Different scaling regimes for geometric Poisson functionals

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Mini-symposium : Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-Ito Chaos Expansions and Stochastic Geometry

Framework

- η_λ: Poisson measure with intensity μ_λ.
- Chaos decomposition: $F_\lambda = F(\eta_\lambda) = \sum_{q=1}^k F_{q,\lambda}$, L^2 variable for each λ.
- Asymptotic regime of renormalized variables

$$\tilde{F}_\lambda = \frac{F_\lambda - \mathbb{E}F_\lambda}{\sqrt{\mathrm{var}(F_\lambda)}}$$

$$\tilde{F}_{q,\lambda} = \frac{F_{q,\lambda} - \mathbb{E}F_{q,\lambda}}{\sqrt{\mathrm{var}(F_{q,\lambda})}}$$

- $\mathcal{N}$: Standard law
- $\mathcal{P}(c)$: Poisson law with parameter c
- d_W: Wasserstein distance.

Finite decompositions and U-statistics

Under proper integrability assumptions (η: Poisson point process):

- k-th order stochastic integral with kernel f:

$$I_k(f) = \int_{X^k} f(\mathbf{x}_k) d(\eta - \mu)^{\otimes k}.$$

- k-th order U-statistic with kernel h:

$$U_k(h) = \sum_{\mathbf{x}_k \subseteq \eta_\lambda} h(\mathbf{x}_k) = \int_{X^k} h(\mathbf{x}_k) d\eta^k(\mathbf{x}_k).$$

- Each k-tuple of points gives a contribution **independent of the other points of** η_λ.

Geometric framework

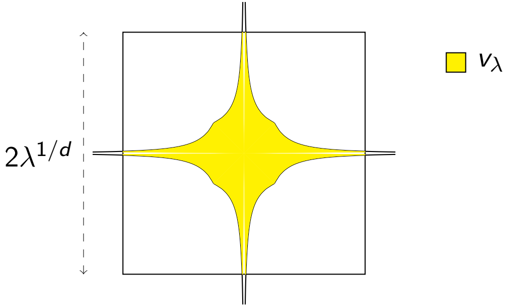
- η_λ : Marked Poisson process.
- ℓ: Lebesgue measure
- $X_\lambda = [-\lambda^{1/d}, \lambda^{1/d}]$
- (M, ν): Marks probability space
- $x = (t, m)$: marked point.
- $\mu_\lambda = 1_{X_\lambda} \ell \otimes \nu$ (Lebesgue measure).
- Kernel scale change $\Rightarrow$ Equivalent to $\mu_\lambda = 1_{X_1} \lambda \ell \otimes \nu$.

Graph model with unbounded connections

- $H_\lambda \subseteq \mathbb{R}^d$ measurable.
- $x, x' \in \eta$ connected if $x - x' \in H_\lambda$.
- If $H_\lambda =$ Unit ball $\Rightarrow$ Unit disk graph (Boolean model).
- F_λ: Number of connections.

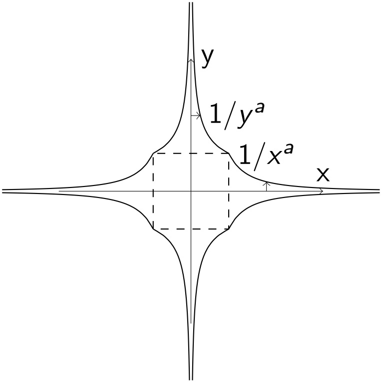
Interaction volume:

$$v_\lambda := \ell(H_\lambda \cap X_\lambda).$$

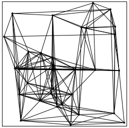


Different regimes: $H_\lambda = \alpha_\lambda H_1$

H_1:

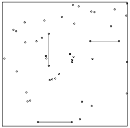


$a = 1, \alpha_\lambda = 1, \lambda = 25$:



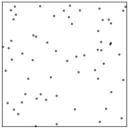
CLT at speed $\lambda^{-1/2}$

$a = 1, \alpha_\lambda = \lambda^{-1/d}, \lambda = 50$:



CLT in $\log(\lambda)^{-1}$

$a = 2, \alpha_\lambda = \lambda^{-1/d}, \lambda = 50$:



no CLT.

- **(R0)** λv_λ is bounded.
- **(R1)**$v_\lambda \to 0$ and $\lambda v_\lambda \to \infty$.
- **(R2)** $v_\lambda \to c > 0$
- **(R3)** $v_\lambda \to \infty$.

Theorem

Assume O-regular variation: $0 < c_1 \leq \frac{v_\lambda}{v_{c\lambda}} \leq c_2 < \infty$ *for* $c > 0$*.*
There are $C_i, C_i' > 0$ *such that*

- **(R3)** *:* $\mathrm{var}(F_\lambda) \sim C_3 \lambda v_\lambda^2$

  CLT: $d_W(\tilde{F}_\lambda, \mathcal{N}) \leq C_3' \lambda^{-1/2}$
- **(R2)** $\mathrm{var}(\widetilde{F}_\lambda, \mathcal{N}) \sim C_2 \lambda$

  CLT: $d_W(\widetilde{F}_\lambda, \mathcal{N}) \leq C_2' \lambda^{-1/2}$

  (Sandard behaviour)
- **(R1)** $\mathrm{var}(F_\lambda(H)) \sim C_1 \lambda v_\lambda$

  CLT: $d_W(\widetilde{F}_\lambda, \mathcal{N}) \leq C_1' (\lambda v_\lambda)^{-1/2}$
- **(R0)** $\tilde{F}_\lambda$ *converges to a Poisson law (or to* 0*).*

Hierarchy of chaoses

As v_λ grows, $F_{1,\lambda}$ becomes more predominant.

- Under **(R3)**
$$\frac{\mathrm{var}(F_{1,\lambda})}{\mathrm{var}(F_{2,\lambda})} \to \infty,$$
the first chaos dominates.
- Under **(R2)**
$$0 < c' \leq \frac{\mathrm{var}(F_{1,\lambda})}{\mathrm{var}(F_{2,\lambda})} \leq C' < \infty$$
- Under **(R1),(R0)**
$$\frac{\mathrm{var}(F_{2,\lambda})}{\mathrm{var}(F_{1,\lambda})} \to \infty$$
the second chaos dominates.

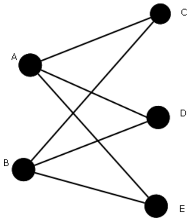
Remark: CLT $\Leftrightarrow \mathrm{var}(F_\lambda) \to \infty$.

Summary:

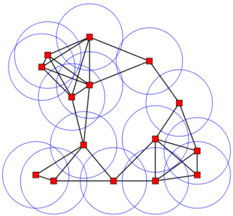
- **(R3) Large interactions**: CLT, first chaos dominates.
- **(R2) Constant size interactions**: CLT, chaoses co-dominate.
- **(R1) Small interactions**: Slow CLT, 2d chaos dominates.
- **(R0) Rare interactions**: Poisson limit, 2d chaos dominates.

Subgraph counting

- G: connected formal graph with cardinality $k \geq 1$.
- η_λ: Homogeneous Poisson process on X_λ.
- $\mathcal{G}_\lambda$: Graph obtained by connecting points $(x, y) \in \eta_\lambda$ with distance $\|x - y\| \leq \alpha_\lambda$. $v_\lambda := \alpha_\lambda^d$.
- $F_\lambda(G)$: Number of occurences of G as a subgraph of $\mathcal{G}_\lambda$.



$\Rightarrow$



- **(R1)**$v_\lambda \to 0$ and $\lambda v_\lambda^{k-1} \to \infty$.
- **(R2)** $v_\lambda \to c > 0$
- **(R3)** $v_\lambda \to \infty$.

Results: Penrose, Peccati, LR.

- **(R1)** $\mathrm{var}(F_\lambda) \sim c_1 \lambda v_\lambda^{k-1}$ CLT: $d_W(\tilde{F}_\lambda, \mathcal{N}) \leq C_1(\lambda v_\lambda^{k-1})^{-1/2}$
- **(R2)** $\mathrm{var}(F_\lambda) \sim c_2 \lambda$ CLT: $d_W(\tilde{F}_\lambda, \mathcal{N}) \leq C_2 \lambda^{-1/2}$
- **(R3)** $\mathrm{var}(F_\lambda) \sim c_3 \lambda v_\lambda^{2k-2}$ CLT: $d_W(\tilde{F}_\lambda, \mathcal{N}) \leq C_3 \lambda^{-1/2}$

Stationary rescaled marked kernels

- $k \geq 2$.

$$F_\lambda = U_k(h_\lambda) = \sum_{y_1,\dots,y_k \in \eta_\lambda} h_\lambda(y_1,\dots,y_k), \quad y_i = (t_i, m_i) \in \mathbb{R}^d \times \mathcal{M}.$$

Assumptions on h_λ:

- $h_\lambda(\cdot) \sim h(\alpha_\lambda\ .\ )$ in some sense.
- α_λ: Scaling regime. ($v_\lambda := \alpha_\lambda^d$: Interaction measure.)
- $h(y_1,\dots,y_k)$ invariant under **spatial translations**.
- h is **rapidly decreasing** away from the diagonal: there exists $\varkappa(y) > 0$ bounded probability density such that for $p = 2, 4$,

$$\int_{\mathcal{M}\times(\mathbb{R}^d\times\mathcal{M})^{k-1}} \frac{|h(0, m_0, y_1, \dots, y_{k-1})|^p}{(\varkappa(y_1)\dots\varkappa(y_{k-1}))^{p-1}} \mu\otimes\nu(dy_1,\dots,dy_{k-1}) < \infty.$$

Tool: Bounds on the contractions

Theorem

For $F = \sum_{q=1}^{k} I_q(f_q) \in L^2$ with $\mathrm{var}(F) = 1$,

$$d_W(F, \mathcal{N}) \leq C(k)(\max \|f_q \star_r^l f_{q'}\|_2 + \max_q \|f_q\|_4)$$

where $1 \leq l \leq r \leq q \leq q'$ and $l \neq q'$

$$f_q \star_r^l f_{q'}(\mathbf{x}_{r-l}, \mathbf{y}_{q-r}, \mathbf{y}'_{q'-r}) = \int f_q(\mathbf{x}_l, \mathbf{x}_{r-l}, \mathbf{y}_{q-r}) f_{q'}(\mathbf{x}_l, \mathbf{x}_{r-l}, \mathbf{y}'_{q'-r}) d\mathbf{x}_l.$$

Theorem

If $h(\mathbf{x}_k)$ and $g(\mathbf{x}_q)$ are stationary and rapidly decreasing, for r, l as above,

$$\|h \star_r^l g\|^2_{L^2(X \times (\mathbb{R}^d)^{k+q-r-l-1})} \leq \ell(X) A(h) A(g)$$

Chaos behaviour

$$F_\lambda = U_k(h_\lambda) = \sum_{q=1}^{k} F_{q,\lambda}.$$

Theorem

*q-**th chaos behaviour of** $U_k(h_\lambda)$:*

$$\mathrm{var}(F_{q,\lambda}) \sim C\lambda v_\lambda^{2k-q-1}$$

$$d_W(\tilde{F}_{q,\lambda}, \mathcal{N}) \leq C' \sqrt{\frac{v_\lambda^{1-q}}{\lambda}} \left(1 + 1_{\{q \neq 1\}} \sqrt{\begin{cases} v_\lambda^q \text{ if } v_\lambda > 1 \\ v_\lambda \text{ otherwise} \end{cases}} \right)$$

First term: kernel 4-th moments. second term: kernel contractions.

- First chaoses win if $v_\lambda \to \infty$.
- Last chaoses win if $v_\lambda \to 0$.
- High order chaoses convergence is slower.

Applications

- $v_\lambda = \lambda$: All points interact $\Rightarrow$ Geometric U-statistics.
- $v_\lambda = 1$: Standard behaviour/Thermodynamic regime.
- v_λ small: rarefaction of interactions: Slow CLT/no CLT.

Examples of functionals with a standard behaviour

Boolean model:

- $\mathcal{M}$: Compact sets (with Fell Borel σ-algebra).
- $\{M_k;\ k \geq 1\}$ IID Random compact sets .
- $\{x_k; k \geq 1\}$ Poisson point process with intensity λ.
- $\eta_\lambda = \{(x_k, M_k)\}$ marked Poisson measure.

$$R_\lambda = \bigcup_{k : x_k \in X_\lambda} (M_k \oplus x_k)$$

- F_λ : U-statistic with stationary kernel

$$h((x, M); (x', M')) = \varphi(x - x')\mathbf{1}_{\{(M \oplus x) \cap (M' \oplus x') \neq \emptyset\}},$$

$$F_\lambda = \sum_{x_i \neq x_j \in \eta_\lambda} \varphi(x_i - x_j)\mathbf{1}_{\{\text{The grains with centers } x_i \text{ and } x_j \text{ touch}\}}.$$

Magnitude assumption on φ

$$\varphi(x-y) \leq \|x-y\|^{\beta}, x, y \in \mathbb{R}^d.$$

Theorem

Assume that in the boolean model the typical grain has diameter R such that for some $\varepsilon > 0, \beta > -d/2$*, and for* $r \geq 1$

$$\mathbb{P}(R \geq r) \leq Cr^{-(2(\beta+d)+1+\varepsilon)}$$

then for some $C, C' > 0$

$$\mathrm{var}(F_\lambda) \sim C\lambda$$
$$d_W(\widetilde{F}_\lambda, \mathcal{N}) \leq C'\lambda^{-1/2}.$$

The optimal condition bears actually upon the decay of

$$\chi(x) = \mathbb{P}(M_1 \cap (M_2 \oplus x) \neq \emptyset), \ x \in \mathbb{R}^d.$$

Number of intersections in a process of line segments.

- $M_k; k \geq 1$ Line segments with random IID lengths.
- $\varphi(x, y) = 1$.

Then F_λ is the number of intersections of segments with centers in X_λ.

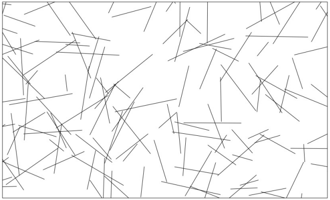


Figure: Zbynek Pawlas

Theorem (Pawlas 2012)

There is standard behaviour if for some $\varepsilon > 0$

$$\mathbb{P}(length(M_1) \geq r) \leq Cr^{-5-\varepsilon}, r \geq 1.$$

sub-hypergraph counting and telecommunications network

Decreusefond et al.

- $\eta_\lambda = \{x_k;\ k \geq 1\}$ Poisson point process with intensity $\lambda\ell$ on the torus.
- k-th order hypergraph $\mathcal{H}_{k,\lambda}$: Data of k-tuples $(x_1, \ldots, x_k) \in \eta_\lambda$ such that $\|x_i - x_j\| \leq \varepsilon$
- Exact mean formulas for $\mathcal{H}_{k,\lambda}$ and related topological quantities.

Our approach:

- $\{m_k;\ k \geq 1\}$ random radii around the points.
- Edge effects: $\eta_\lambda = \{y_k = (x_k, m_k);\ k \geq 1\}$ on $X_\lambda = [-\lambda^{1/d}, \lambda^{1/d}]$.
- $\widetilde{\mathcal{H}}_{k,\lambda}$: k-tuples $(y_1, \ldots, y_k)$ such that for all $i \neq j$, $\|x_i - x_j\| \leq m_i$.

Results(R=typical radius) :

Theorem

If $\mathbb{E}R^{4d+\varepsilon} < \infty$,

$$\mathrm{var}(F_{k,\lambda}) \sim C\lambda$$

$$d_W(\tilde{\mathcal{H}}_{k,\lambda}, \mathcal{N}) \leq C'\lambda^{-1/2}$$

Geometric U-statistics: $\alpha_\lambda = \lambda^{1/d}$

Points interact regardless of the distance:

$$F_\lambda = \sum_{\mathbf{y}_k \in \eta_\lambda} h(\lambda^{-1/d} y_1, \dots, \lambda^{-1/d} y_k).$$

General form of a **Geometric U-statistic**:

- (X, μ) loc. compact measured space
- $\mu_\lambda = \lambda\mu$
- $h(y_1, \dots, y_k)$: kernel on X^k.

$$F_\lambda = \sum_{\mathbf{y}_k \in \eta_\lambda} h(x_1, \dots, x_k).$$

Examples[Reitzner and Schulte]

- Number of k-tuples of points in convex position $\Rightarrow$ Approximation of Sylvester's constant in a convex body .
- Intersections of flats in a compact window.

Results

$$F_\lambda = \sum_{q=1}^{k} \varkappa_{k,q} F_{q,\lambda}$$

$$\text{and } F_{q,\lambda} = \lambda^{k-i} I_q(h_q)$$

Kernel projections:

$$h_q(\mathbf{y}_q) = \int_{X^q} h(\mathbf{y}_q, \mathbf{y}_{k-q}) d\mu(\mathbf{y}_q)$$

Asymptotic behaviour:

Theorem

$$\mathrm{var}(F_{q,\lambda}) \sim C\lambda^{2k-q}$$
$$F_{q,\lambda} \to G_q(h_q)$$

where $G_q(h_q)$ is a Gaussian chaos of order q.

Summary for geometric U-statistics

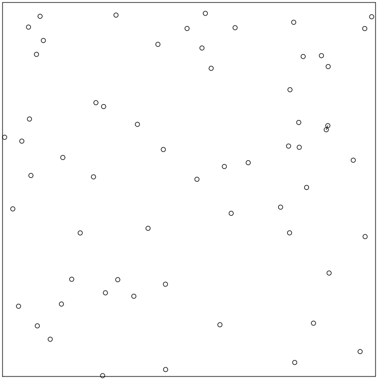
$$q_0 = \min\{q : \|h_q\|_2 \neq 0\}.$$

F_λ behaves like $F_{q_0,\lambda}$.

- Reitzner and Schulte. $q_0 = 1 \Rightarrow$ CLT at speed $\lambda^{-1/2}$
- LR and Peccati: $q_0 \geq 2 \Rightarrow$ no CLT, convergence to $G_{q_0}(h_{q_0})$.
- Peccati and Thaele: $q_0 = 2$: Speed of convergence to G_2, a Γ random variable.

Poisson regime: $\lambda\alpha_\lambda \to c$

Peccati (2011) : sufficient conditions for the convergence to a Poisson law in terms of the contractions.



Mixed chaos behaviour

Multi-dimensional CLTs: Peccati, Zengh, Minh, Schulte, Thaele, Last, Penrose, Reitzner, LR ...

Peccati , Bourguain 2012: Portmanteau inequalities $\Rightarrow$ Mixed limit theorems.

Example. Disk graph with influence volume v_λ such that $\lambda v_\lambda^{3-1} \to 0$ and $\lambda v_\lambda^{2-1} \to \infty$. Consider

$$F_{2,\lambda} = \# segments$$
$$F_{3,\lambda} = \# triangles.$$

Then

$$(\tilde{F}_{2,\lambda}, \tilde{F}_{3,\lambda}) \to (\mathcal{N}, \mathcal{P})$$

where $\mathcal{N}$ and $\mathcal{P}$ are independent.