.$$

Proposition 16. *Let K a projection canonical kernel. Then K is reproducing.*

Proof.

$$\begin{aligned}\int_{\mathbb{R}^d} K(x, z)K(z, y)dz &= \sum_{k,j=1}^{n} \int \varphi_k(x)\bar{\varphi}_k(z)\varphi_j(z)\bar{\varphi}_j(y)dz \\ &= \sum_{k,j} \underbrace{\langle \varphi_k, \varphi_j \rangle}_{\delta_{k=j}} \varphi_k(x)\bar{\varphi}_j(y) \\ &= \sum_{k=1}^{n} \varphi_k(x)\bar{\varphi}_k(y).\end{aligned}$$

Hence indeed it is $K(x, y)$. □

This property is more conceptually seen as a projection property in the L^2 space. If (6.2) is satisfied, for $f \in L^2(\mathbb{R}^d)$ with compact support,

$$\mathsf{L}_K f := (x \mapsto \int f(y)K(x, y)dy)$$

is the projection of f onto the space spanned by the φ_k:

$$L_K f(x) = \sum_k \int \varphi_k(x) f(y) \bar{\varphi}_k(y) dy = \sum_k \langle f, \varphi_k \rangle \varphi_k.$$

For a general reproducing kernel that induces an operator on $L^2(\mathbb{R}^d)$

$$\mathsf{L}_K(\mathsf{L}_K f)(x) = \int K(x,z)(\int f(y)K(z,y)dy)dz = \int f(y) \int K(x,z)K(z,y)dz = \int f(y)K(x,y)dy = \mathsf{L}_K f(x).$$

As a counterexample, the Poisson kernel $K(x,y) = \mathbf{1}\{x = y\}$ neither satisfies (6.2) nor is reproducing.

This reproducing property is especially useful if P has a fixed number of points n. In general, the fact that the defining DPP property (6.1) holds for $\rho_{\mathsf{P}}^{(n)}$ does not imply automatically that it holds for $\rho_{\mathsf{P}}^{(k)}, k \leqslant n$, except for reproducing kernels:

Proposition 17 (Dyson identity [?])**.** *For a CPK K on $\mathbb{R}^d \times \mathbb{R}^d$ with n terms: for $k < n$,*

$$\int \rho_K^{(n)}(x_1, \dots, x_k, x_{k+1}) dx_{k+1} \dots dx_n = (n-k)! \rho_K^{(k)}(x_1, \dots, x_k).$$

Hence the reproducing property saves us a lot of effort in the proof that a point process is determinantal, as we will see with the GUE and Ginibre ensembles.

Exercise 8.
- Let $K(x,y) = \hat{\varphi}(x-y)$ where $\varphi = 1_A$ for some bounded symmetric A. Prove that P_K is reproducing.
- Prove that the sine kernel $K(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$ is reproducing. This is the kernel of the limit GUE process.

Lemma 8. *Let P_n a point process with n points. Then for $1 \leqslant k < n$, we have the projection*

$$\mu_{\mathsf{P}_n}^{(k)} = \frac{1}{(n-k)!} \mu_{\mathsf{P}_n}^{(n)}(\cdot, \dots, \cdot, \mathbb{R}^d \times \dots \times \mathbb{R}^d)$$

Proof. For each test function f symmetric in k arguments let

$$\tilde{f}(x_1, \dots, x_n) = \sum_{\text{distinct } (i_1, \dots, i_k)} f(x_{i_1}, \dots, x_{i_k})$$

Then

$$\begin{aligned}
\mu^k(f) =& \mathbf{E} \sum_{(x_1, \dots, x_k)^{\neq}} f(x_1, \dots, x_k) \\
=& \mathbf{E} \tilde{f}(Z_1, \dots, Z_n) \\
=& \frac{1}{n!} \mathbf{E} \left[\sum_{(x_1, \dots, x_n) \neq} f(x_1, \dots, x_n) \right] \\
=& \frac{1}{n!} \int \tilde{f}(x_1, \dots, x_n) \mu^n(dx_1 \dots dx_n) \\
=& \frac{1}{n!} \sum_{(i_1, \dots, i_k) \neq} \int f(x_{i_1}, \dots, x_{i_k}) \mu^n(dx_1 \dots dx_n) \\
=& \frac{1}{n!} \underbrace{\#\{\text{ordered } \ k\text{-tuples}\}}_{\binom{k}{n} \times k!} \times \int f(x_1, \dots, x_k) \tilde{\mu}^{(k)}(dx_1 \dots dx_k)
\end{aligned}$$

where

$$\tilde{\mu}^{(k)}(dx_1, \dots, dx_k) := \int \mu^n(dx_{k+1} \dots dx_n)$$

using that μ^n is symmetric, i.e. invariant under permutation of its arguments. Seeing that the prefactor is $(n-k)!^{-1}$, it completes the proof. □

Proof of Proposition 17. It suffices to show it for $k = n-1$ by induction. We use the projection property:

$$\rho_{\mathsf{P}}^{(n-1)}(z_1, \dots, z_{n-1}) = \frac{1}{1!} \int_{\mathbb{R}^d} \rho_{\mathsf{P}}^{(n)}(z_1, \dots, z_n) dz_n = \int_{\mathbb{R}^d} \rho_K^{(n)}(z_1, \dots, z_n) dz_n.$$

We must come back to the representation of the determinant with permutations:

$$\det(M) = \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}.$$

We decompose the set of permutations Σ_n in Σ_n^1 for which $\sigma(n) = n$ and Σ_n^* the complement. For $\sigma \in \Sigma_n^1$, let $\tilde{\sigma} \in \Sigma_{n-1}$ the restriction of σ to $[\![1, n-1]\!]$. For $\sigma \in \Sigma_n^*$, call $c_n(\sigma)$ the cycle containing n, and $\tilde{\sigma} \in \Sigma_{n-1}$ bypassing n (i.e. $\tilde{\sigma}(\sigma^{-1}(n)) := \sigma(n)$, others values do not change). Denote by $\varepsilon(\sigma)$ the signature of a permutation σ. Then for fixed $z_1, \dots, z_{n-1} \in \mathbb{R}^d$, by the projection property of factorial moment measures for a system with a.s. n points:

$$\begin{aligned}
\int_{\mathbb{R}^d} \rho_{\mathsf{P}}^{(m)}(z_1, \dots, dz_n) dz_n &= \int_{\mathbb{R}^d} \det((K(z_i, z_j))_{i,j \leqslant n}) dz_n \\
&= \sum_{\sigma \in \Sigma_n^1} \varepsilon(\sigma) \prod_{i \leqslant n-1} K(z_i, z_{\sigma(i)}) \int K(dz_n, dz_n) dz_n + \sum_{\sigma \in \Sigma_n^*} \varepsilon(\sigma) \prod_{i \notin c_n(\sigma)} K(z_i, z_{\sigma(i)}) \int \prod_{i \in c_n(\sigma)} K(z_i, z_{\sigma(i)}) dz_n \\
&= I \sum_{\sigma \in \Sigma_n^1} \varepsilon(\tilde{\sigma}) \prod_{i \leqslant n-1} K(z_i, z_{\tilde{\sigma}(i)}) \\
&\qquad + \sum_{\sigma \in \Sigma_n^*} \varepsilon(\sigma) \prod_{i \notin c_n(\sigma)} K(z_i, z_{\sigma(i)}) \prod_{i \in c_n(\sigma) \setminus \{n, \sigma^{-1}(n)\}} K(z_i, z_{\sigma(i)}) \times \int K(z_{\sigma^{-1}(n)}, dz_n) K(dz_n, z_{\sigma(n)}) dz_n \\
&= I \sum_{\tilde{\sigma} \in \Sigma_{n-1}} \varepsilon(\tilde{\sigma}) \prod_{i \leqslant n-1} K(z_i, z_{\tilde{\sigma}(i)}) \\
&\qquad + \sum_{\sigma \in \Sigma_n^*} \underbrace{\varepsilon(\sigma)}_{-\varepsilon(\tilde{\sigma})} \prod_{i \notin c_n(\sigma)} K(z_i, z_{\tilde{\sigma}(i)}) \underbrace{\prod_{i \in c_n(\sigma) \setminus \{n, \sigma^{-1}(n)\}} K(z_i, z_{\sigma(i)}) \times K(z_{\sigma^{-1}(n)}, z_{\sigma(n)})}_{= \prod_{i \in c_n(\sigma) \setminus \{n\}} K(z_i, z_{\tilde{\sigma}(i)})} \\
&= I \det((K(z_i, z_j))_{i,j \leqslant n-1}) - \sum_{\sigma' \in \Sigma_{n-1}} \#\{\sigma \in \Sigma_n^* : \tilde{\sigma} = \sigma'\} \varepsilon(\sigma') \prod_{i \leqslant n-1} K(z_i, z_{\sigma'(i)}).
\end{aligned}$$

To conclude, notice that for each σ', there are $n-1$ ways to choose where to insert index n in permutation σ' to obtain $\tilde{\sigma} \in \Sigma_n^*$, in particular it does not depend on σ'. It gives the conclusion that

$$\int \rho_K^{(n)}(z_1, \dots, z_n) dz_n = (I - (n+1)) \rho_K^{n-1}(z_1, \dots, z_{n-1}).$$

We only used the reproducing property, hence this relation works for every $k < n$ with $I - k$ instead of $I - (n+1)$, with the same value for I:

$$I = \int K_n(z, z) dz = \sum_{k=0}^{n-1} \int |\varphi_k(z)|^2 dz = n \times 1 = n.$$

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Then one can interate on k :

$$\begin{aligned}\int_{(\mathbb{R}^d)^{n-k}} \rho_K^{(n)}(z_1,\dots,z_n)dz_{k+1}\dots dz_n =&(I-(n-1))(I-(n-2))\dots(I-k)\rho_K^{(k)}(z_1,\dots,z_k)\\ =&(n-k)!\rho_K^{(k)}(z_1,\dots,z_k)\end{aligned}$$

□

It implies that up to the constant λ_n Ginibre is the K_n-DPP:

$$\rho_{\mathsf{P}_n}^{(k)}(z_1,\dots,z_k) = \frac{1}{(n-k)!}\int \lambda_n \rho_K^{(n)}(z_1,\dots,z_n)dz_{k+1}\dots dz_n = \lambda_n \rho_K^{(k)}(z_1,\dots,z_n).$$

For $k=1$, integrating gives

$$n = \mathbf{E}\mathsf{P}_n(\mathbb{R}^d) = \int \rho_{\mathsf{P}_n}^1 = \lambda_n \int K_n(z,z) = n\lambda_n.$$

Hence $\lambda_n = 1$. This concludes the proof that Ginibre is the K_n-DPP.

Exercise 9. 1. Prove Cauchy-Binet formula

$$\det(\Phi\Phi^*) = \sum_{I\subset\{1,\dots,n\},\,|I|=m} |\det(\Phi_I)|^2,$$

2. Give another proof of Dyson's identity.

6.2 Ginibre circular law

We can use the DPP representation to prove the circular law:

Theorem 13. *The Ginibre process $\mathsf{P}_n = \{Z_1,\dots,Z_n\}$ satisfies the circular law: define*

$$\tilde{\mathsf{P}}_n = \frac{1}{n}\sum_{i=1} \delta_{Z_i/\sqrt{n}}$$

then a.s. for A bounded and $\mathscr{L}^d$-regular

$$\tilde{\mathsf{P}}_n(A) \to \mu_{\text{circ}}(A) = \frac{1}{\pi}\mathscr{L}^d(A\cap B(0,1)).$$

Remark that this time we must dilate points. As before, we have two steps: for $A\subset\mathbb{R}^d$ bounded

1. $\mathbf{E}\tilde{\mathsf{P}}_n(A) \to \frac{1}{\pi}\mathscr{L}^d(A\cap B(0,\pi^{-1/2}))$.
2. $\text{Var}\left(\tilde{\mathsf{P}}_n(A)\right) = O(n^{-2})$ for A with negligible boundary.

In fact we do not prove that for set indicators, but rather for smoother functions. We prove that for ψ Lipschitz with compact support,

$$\text{Var}\left(\tilde{\mathsf{P}}_n(\psi)\right) = O(n^{-2}).$$

We use the following lemma, that we admit:

Lemma 9. *The Lipschitz functions with compact support form a convergence-determining class: there exists a countable family $\psi_q, q \geqslant 1$ of such functions such that if some measures μ_n, μ satisfy $\mu_n(\psi_q) \to \mu(\psi_q)$ for all $q \geqslant 1$, then $\mu_n \to \mu$ weakly.*

Then we have a.s. that $\forall q, \tilde{\mathsf{P}}_n(\psi_q) \to \mu_{\text{circ}}(\psi_q)$, which implies that $\tilde{\mathsf{P}}_n \to \mu_{\text{circ}}$ weakly a.s.. It implies in particular that for A bounded with negligible boundary, $\tilde{\mathsf{P}}_n(A) \to \mu_{\text{circ}}(A)$ a.s.

Proof. For the first order, since we have a DPP, we have access to the intensity measure

$$\mathbf{E}\tilde{\mathsf{P}}_n(A) = \frac{1}{n}\mathbf{E}\mathsf{P}_n(\sqrt{n}A) = \frac{1}{n}\mu^{(1)}_{\mathsf{P}_n}(\sqrt{n}A) = \frac{1}{n}\int_{\sqrt{n}A} K_n(z,z)dz = \int_A K_n(\sqrt{n}z, \sqrt{n}z)dz.$$

We have

$$\pi K_n(\sqrt{n}z, \sqrt{n}z) = \sum_{k=0}^{n-1} \frac{(\sqrt{n}|z|)^{2k}}{k!}\exp(-n|z|^2) = \mathbf{P}(\text{Poiss}(n|z|^2) \leqslant n-1) =: p_{n,z}.$$

Since $0 \leqslant p_{n,z} \leqslant 1$ on A, we just need to prove pointwise convergence. At fixed z, $\text{Poiss}(n|z|^2)$ is the sum of n iid Poisson variables with parameter $|z|^2$, hence the LLN gives the convergence in law

$$\frac{1}{n}\text{Poiss}(n|z|^2) \to \mathbf{E}\left[\text{Poiss}(|z|^2)\right] = |z|^2.$$

In particular,

$$\mathbf{P}(\text{Poiss}(n|z|^2) \leqslant n-1) \to \mathbf{P}(|z|^2 \leqslant 1) = \mathbf{1}\{|z| \leqslant 1\}$$

gives with Lebesgue's domination theorem

$$\mathbf{E}\tilde{\mathsf{P}}_n(A) \to \frac{1}{\pi}\int_A \mathbf{1}\{|z| \leqslant 1\}dz = \mu_{\text{circ}}(A).$$

□

For a.s. convergence,we have the more general result:

Proposition 18. *For a projector kernel K and a K-DPP P,*

$$Var(\mathsf{P}_n(\psi)) = \frac{1}{2}\int_{(\mathbb{R}^d)^2} (\psi(z) - \psi(w))^2 K(z,w)^2 dzdw.$$

Let P_n a reproducing K_n-DPP with n points, then $Var\left(\frac{1}{n}\mathsf{P}_n(\psi)\right) = O(1/n)$. This gives convergence in probability for Ginibre.

Furthermore for Ginibre, for ψ Lipschitz with compact support,

$$Var\left(\frac{1}{n}\mathsf{P}_n(\psi)\right) = O(1/n^2)$$

hence we have a.s. convergence $\tilde{\mathsf{P}}_n \to \mu_{\text{circ}}$ vaguely.

Proof. Recall that by the DPP property

$$\rho^{(2)}(z,w) = K(z,z)^2 - K(z,w)K(w,z) = K(z,z)K(w,w) - |K(z,w)|^2.$$

Let us compute the variance for ψ a test function

$$\begin{aligned}\mathrm{Var}\left(\mathsf{P}_n(\psi)\right) =&\mathbf{E}\sum_{z\in\mathsf{P}_n}\psi(z)^2+\mathbf{E}\sum_{z\neq w\in\mathsf{P}_n}\psi(z)\psi(w)-\left|\mathbf{E}\sum_{z\in\mathsf{P}_n}\psi(z)\right|^2\\ =&\int_{\mathbb{C}}\psi(z)^2K_n(z,z)+\int_{\mathbb{C}^2}\psi(z)\psi(w)(K_n(z,z)K_n(w,w)-|K_n(z,w)|^2)dzdw-\left(\int\psi(z)K_n(z,z)dz\right)^2\end{aligned}$$

Two terms cancel out, then use reproducing property $K(z,z)=\int|K(z,w)|^2dw$

$$=\int_{\mathbb{C}^2}\psi(z)^2\int_C|K(z,w)|^2dzdw-\psi(z)\psi(w)|K(z,w)|^2dzdw$$

then use

$$\int\psi^2(z)K(z,w)=\frac{1}{2}\int\psi^2(z)K(z,w)dz+\int\psi^2(w)K(z,w)dw.$$

For the general $O(1/n)$ bound for $\tilde{\mathsf{P}}_n(\psi)=n^{-1}\mathsf{P}_n(\psi(\cdot/\sqrt{n}))$ with n points, for ψ bounded

$$\begin{aligned}\int|\psi(z/\sqrt{n})-\psi(w/\sqrt{n})|^2|K_n(z,w)|^2dzdw \leqslant&2\|\psi\|_\infty^2\int_{(\mathbb{R}^d)^2}|K(z,w)|^2dzdw\\ =&2\|\psi\|_\infty^2\int_{(\mathbb{R}^d)}K_n(z,z)dz=2\|\psi\|_\infty^2\mathbf{E}\mathsf{P}_n(\mathbb{R}^d)=2\|\psi\|_\infty^2n.\end{aligned}$$

Therefore $\mathrm{Var}\left(\mathsf{P}_n(\psi)\right)=O(n)$ and

$$\mathrm{Var}\left(\tilde{\mathsf{P}}_n(\psi)\right)=O(\frac{1}{n^2}n)\to 0$$

which shows convergence in proba by Byenaimé/Tchebyshev.

For Ginibre we need a further cancellation. The important fact is that $K_n(z,w)$ decays fast off the diagonal. Assume ψ is L-Lipschitz with compact support C

$$\mathrm{Var}\left(\mathsf{P}_n(\psi(\cdot/\sqrt{n}))\right)\leqslant\frac{L^2}{n}\int_{(\sqrt{n}C)^2}(z-w)^2K_n(z,w)^2dzdw$$

For Ginibre,

$$|K_n(z,w)|=|\sum\frac{1}{\pi k!}z^k\bar{w}^ke^{-(|z|^2+|w|^2)/2}\leqslant\frac{1}{\pi}|e^{z\bar{w}}e^{-(|z|^2+|w|^2)/2}|=\frac{1}{\pi}|e^{\bar{z}w}e^{-(|z|^2+|w|^2)/2}|=\frac{1}{\pi}\exp(-|w-z|^2/2).$$

Hence

$$\mathrm{Var}\left(\mathsf{P}_n(\psi)\right)\leqslant\frac{L^2}{n}\int_{(\sqrt{n}C)^2}|(z-w)|^2\exp(-|(z-w)|^2)dzdw\leqslant\frac{L^2}{n}\int_{\sqrt{n}C}\int_{\mathbb{C}}|v|^2\exp(-|v|^2/2)dv=\frac{1}{n}O(\mathscr{L}^d(\sqrt{n}C))=O(1).$$

We therefore finally have

$$\mathrm{Var}\left(\tilde{\mathsf{P}}_n(\psi)\right)=O(n^{-2})$$

hence for each φ, $\tilde{\mathsf{P}}_n(\psi)\to\mu_{\mathrm{circ}}(\psi)$ a.s.. Hence a.s. it holds for a countable family $\psi_q, q\geqslant 1$. We can choose a convergence determining family ψ_q, therefore we have a.s. $\tilde{\mathsf{P}}_n\to\mu_{\mathrm{circ}}$.

□

6.3 Infinite Ginibre ensemble

Convergence: remark that

$$K_n(z,w) = \sum_{k=0}^{n-1} \frac{z^k \bar{w}^k}{\pi k!} e^{-|z|^2/2-|w|^2/2} \to \frac{1}{\pi} e^{z\bar{w}-\frac{|z|^2+|w|^2}{2}}.$$

Theorem 14. *The Ginibre process* P_n *converges weakly in law to a stationary point process* P *which is a DPP with reproducing kernel*

$$K(z,w) = \frac{1}{\pi} e^{z\bar{w}-\frac{|z|^2+|w|^2}{2}}.$$

Furthermore P *is stationary and isotropic (i.e. its law is invariant under Euclidean rotations).*

Proof. As we already saw, since K is locally bounded, it suffices to prove that for each fixed m, $\mu_{\mathsf{P}_n}^{(m)}$ converges to $\mu_{\mathsf{P}}^{(m)}$ on each bounded A, more precisely

$$\mu_{\mathsf{P}_n}^{(m)}(A^m) = \int_{A^m} \det K_n \to \mu_{\mathsf{P}}^{(m)}(A) = \int_{A^m} \det K.$$

For $(\mathscr{L}^d)^m$-a.e. $(x_1,\dots,x_m)$, we have pointwise convergence. Also A is contained in a ball $B(0,r)$. We have

$$|K(z,w) - K_n(z,w)| \leqslant \sum_{k=n}^{\infty} \frac{r^{2k}}{\pi k!} \to 0.$$

Hence

$$|\det(K_n(z_i,z_j)) - \det(K(z_i,z_j))| \leqslant \sum_{\sigma} m M_A^{m-1} \sup_{i,j} |K_n(z_i,z_j) - K(z_i,z_j)| \leqslant C_{m,A} \sum_{k=n+1}^{\infty} \frac{r^{2k}}{k!}.$$

Therefore we have uniform convergence.

The reproducing property passes to the limit: for $z,w \in \mathbb{C}$

$$\int K(z,u)K(u,w)^2 du = \int \lim_n |K_n(z,u)K_n(u,w)du = \lim_n \int K_n(z,u)K_n(u,w) = \lim_n K_n(z,w) = K_n(z,w).$$

to switch $\lim_n$ and $\int_{\mathbb{C}}$ we use Lebesgue's theorem with the domination

$$|K_n(z,u)| \leqslant \exp(-|z-u|^2).$$

□

Hence by Proposition 15, $\mathsf{P}_n^{\mathrm{Gin}}$ converges to $\mathsf{P}^{\mathrm{Gin}}$, the DPP with kernel K, weakly in the vague topology.

End of 7th session

The invariance of $\mathsf{P} = \mathsf{P}^{\mathrm{Gin}}$ under rotations is inherited from the invariance of $\mathsf{P}_n = \mathsf{P}_n^{\mathrm{Gin}}$ under rotations, because the random complex Ginibre matrix M_n is invariant under the action of the orthogonal group: for $O \in \mathscr{O}_n(\mathbb{R})$,

$$O\mathsf{P}_n \overset{(d)}{=} \mathsf{P}_n$$

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hence for f continuous with bounded support

$$O\mathsf{P}_n(f) = \mathsf{P}_n(f(O^{-1}\cdot)) \overset{(d)}{=} \mathsf{P}_n(f)$$

and since f and $f(O^{-1}\cdot)$ are continuous with bounded support,

$$O\mathsf{P}(f) \overset{(d)}{=} \mathsf{P}(f),$$

meaning $O\mathsf{P} \overset{(d)}{=} \mathsf{P}$.

Finally, for the stationarity of P,

$$K(z+v, w+v) = e^{i\varphi(z,v)}K(z,w)e^{-i\varphi(w,v)}$$

for some real function φ. The kernel K is not invariant under the action of shifts, but it does not prevent P to be stationary because for each $m \in \mathbb{N}$,

$$\rho_{\mathsf{P}}^{(m)}(z_1 + w, \ldots, z_m + w) = \det(K(z_i + w, z_j + w)_{i,j \leqslant m}) = \sum_\sigma \varepsilon(\sigma) \prod_i K(z_i + w, z_{\sigma(i)} + w)$$

where the sum is over permutations of $\{1, \ldots, m\}$, and with $i' = \sigma(i)$

$$\prod_i K(z_i + w, z_{i'} + w) = \prod_i e^{i\varphi(z_i,w)}K(z_i, z_{i'})e^{-i\varphi(z_{i'},w)} = \exp\left(i\sum_i \varphi(z_i, w) - i\sum_i \varphi(z_{i'}, w)\right)\prod_i K(z_i, z_{i'})$$

and the first exponential equals 1. Therefore the kernels $\rho_{\mathsf{P}}^{(m)}$ are invariant under the action of shifts, which proves by Proposition 13 that the point processes $\tau_w\mathsf{P}^{\text{Gin}}$ and P^{Gin} have the same law, i.e. P^{Gin} is stationary.

6.4 GUE as a DPP

Theorem 15. *The GUE ensemble is a DPP with kernel*

$$K_n(x,y) = \sum_{k=0}^{n-1} H_k(x)H_k(y)$$

where the H_k are the Hermite polynomials defined as: the degree of H_k is k, and they form the orthonormal family

$$\langle H_k, H_j\rangle = \int_{\mathbb{R}} H_k(x)H_j(x)\exp(-x^2/2)dx = \delta_{k=j}.$$

It converges to the "Sine process", a stationary DPP with reproducing kernel

$$K(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} = \operatorname{sinc}(\pi(x-y)).$$

The GUE has a very similar density

$$\prod_{i<j}(\lambda_i - \lambda_j)^2 \prod_i \exp(-\lambda_i^2/2),$$

but this time it is on $\mathbb{R}$. **What changes in the proof?**

The φ_k are not orthogonal anymore:

$$\int_{\mathbb{C}} z^2 \exp(-|z|^2) dz = 0 \text{ but } \int_{\mathbb{R}} x^2 \exp(-x^2/2) dx \neq 0,$$

and for $k-j \neq 0$ and even,

$$\int_{\mathbb{R}} x^{k-j} \exp(-x^2/2) dx \neq 0.$$

We can still get the DPP form almost for free.

Let us go back to the Vandermonde determinant. For $\mathsf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$\prod_{i<j} (z_i - z_j) = \det((z_i^{k-1})_{1 \leqslant i,k \leqslant n}).$$

We can multiply each column by the scalar α_{k-1}, but we can also add linear combinations of previous columns: $M(\mathsf{x}) = (\alpha_{k-1} x_i^{k-1} + Q_{k-1}(x_i))_{1 \leqslant i,k \leqslant n}$ where Q_k is a polynomial with degree $< k-1$. Let

$$H_k(x) = \varphi_k(x) + Q_k(x)$$

Then

$$\prod_{i<j} |x_i - x_j|^2 \propto |\det(M(\mathsf{z}))|^2 = \det(M(\mathsf{x}) M(\mathsf{x})^T) = \det(\tilde{K}_n(z_i, z_j))$$

with

$$\tilde{K}_n(x,y) := \sum_{k=0}^{n-1} H_k(x) H_k(y).$$

With the Gram Schmidt orthonormalisation procedure, we can recursively choose the α_k and Q_k to have an orthonormal family for the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) e^{-x^2/2} dx,$$

i.e. $H_0 = c_{00}, H_1 = c_{11} x + c_{10}, \ldots$ satisfy

$$1 = \langle H_0, H_0 \rangle = c_0^2 \int_{\mathbb{R}} \exp(-x^2/2) dx = c_0^2 \sqrt{2\pi}$$

therefore $c_0 = (2\pi)^{-1/4}$. Then H_1 with degree 1 is uniquely defined by

$$\int c_0 (c_{11} x + c_{10}) \exp(-x^2/2) dx = 0$$

$$\int (c_{11} x + c_{10})^2 \exp(-x^2/2) dx = 1,$$

etc... $\langle H_k, H_j \rangle = \delta_{k=j}$. The $H_k(x)$ are the *renormalised Hermite polynomials.*

For the GUE we have

$$\sum_{k=0}^{n} H_k(x) H_k(y) \to \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

This limit kernel is immediately seen to be invariant under translations, hence the limit process is stationary.

The convergence is based on the Christoffel-Darboux formula

$$\sum_{k=0}^{n-1} \psi_k(x) \psi_k(y) = \sqrt{n/2} (\psi_n(x) \psi_{n-1}(y) - \psi_{n-1}(x) \psi_n(y)) / (x-y).$$

Chapter 7

General hyperuniformity and Zeros of the planar GAF

As a concluding section, let us study a common feature of obtained processes P for Ginibre and GUE DPPs. Hyperuniformity is assessed with the behaviour of the variance of rescaled linear statistics: define for ψ bounded with compact support, for $R > 0$,

$$\mathsf{P}(\psi) = \sum_{x \in \mathsf{P}} (\psi)$$
$$\psi_R(x) = \psi(x/R).$$

For instance the number of points in $B(0,R)$ is obtained with $\psi(x) = 1_{B(0,1)}(x)$,

$$\#\mathsf{P} \cap B(0,R) = \mathsf{P}(1_{B(0,R)}) = \mathsf{P}(\psi_R).$$

By stationarity, the variance does not depend on the centring:

$$\mathrm{Var}\left(\mathsf{P}(B(x,R))\right) = \mathrm{Var}\left(\mathsf{P}(B(0,R))\right).$$

Definition 13. *Let $\alpha > 0$. A stationary point process on $\mathbb{R}^d$ is said to be α-hyperuniform if for some non-negative $f \in \mathsf{C}_c^\infty(\mathbb{R}^d)\backslash\{0\}$,*

$$Var\left(\mathsf{P}(f_R)\right) = O(R^{d-\alpha}).$$

This has to be put in perspective with the infinite stationary Poisson process for which

$$\mathrm{Var}\left(\mathsf{P}(f_R)\right) = \int f_R^2 \asymp R^2.$$

This cannot happen "by chance", it shows there is some cancellation phenomenon coming from long distance interaction.

Theorem 16. *The infinite Ginibre process is 2-hyperuniform on $\mathbb{C} \equiv \mathbb{R}^2$ and the Sine process is 1-hyperuniform on $\mathbb{R}$. We have furthermore the number variance over balls*

$$Var\left(\mathsf{P}(B(0,R))\right) = \begin{cases} O(R^{d-1}\ln(R)) & \text{if } \alpha = 1 \\ O(R^{d-\min(\alpha,1)}) & \text{if } \alpha \neq 1. \end{cases}$$

Exercise 10. 1. Let P a stationary DPP with a reproducing kernel K satisfying for some $C > 0, \alpha \in (0, 2)$

$$|K(x,y)|^2 \leqslant C(1 + \|x - y\|)^{-\alpha - d}.$$

Then P is α-hyperuniform. If this is true for some $\alpha > 2$, then P is 2-hyperuniform.

2. We have the number variance over balls

$$\mathrm{Var}\,(\mathsf{P}(B(0,R))) = \begin{cases} O(\ln(R)) \text{ if } \alpha = 1 \\ O(R^{d-\min(\alpha,1)}) \text{ if } \alpha \neq 1. \end{cases}$$

Proof. We will use the radial change of variable on $\mathbb{R}^d$: for f bounded with compact support on $\mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}^d} f(\|x\|)dx &= \int_{\mathbb{S}^{d-1}} \int_0^\infty f(\rho)\rho^{d-1} d\rho\sigma(d\theta) \\ &= \underbrace{\mathscr{H}^{d-1}(\mathbb{S}^{d-1})}_{\kappa_d} \int_0^\infty f(\rho)\rho^{d-1} d\rho. \end{aligned}$$

It also works on $\mathbb{C}$ with $d = 2$.

Let $\psi \in \mathsf{C}_c^\infty(\mathbb{R}^d)\backslash\{0\}$ non-negative with support on $B(0,c)$. Hence ψ_R is supported by $B(0,cR)$.

The starting point is that the Ginibre kernel satisfies

$$|K(x,y)|^2 = \exp(-|x-y|^2) \leqslant C(1 + \|x - y\|)^{-5}$$

(it could be actually any power > 4).

Since P is a reproducing DPP, we have the following computation on $\mathbb{C}$ with $d = 2, a = 5$:

$$\begin{aligned} \mathrm{Var}\,(\mathsf{P}(\psi_R)) &= \int_{B(0,cR)^2} |\psi(x/R) - \psi(y/R)|^2 |K(x,y)|^2 dxdy \leqslant \frac{\|\nabla\psi\|}{R^2} \int_{B(0,cR)^2} \|x-y\|^2 K(x,y)^2 dxdy \\ &\leqslant \frac{\|\nabla\psi\|}{R^2} \int_{B(0,cR)^2} (1 + \|x-y\|)^{2-a} dxdy \\ &\leqslant CR^{-2} \int_{B(0,cR)} \int_{B(x,2cR)} (1 + \|x-y\|)^{2-a} dydx \\ &\leqslant CR^{-2} \int_{B(0,cR)} \int_0^{2cR} (1+\rho)^{2-a} \rho^{d-1} d\rho dx \\ &\leqslant CR^{-2} \int_{B(0,cR)} d\rho dx \times \int_0^\infty (1+\rho)^{1+d-a} d\rho \text{ using } a > 4 \\ &\leqslant CR^{-2} R^d R^{2-\alpha} = CR^{d-2}. \end{aligned}$$

That proves 2-HU.

For GUE, the same computation on $\mathbb{C}$ with $d = 1, a = 2$, using

$$(\mathrm{sinc}(\|x-y\|))^2 \leqslant \begin{cases} \frac{1}{\|x-y\|} \text{ if } \|x-y\| \geqslant 1 \\ \frac{\|x-y\| + o(\|x-y\|)}{\|x-y\|} \text{ if } \|x-y\| < 1 \end{cases} \leqslant c(1 + \|x-y\|)^{-2}$$

gives

$$\mathrm{Var}\,(\mathsf{P}(\psi_R)) \leqslant CR^{-2} \int_{[-cR,cR]} \int_{[-2cR,2cR]} (1+\rho)^0 d\rho dx \leqslant C.$$

For the number variance over balls,

$$
\begin{aligned}
\operatorname{Var}(\mathsf{P}_n(\psi_R)) \leqslant & \int_{B(0,R)\times B(0,R)^c} (1+\|x-y\|)^{-a} dxdy \\
\leqslant & C\int_{B(0,R)}\int_{B(x,R-\|x\|)^c} (1+\|x-y\|)^{-a} dxdy \\
\leqslant & C\int_{B(0,R)}\int_{R-\|x\|}^{\infty} (1+\rho)^{-a}\rho^{d-1} d\rho dx \\
\leqslant & C\int_{B(0,R)} (1+R-\|x\|)^{d-a} dx \text{ using } \alpha > 0 \\
\leqslant & C\int_0^R (1+R-\rho)^{d-a}\rho^{d-1} d\rho \\
\leqslant & CR^{d-1}\int_0^R (1+r)^{d-a} dr \text{ with } r = R-\rho \\
\leqslant & R^{d-1}\begin{cases} O(1) \text{ if } a > d+1 \\ O(\ln(R)) \text{ if } a = d+1 \\ O(R^{1-\alpha}) \end{cases}
\end{aligned}
$$

□

7.1 The planar Gaussian analytic function

Let $G_k, k \geqslant 0$ i.i.d $\mathscr{N}_{\mathbb{C}}(0,1)$ variables. Recall the Weyl random polynomials

$$\mathsf{F}_n(z) = \sum_{k=0}^{n} G_k \frac{z^k}{\sqrt{k!}}.$$

and the planar GAF

$$\mathsf{F}(z) = \sum_{k=0}^{\infty} G_k \frac{z^k}{\sqrt{k!}}.$$

Well defined a.s. because

$$\mathbf{E}|F(z)|^2 \leqslant \sum_{k=0}^{\infty} \frac{|z|^{2k}}{k!} \leqslant \exp(|z|^2) < \infty.$$

The zero set is a random element of $\mathscr{N}$ denoted by

$$\mathscr{Z}_{\mathsf{F}} = \{z : F(z) = 0\}.$$

Theorem 17. *The process $\mathscr{Z}_{\mathsf{F}}$ is stationary, isotropic and 4-hyperuniform. In particular for f smooth*

$$Var(\mathscr{Z}_{\mathsf{F}}(R)) = O(R^{-2}).$$

The proof relies on the theory of Gaussian analytic functions, andmore deeply of CGVs. See [?, ?, ?, ?] or the surveys [?], [?, Section 3.2].

The 4-HU comes from the following lemma, proved later.

Lemma 10. *There is a constant C such that for f in the class $\mathscr{C}_c^2(\mathbb{C})$ of $\mathscr{C}^2$-smooth functions with compact support,*

$$Var(\mathsf{P}(f)) \leqslant C\|\Delta f\|^2_{L^2(\mathbb{C})}.$$

This entails 4-HU because as $R \to \infty$,

$$\mathrm{Var}\,(\mathsf{P}(f_R)) = O(R^{-2}) = O(R^{2-\alpha}) \text{ with } \alpha = 4.$$

Definition 14. *A CGV is a vector of the form $M\alpha$ where α consists in i.i.d $\mathscr{N}_{\mathbb{C}}(1)$ variables and M is a deterministic complex matrix.*

Definition 15. *A GAF is a centred random function which is a.s. analytic and whose FIDIs are CGVs.*

Proposition 19. *The law of a centred GAF F is characterised by its complex covariance function*

$$\mathsf{C}(z,w) = \mathbf{E}\left[\mathsf{F}(z)\bar{\mathsf{F}}(w)\right].$$

The covariance of the planar GAF is

$$\mathsf{C}(z,w) = e^{z\bar{w}}.$$

Proof. The fact that complex covariance characterises the law of CGVs can be proved with the characteristic function with the following ingredients: for a real centred Gaussian vector $V \in \mathbb{R}^n$ with covariance Σ, for $t \in \mathbb{R}^n$

$$\mathbf{E}\left[\exp(i\langle t, V\rangle)\right] = \exp\left(-\frac{1}{2}t\Sigma^{-1}t^T\right).$$

Writing it as power series, the identity can be extended to

$$\mathbf{E}\left[\exp\left(\langle t, V\rangle\right)\right] = \mathbf{E}\left[\exp\left(i\langle -it, V\rangle\right)\right] = \exp(\frac{1}{2}(-it)\Sigma^{-1}(t)^T). = \exp(\frac{1}{2}t\Sigma^{-1}t^T).$$

it can be extended to complex $z = t_R + it_I \in \mathbb{C}^n$ and then to complex $V = V_R + iV_I$. The complex covariance is

$$S = \mathbf{E}VV^* = \mathbf{E}(V_R + iV_I)(V_R^T - iV_I) = \mathbf{E}V_RV_R^T - \mathbf{E}V_IV_I^T + i\mathbf{E}V_RV_I + \mathbf{E}V_IV_R.$$

□

Proof of stationarity. We have the computation

$$\mathsf{C}(z+v, w+v) = e^{\varphi(z,v)}\mathsf{C}(z,w)e^{-\varphi(z,v)}.$$

with $\varphi(z,v) = e^{\frac{1}{2}|v|^2}\exp(z\bar{v})$, which is holomorphic in z.

This is the covariance of $\mathsf{F}(z+v), \mathsf{F}(w+v)$. Therefore the field

$$\tilde{f}(z) = e^{-i\varphi(z,v)}\mathsf{F}(z+v)$$

has complex covariance

$$e^{-i\varphi(z,v)}\mathsf{C}(z+v, w+v)e^{i\varphi(z,w)} = \mathsf{C}(z,w)$$

identical as F. Why is $\tilde{\mathsf{F}}$ a GAF? Its FIDIs are the

$$(\tilde{f}(z_1), \dots, \tilde{f}(z_m)) = e^{\frac{1}{2}|v|^2}(e^{\varphi(z_1,v)}\mathsf{F}(z_1), \dots, e^{\varphi(z_m,v)}\mathsf{F}(z))$$

it is the product of a matrix with a CGV, hence it is a CGV.

Since they both are GAFs, it means that $\mathsf{F} \overset{(d)}{=} \tilde{\mathsf{F}}$. Since $\tau_v\mathsf{F}$ and $\tilde{\mathsf{F}}$ have the same zero sets, it means

$$\mathscr{Z} \overset{(d)}{=} \tau_v\mathscr{Z}$$

Since this is true for all $z \in \mathbb{C}$, $\mathscr{Z}$ is stationary. □

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Proposition 20. *The Weyl polynomials and planar GAF are GAFs.*

Proof. They are clearly both analytic. We must prove their FIDIs are CGVs. Recall

$$\mathsf{F}_n(z) = \sum_k a_k(z)\alpha_k z^k = \alpha a(z)$$

where $\alpha = (\alpha_0, \ldots, \alpha_n)$ consists of i.i.d. $\mathscr{N}_{\mathbb{C}}(0,1)$ variables and $a_k(z) := k!^{-1/2} z^k$.

$$(\mathsf{F}_n(z_1), \ldots, \mathsf{F}_n(z_m)) = \alpha \times (a(z_1), \ldots, a(z_m))$$

it is the product of a CGV and a matrix, hence it is a CGV.

Then

$$(\mathsf{F}(z_1), \ldots, \mathsf{F}(z_m)) = \lim_n (\mathsf{F}_n(z_1), \ldots, \mathsf{F}_n(z_m))$$

and the L^2 limit of CGVs is a CGV. □

Proof of Lemma 10. Morally, the proof follows from the a.s. analycity of F and the stationarity of P. It relies on two claims: a (non-random) analytic function F on $D \subset \mathbb{C}$ with zero set P satisfies

$$P(f) = \frac{1}{2\pi} \int_D \Delta f(z) \ln |F(z)| dz \tag{7.1}$$

and the zero set P of a GAF F satisfies for $f \in \mathscr{C}_c^2(\mathbb{C})$

$$\mathrm{Var}\left(\mathsf{P}(f)\right) = \frac{1}{4\pi^2} \int \Delta f(z) \Delta f(w) \mathrm{Cov}\left(\ln |\widetilde{\mathsf{F}}(z)|, \ln |\widetilde{\mathsf{F}}(w)|\right) dz dw, \tag{7.2}$$

where $\widetilde{\mathsf{F}}(z) = \mathrm{Var}\left(\mathsf{F}(z)\right)^{-1/2} \mathsf{F}(z)$. For the first claim, the starting point is the harmonicity of the log on the complex plane: $\Delta \ln(|\cdot|) = \frac{1}{2\pi}\delta_0$ in the distributional sense, i.e. for $f \in \mathscr{C}_c^2(\mathbb{C})$,

$$f(0) = \frac{1}{2\pi} \int \Delta f(z) \ln(|z|) dz. \tag{7.3}$$

Hereafter, fix f and denote by Λ its support. A non null holomorphic function F has finitely many zeros z_i in Λ, and the logarithm has an analytic determination on Λ, hence F can be written

$$F(z) = e^{g(z)} \prod_i (z - z_i) dz, z \in \Lambda,$$

for some analytic function g. Therefore with $P = \sum_i \delta_{z_i} \in \mathscr{N}(\mathbb{C})$, the smooth linear statistic can be expressed

$$P(f) = \sum_{i=1}^n f(z_i) = \frac{1}{2\pi} \sum_{i=1}^n \int \Delta f(z) \ln(|z_i - z|) dz = \frac{1}{2\pi} \int \Delta f(z) \ln |F(z) e^{-g(z)}| dz = \frac{1}{2\pi} \int \Delta f(z) \ln |F(z)| dz$$

which yields (7.1), exploiting the fact that the real part of an analytic function g is harmonic:

$$\int \Delta f(z) \mathscr{R} g(z) dz = 0.$$

Let us apply this to a GAF F. Denoting by $\kappa(z)^2 = \mathbf{E}|\mathsf{F}(z)|^2$, the variable $\tilde{\mathsf{F}}(z) = \mathsf{F}(z)/\kappa(z)$ is complex Gaussian with constant variance, hence $\ell := \mathbf{E}\ln|\tilde{\mathsf{F}}(z)|$ is constant as well, and

$$2\pi\mathbf{E}\mathsf{P}(f) = \int_\Lambda \Delta f(z)\mathbf{E}\ln|\mathsf{F}(z)|dz = \int_\Lambda \Delta f(z)\mathbf{E}\ln|\tilde{\mathsf{F}}(z)|dz + \int \Delta f(z)\ln(\kappa(z))dz = 0 + \int_\Lambda \Delta f(z)\ln(\kappa(z))dz$$

$$\begin{aligned}4\pi^2\mathbf{E}|\mathsf{P}(f)|^2 &= \int_{\Lambda^2} \Delta f(z)\Delta f(w)\mathbf{E}[\ln|\mathsf{F}(z)|\ln|\mathsf{F}(w)|]dzdw \\ &= \int_{\Lambda^2} \Delta f(z)\Delta f(w)\mathbf{E}[\ln|\tilde{\mathsf{F}}(z)|\ln|\tilde{\mathsf{F}}(w)|]dzdw + \int \Delta f(z)\Delta f(w)\ln(\kappa(z))\ln(\kappa(w))dzdw + 0 + 0 \\ &= \int \Delta f(z)\Delta f(w)\mathrm{Cov}\left(\ln|\tilde{\mathsf{F}}(z)|, \ln|\tilde{\mathsf{F}}(w)|\right)dzdw + 0 + 0 + \int \Delta f(z)\Delta f(w)\ln(\kappa(z))\ln(\kappa(w))dzdw\end{aligned}$$

which yields (7.2). Let us apply this to F. The final idea is that $|\tilde{\mathsf{F}}^{\mathrm{Pl}}(z)|, |\tilde{\mathsf{F}}^{\mathrm{Pl}}(w)|$ have a small correlation when z, w are far away. We have the general inequality ([**?**, Lemma 3.5.2]) for $\mathscr{N}_{\mathbb{C}}(0,1)$ variables Z, Z', $\mathrm{Cov}(\ln|Z|, \ln|Z'|) \leqslant \frac{1}{2}|\mathbf{E}Z\overline{Z'}|^2$, hence

$$\mathrm{Cov}\left(\ln|\tilde{\mathsf{F}}^{\mathrm{Pl}}(z)|, \ln|\tilde{\mathsf{F}}^{\mathrm{Pl}}(w)|\right) \leqslant \frac{1}{2}|\mathbf{E}\tilde{\mathsf{F}}^{\mathrm{Pl}}(z)\overline{\tilde{\mathsf{F}}^{\mathrm{Pl}}(w)}|^2.$$

By (**??**), the right hand side is $\frac{1}{2}e^{-|z-w|^2} =: \sigma(z-w)$. Hence by Cauchy-Schwarz inequality

$$4\pi^2\mathrm{Var}(\mathsf{P}(f)) = \int \Delta f(z)\Delta f(w)\sigma(z-w)dzdw \leqslant \|\Delta f\|_{L^2(\mathbb{C})}\|\Delta f \star \sigma\|_{L^2(\mathbb{C})} \leqslant \|\Delta f\|^2_{L^2(\mathbb{C})}\|\sigma\|^2_{L^1(\mathbb{C})}$$

which concludes the proof.

□

Exam Topics

1. Conditions under which a random $n \times n$ matrix has n zeros a.s. (in $\mathbb{R}$ or $\mathbb{C}$)
2. Semi-circular law for the GOE and GUE ensembles.
3. Relations between Gibbs measures and random matrices eigenvalues.
4. Characterisation of the law of a point process, convergence between point processes.
5. Definitions and properties of Determinantal point processes.
6. GUE and Ginibre processes as DPPs.
7. Infinite GUE and Ginibre processes and hyperunformity.