Rearrangements of Gaussian fields

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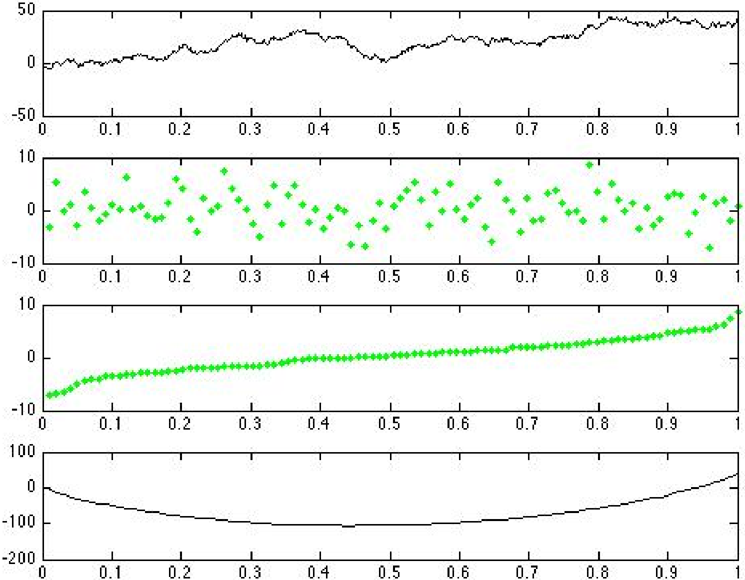
Asymptotic rearrangement of the Brownian motion



Theorem (Davydov, Zitikis 2004)

X: Brownian motion.
X_n: Piece-wise linear interpolation of X on $\{0, 1/n, \ldots, 1\}$.
$\mathfrak{C}X_n$: Convex rearrangement of X_n.
Then

$$\sup_{x\in[0,1]} \left| \frac{1}{\sqrt{n}} \mathfrak{C}X_n(x) - L(x) \right| \to 0,$$

L: Lorenz curve.

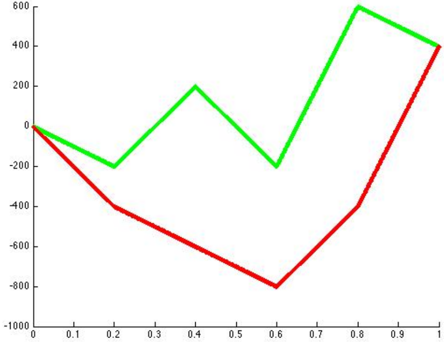
Other asymptotic convex rearrangements in Davydov & Vershik 1998.
X^H: fBm with Hurst parameter H. Then

$$n^{H-1}\mathfrak{C}X_n^H \to L.$$

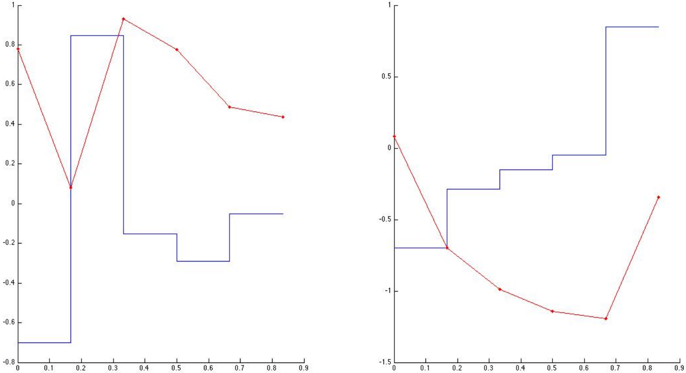
(L is the limit rearrangement for many Gaussian processes with stationary increments)

Convex rearrangement

green: Piecewise linear function f.
Lower part (red): *convex rearrangement of f*, denoted by $\mathfrak{C}f$.



Rearrangement of the derivative



It corresponds to rearranging the derivative in a monotone way. If f' is the derivative of f, and $(\mathfrak{C}f)'$ the derivative of $\mathfrak{C}f$, we have

$$\lambda_1 f'^{-1} = \lambda_1 (\mathfrak{C}f)'^{-1}.$$

The proof can be decomposed in two steps:

1: The probabilistic result:

Consider the image measure

$$\mu_n = \lambda_1(n^{-1/2}\nabla X_n)^{-1}.$$

Then $\mu_n \Rightarrow \gamma_1$ a.s..

(λ_1: 1-dim. Lebesgue, γ_1: Normal distrib., $\Rightarrow$: weak convergence.)

2: The measure theory result:

Theorem

If a sequence of convex functions $\{g_n : n \geq 1\}$ satisfies

$$\lambda_1(g_n^{-1}) \Rightarrow \mu$$

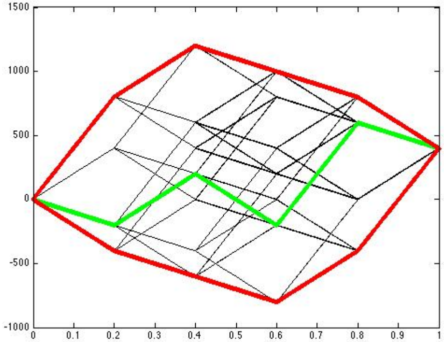
for some measure μ with finite first moment, then $g_n \to g$, with g convex and $\mu = \lambda_1 g^{-1}$.

Associated convex body of a 1-dimensional function

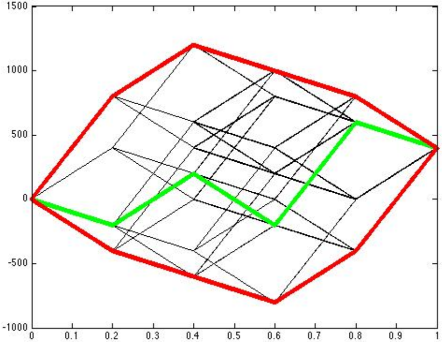
Resource distributed to a population of size N.

- Member labelled k receives r_k.
- Cumulative income function: $f(n) = \sum_{k \leq n} r_k$.

f is extended to a piece-wise linear function on $[0, N]$.



The area of the convex body can measure the inequalities over this particular resource (consider the equality case, where r_k is equal for all k)



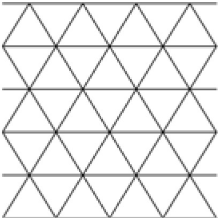
Gaussian fields

X: Centered Gaussian field, with covariance function

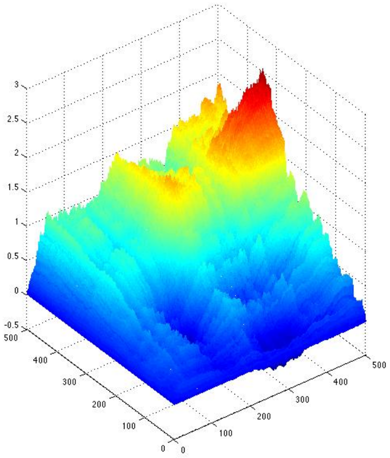
$$\sigma(z,\zeta) = \mathbb{E}X(z)X(\zeta),\ z,\zeta \in [0,1]^d.$$

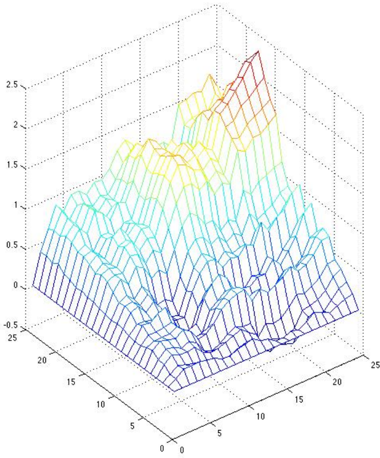
X_n : Approximations of a Gaussian field X on $[0,1]^d$.
X_n is obtained by interpolation of X on a triangulation $\mathcal{T}_n$.
There are regular simplices $T_1, \ldots, T_k$, and a discrete group Γ of $\mathbb{R}^d$ such that

$$\mathcal{T}_n = \{\frac{1}{n}(\gamma + T_j) : \gamma \in \Gamma, 1 \leq j \leq k\}.$$



Brownian sheet approximation





Results

Define

$$\mu_n = \lambda_d (b_n \nabla X_n)^{-1}$$

and

$$\sigma^{(2)}_{z,\zeta}(u,v) = \sigma(z,\zeta) + \sigma(z+u,\zeta+v) - \sigma(z+u,\zeta) - \sigma(z,\zeta+v),$$

the *second order local increment of* σ.

Theorem

Assume the following: For all u, v *in* $\mathbb{R}^d$

$$(nb_n)^2 \sigma^{(2)}_{z,z}(n^{-1}u, n^{-1}v) \to \sigma^{diag}_z(u,v)$$

uniformly in $z \in [0,1]^d$*.*
Then there is a deterministic measure μ *such that, for all Borel set* B*,*

$$\mathbb{E} \int_{[0,1]^d} \mathbf{1}_{\{b_n \nabla X_n(z) \in B\}} dz = \mathbb{E}(\mu_n(B)) \to \mu(B).$$

examples

Multifractional Brownian field:

$$\sigma(z,\zeta) = \|z\|^\alpha + \|\zeta\|^\alpha - \|z-\zeta\|^\alpha,\ \alpha \in (0,2)$$

$$\begin{cases} \sigma^{(2)}_{z,z}(u,v) = \|u\|^\alpha + \|v\|^\alpha - \|u-v\|^\alpha = \sigma_z^{diag}(u,v), \\ b_n = n^{\alpha/2-1} \end{cases}$$

Brownian sheet:

$$\sigma(z,\zeta) = \prod_i \min(z_i,\zeta_i).$$

$$\begin{cases} \sigma_z^{\mathrm{diag}}(u,v) = \langle l(z), u\wedge v - u\wedge 0 - v\wedge 0\rangle, \\ b_n = \sqrt{n} \end{cases}$$

with

$$l(z) = (z_2\dots z_d, z_1 z_3\dots z_d, \dots, z_1\dots z_{d-1}).$$

φ_n: Characteristic function of μ_n.

Theorem

Let $h \in \mathbb{R}^d$.

$$\mathbb{E}|\varphi_n(h) - \mathbb{E}\varphi_n(h)|^4 \leq C \left((n/b_n)^2 \sum_{S,S' \in \mathcal{T}_n} \mathrm{vol}(S)\mathrm{vol}(S')|\sigma^{(2)}_{z,\zeta}(n^{-1}u, n^{-1}v)| \right)^2$$

(u, v are the directions of edges of resp. S and S'.)

For the Multivariate Brownian field and the Brownian sheet, the right hand member is in $O(n^{-2})$, whence (Borel-Cantelli),

$$\mu_n \Rightarrow \mu$$

a.s..

Remarks:

- μ is deterministic,
- the convergence happens on each sample path.

New consistent estimators for parameters $\sigma(z, \zeta)$:

- Regularity parameters (Hurst Index),
- Directional parameters (Privileged axes)

Multidimensional rearrangement

Let $f : [0,1]^d \to \mathbb{R}$, differentiable a.e. such that

$$\int_{[0,1]^d} \|\nabla f(x)\| dx < +\infty.$$

A convex function C is a convex rearrangement of f if

$$\lambda_d \nabla f^{-1} = \lambda_d \nabla C^{-1}.$$

Theorem (Brenier, 91)

Every function f with finite gradient mass has a convex rearrangement $\mathfrak{C}f$. The convex rearrangement is unique up to a constant.

Asymptotic rearrangement

- f: "irregular function"
- f_n: Functions with finite gradient mass, the f_n converge to f. Is there a function C, and positive numbers $\{b_n;\ n \geq 1\}$, such that

$$b_n \mathfrak{C} f_n \to C?$$

If yes, C is an *asymptotic convex rearrangement*.

Theorem

$\{f_n;\ n \geq 1\}$: Functions with finite gradient mass,
$\{b_n;\ n \geq 1\}$: Positive numbers.
The following assertions are equivalent

(i) *Weak convergence* $\lambda_d \nabla (b_n f_n)^{-1} \Rightarrow \mu$.

(ii) $b_n \mathfrak{C} f_n(z) \to C(z)$, *for* $z \in \mathrm{int}([0,1]^d)$,

(iii) $\nabla (b_n \mathfrak{C} f_n)^{-1} \to \nabla C$ *in the* L^1 *sense on every sub-compact, whence* $C \in \mathfrak{C} f$.

In this case: $\mu = \lambda_d \nabla C^{-1}$.

Asymptotic rearrangement of the Brownian sheet

$$n^{-1/2}\mathfrak{C}X_n(z) \to C(z) \quad a.s., \quad z \in (0,1)^2,$$