Rigidity of point processes

Raphaël Lachièze-Rey (Inria Paris and Université Paris Cité)

Rigidity of lattices

- Let U uniform on $[0,1]^d$, and the shifted lattice
$$\mathbf{P} = \{k + U : k \in \mathbb{Z}^d\}.$$
This random point process is **maximally rigid** on any bounded $A \subset \mathbb{R}^d$:
$$\mathbf{P}_A := \mathbf{P} \cap A \in \sigma(\mathbf{P}_{A^c}).$$

- Let now a perturbed (shifted) lattice
$$\mathbf{P}' = \{k + U + \underbrace{\varepsilon_k}_{i.i.d.} : k \in \mathbb{Z}^d\}.$$
It is **number rigid** on A:
$$\#\mathbf{P}'_A \in \sigma(\mathbf{P}'_{A^c}).$$

The infinite Ginibre ensemble

The infinite Ginibre ensemble $\mathscr{G}$ can be defined in (at least) three ways:

- As the limit of the n-Ginibre ensembles, i.e. the eigenvalues of a random matrix with i.i.d. complex Gaussian entries $(G_{i,j})_{,1\leqslant i,j\leqslant n}$.
- As the stationary Determinantal Point Process (DPP) with spatial correlation $\rho(x) = 1 - e^{-\|x\|^2}$ (up to scaling)
- As the infinite stationary 2D Coulomb system, i.e. with interacting potential $\ln(\|x\|)$.

Then $\mathscr{G}$ is **number rigid** on $A = B(0,1)$:

$$\#\mathscr{G}_A \in \sigma(\mathscr{G}_{A^c})$$

[Ghosh, Peres 2017]

Zeros of the planar Gaussian Analytic Function

The random function

$$F(z) = \sum_k a_k \frac{1}{\sqrt{k!}} z^k, z \in \mathbb{C},$$

with a_k i.i.d. standard complex Gaussian, is analytic and the law of its zero set

$$\mathscr{Z} = \{z : F(z) = 0\}$$

is invariant under translation (and F's law is not), it is the unique zero set of a random GAF having this property up to scaling.

[Ghosh, Peres '17] show that $\mathscr{Z}$ is number rigid and 1-rigid:

$$\underbrace{\sum_{z \in \mathscr{Z} \cap A} z}_{\text{1t moment/barycenter}} \quad \in \sigma(\mathscr{Z}_{A^c}).$$

Rigidity bibliography

- More general studies of the rigidity of DPPs (Bufetov, Dabrowski, Qiu, ...)
- Rigidity of Coulomb gases (Dereudre, Leblé, Najnudel, Chaibi, Chatterjee, ...)
- k-Rigidity of other GAFs **[Ghosh, Krishnapur '21]** where k-rigidity of $\mathbf{P}$ means that for $\sum_{i=1}^{d} |k_i| \leqslant k$

$$\int_A x_1^{k_1} \dots x_d^{k_d} d\mathbf{P}(x) \in \sigma(\mathbf{P}_{A^c})$$

- Rigidity of discrete DPPs by Lyons and Steif in the early '00s
- Older result of rigidity **[Aizenman, Martin '80]**
- Notion of tolerance **[Holroyd, Soo '13]**

Uses of ridigiy

- Use in continuous percolation by **[Ghosh, Krishnapur, Peres '16]**
- Used in signal theory and signal reconstruction, related to the completeness question (Bardenet, Ghosh, ...)
- Relation with diffusive dynamics of particle systems **[Osada '24]**
- Lyons and Steif used it to prove phase uniqueness for some discrete models from statistical physics.

Topics

- NSC for k-rigidity?
  (i.e. moments of order $\leqslant k$ of $\mathbf{P}$ determined by $\mathbf{P}_{A^c}$?)
- What is the relation with Hyperuniformity? Why are there (almost) no example in dimension $d \geqslant 3$?

$$\begin{aligned}\text{HU of } \mathbf{P} :\Leftrightarrow & \operatorname{Var}(\#\mathbf{P} \cap B(0,R)) = o(R^d) \\ \Leftrightarrow & \operatorname{Var}(I_{\mathbf{P}}(\gamma_R)) = o(R^d) \\ & \text{with } \gamma_R = \gamma(\cdot/R), \gamma = 1_{B(0,1)} \\ \Leftrightarrow & \operatorname{Var}(I_{\mathbf{P}}(\gamma_R)) = o(R^d) \\ & \text{for some well chosen Schwartz function } \gamma\end{aligned}$$

- Are there point processes which are k-rigid but not $(k+1)-$rigid?

Relation with hyperuniformity

- Let γ smooth with $\gamma(0) = 1$, $\gamma_R(x) := \gamma(x/R), R > 0$

$$
\begin{aligned}
\#\mathbf{P} \cap B(0,1) \approx & \sum_{x \in \mathbf{P} \cap B(0,1)} \gamma_R(x) \\
= & \mathbf{E}(I_{\mathbf{P}}(\gamma)) - \sum_{x \in \mathbf{P} \cap B(0,1)^c} \gamma_R(x) + \text{“}O\left(\mathrm{Var}\left(I_{\mathbf{P}}(\gamma_R)\right)\right)''
\end{aligned}
$$

- “Strong HU” should imply number rigidity.

Structure factor

- Let M a random (unit intensity) L^2 wide-sense stationary measure ($\mathbf{E}(\mathsf{M}(A)^2) < \infty$ for A bounded)
- The **covariance measure** $\mathscr{C}$ and its Fourier transform the **structure factor** $\mathcal{S} = \mathcal{F}\mathscr{C}$ are characterised on Schwartz functions by

$$\begin{aligned} \mathrm{Var}\left(I_{\mathsf{M}}(f)\right) &= \int_{\mathbb{R}^d} f(y) f(x+y) \mathscr{C}(dx) dy, \\ &= \int |\hat{f}|^2 d\mathcal{S} \end{aligned}$$

- **Poisson process:** $\mathscr{C} = \delta_0, \ \ \mathcal{S}(du) = du$
- **stationary DPP with** L^2 **kernel** $\kappa(x-y) := |K(x,y)|^2$**:** $\mathcal{S}(du) = (1 - \widehat{\kappa^2}(u))du$
- **Continuous Gaussian field** $\mathsf{M}(dx) = F(x)dx$: $\mathscr{C}(dx) = \mathrm{Cov}\left(F(0), F(x)\right) dx$
- **Discrete field** $\mathsf{M}(\{k\}) = \mathsf{X}_k, k \in \mathbb{Z}^d : \mathscr{C}(\{k\}) = \mathrm{Cov}\left(\mathsf{X}(0), \mathsf{X}(k)\right)$

- If $\mathscr{C}$ decays fast and M is HU: $\mathcal{S}(du) = (\underbrace{\mathcal{S}(0)}_{=0} + O(\|u\|^2))du$
- **[Ghosh, Lebowitz '18]**: There is number rigidity if $\mathscr{C}$ is a measure with density c such that for $t \in \mathbb{R}^d$

$$|c(t)| \leqslant (1 + |t|)^{-2} \text{ if } d = 1 \ \ \text{(implies } \mathcal{S} \text{ has Lipschitz density)}$$
$$|c(t)| \leqslant (1 + \|t\|)^{-4-\varepsilon} \text{ if } d = 2 \ \ \text{(implies } \mathcal{S}(du) = O(\|u\|^2)du) \ .$$

- **[Bufetov, Dabrowski, Qiu '18]** In dimension 1, number-rigidity if

$$\sup_{N \geqslant 1} N \sum_{|n| \geqslant N} \mathrm{Cov}\left(\mathsf{M}([0,1]), \mathsf{M}([n, n+1])\right) < \infty.$$

$$\mathcal{S}(du) = \underbrace{\mathsf{s}(u)}_{\text{spectral density}} du + \underbrace{\mathcal{S}_s(du)}_{\text{singular part}}$$

Theorem (Lr '24)

A wide-sense stationary locally L^2 random measure M is k-rigid if its spectral density s has a zero of order k, i.e.: for every complex polynomial Q, if for some $\varepsilon > 0$,

$$\int_{B(0,\varepsilon)} \frac{|Q(u)|^2}{\mathsf{s}(u)} du < \infty,$$

then Q does not have terms of order k.

Corollary: Number rigidity $(k = 0)$ if $\int \mathsf{s}^{-1}(u)du = \infty$ (with $Q \equiv 1$)

- $d = 1$: $\mathsf{s}(u) \leqslant c|u|$ (Lipschitz in 0) OK
- $d = 2$: $\mathsf{s}(u) = O(\|u\|^2)$ OK

Converse statement?

- Yes for **Linear rigidity:** if $\#\mathbf{P} \cap B(0,1) = \lim_n \int_{B(0,1)^c} \gamma_n d\mathsf{M}$ in L^2.
- Most known rigidities are linear, at the exception of some examples (e.g. **[Peres , Sly '14]**, **[Klatt, Last '22]**, **[Lr '24]**?)

Definition

Say that s is **simple** if

- s is isotropic (invariant under rotations) **example**: $\mathscr{Z}$, Coulomb systems?
- or s has finitely many 0's of finite order and

$$\mathsf{s}(u) \geqslant c(1 + \|u\|)^{-p}$$

  ε-away from the zeros (for some $\varepsilon, p > 0$) **example**: DPPs with L^2 kernel
- or $\mathsf{s}(u) = \mathsf{s}_1(u_1) \dots \mathsf{s}_d(u_d)$ is separable. **Example:** tensor kernels

NSC

Theorem (Lr' 24)

Assume s *is simple. For any* $k \in \mathbb{N}$, *the following are equivalent*

- s *is linearly* k*-rigid on some compact* A *with non-empty interior*
- s *is linearly* k*-rigid on all compact* A *with non-empty interior*
- s *is linearly* k'*-rigid for* $0 \leqslant k' \leqslant k$ *on all compact* A *with non-empty interior*
- s *has a zero of order* k *in* 0.

- **DPPs:** $\mathsf{s}(u) = (1 - \widehat{\kappa^2})$ only vanishes in 0 and is simple (Riemann-Lebesgue's lemma).
- Implies in particular that $\mathscr{G}$ is not 1-rigid and and $\mathscr{Z}$ is not 2-rigid (**[Ghosh, Peres '17]**)
- **Remark:** $\mathsf{s} \leqslant \mathsf{s}'$ and s' is k-rigid (linearly) implies s is k-rigid

Consequences on quasicrystals

- In our language, a quasicrystal is a purely atomic tempered measure $\mathbf{P}$ which Fourier transform $\mathcal{F}\mathbf{P}$ in the sense of distributions is purely atomic with a dense set of atoms
- **[Bjorklund, Hartnick '24]** study the class of *cut-and-project* processes, stationary point processes: *Are there cut-and-project processes which are not number rigid ?*
- $\Rightarrow$ Yes: all point processes for which the continuous part of the spectral measure vanishes is k-rigid for any k (they are **maximally rigid**)

Consequences on Coulomb systems?

- Sine_β processes are linearly number rigid **[Chaibi, Najnudel '18]** hence
$$\int_{\mathbb{R}} \frac{1}{\mathsf{s}(u)} du = \infty.$$
but not 1-rigid **[Dereudre et al. '21]**.
- $2D$ Riesz gases are not number rigid **[Dereudre, Vasseur '21]**.

Proposition

- *Let $\mathbf{P}$ (weakly) HU stationary isotropic point process which is not number rigid in dimension $d = 1, 2$: then*
$$\int \|t\|^d |\mathscr{C}|(dt) = \infty$$
- *Let $\mathbf{P}$ (weakly) HU in dimension 1 which is not 1-rigid: then*
$$\int |t|^3 |\mathscr{C}|(dt) = \infty \ or \ \int t^2 \mathscr{C}(dt) \neq 0$$

A p-rigid process

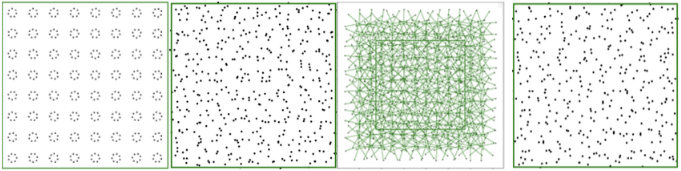
- Let $\mathbf{U}_p \subset \mathbb{C}$ the set of p-th roots of unity ($p \in \mathbb{N}^*$)
- Let $r > 0$ and

$$\mathbf{P}_p = \{k + U + r \underbrace{\theta_k}_{i.i.d.rotations} \mathbf{U}_p, k \in \mathbb{Z}^2\}$$

- If p is a prime number, the structure factor satisfies

$$\mathsf{s}(u) \leqslant c\|u\|^{2p}$$

- $\Rightarrow$ $\mathbf{P}_p$ is p-rigid but not $(p+1)$-rigid on $B(0, r+1)$.



$r = 0.2, p = 7$ $r = 1.3, p = 7$ Making of $r = 1.3, p = 6$

Szëgo and Kolmogorov theorems on time series

Let $\mathsf{X}_k, k \in \mathbb{Z}$ a stationary process, assume

$$\mathscr{C}(k) := \mathrm{Cov}\,(\mathsf{X}_0, \mathsf{X}_k) \in L^1,$$

$$\mathsf{s}(u) := \hat{\mathscr{C}}(u) = \sum_k \mathscr{C}(k) e^{-iuk}$$

[Szegö '21]: X is *predictable*, i.e.

$$\mathsf{X}(0) \in \sigma(\mathsf{X}(k), k < 0)$$

if s has a "very deep zero", i.e.

$$\int_{\mathbb{T}} \log |\mathsf{s}(u)| du = -\infty.$$

[Kolmogorov '41]: $X(0) \in \sigma(X(k), k \neq 0)$ if s has a "weak zero" (HU):

$$\int_{\mathbb{T}} \mathsf{s}(u)^{-1} du = \infty.$$

Proof of Kolmogorov's result

For a linear statistic

$$I_\gamma := \sum_k \gamma_k \mathsf{X}_k,$$

the variance is

$$var(I_\gamma) = \int_{\mathbb{T}} |\hat{\gamma}(u)|^2 \mathsf{s}(u) du.$$

Then $\mathsf{X}(0) = I_{\delta_0} = \lim_n I_{\gamma_n}$ a.s. and in L^2 for some γ_n vanishing on $\{0\}$ if and only if

$$0 = \inf_{\gamma:\gamma(0)\neq 0} \int |\underbrace{1}_{\hat{\delta}_0} - \hat{\gamma}(u)|^2 \mathsf{s}(u) du \Leftrightarrow 1 \in H := \text{ span }_{L^2(\mathsf{s})}(\hat{\gamma} : \gamma(0) = 0)$$

$1 \in H \Leftrightarrow \langle 1, \varphi \rangle_{L^2(\mathsf{s})} = 0$ for all $\varphi \in H^\perp$.

$$\varphi \in H^\perp \Leftrightarrow \int \hat{\gamma}(u)\varphi(u)\mathsf{s}(u)du = 0 \text{ for } \gamma(0) = 0$$
$$\Leftrightarrow \text{ spectrum}(\varphi\mathsf{s}) \subset \{0\}.$$
$$\Leftrightarrow \varphi\mathsf{s} = c.$$

Assume $\varphi := 1/\mathsf{s} \in L^2(s)$: no orthogonality:

$$\langle 1, \varphi \rangle_{L^2(\mathsf{s})} = \int_{\mathbb{T}} 1 \frac{1}{s} \mathsf{s}(u)du = \int_{\mathbb{T}} 1 du \neq 0.$$

Therefore, $1 \in H^\perp$ iff $1/s \notin L^2(\mathsf{s})$, iff

$$\int \mathsf{s}(u)^{-1} du = \int \frac{1}{\mathsf{s}(u)^2} \mathsf{s}(u) du = \infty.$$

Generalisation

Proposition (Lr' 24)

- $\{\mathsf{X}_k, k \in \mathbb{Z}^d\}$ *is maximally rigid on* $\{-m, \dots, m\}^d$ *iff there is no trigonometric polynomial* φ *of order* m *in* $L^2(\mathsf{s}^{-1})$
- *It is* k*-rigid if all such* φ *satisfy* $\partial^p \varphi = 0$ *for* $|p| = k$

Related to **[Lyons, Steif '03]**

Remark: There exists X which is 1-rigid but not 0-rigid on $A = \{-1, 0, 1\}$ in dimension $d = 1$ (not possible for "simple" s in the continuous setting)

Number rigidity of a point process

Linear number rigidity on $A = B(0,1)$

$$\Leftrightarrow \inf_{\gamma \subset A^c} \mathrm{Var}\left(I_{\mathbf{P}}(1_A) - I_{\mathbf{P}}(\gamma)\right) = 0$$

$$\Leftrightarrow \inf_{\gamma \subset A^c} \int |\widehat{1_A} - \hat{\gamma}|^2 d\mathcal{S} = 0$$

$$\Leftrightarrow \langle \widehat{1_A}, \varphi \rangle_{L^2(\mathcal{S})} = 0 \text{ for all } \varphi \in H^\perp$$

where $H = \overline{\mathrm{span}}_{L^2(\mathcal{S})}(\hat{\gamma} : \gamma \subset A^c)$: for $\varphi \in L^2(\mathsf{s})$

$$\varphi \in H^\perp \Leftrightarrow \int \hat{\gamma}(u)\varphi(u)\mathcal{S}(du) = 0 \text{ for } \gamma \subset A^c$$

$$\Leftrightarrow \text{spectrum}(\varphi\mathcal{S}) \subset A$$

$$\Leftrightarrow \varphi\mathcal{S} = \varphi\mathsf{s} =: \psi \text{ is analytic of type 1} \quad \text{(Schwartz Paley Wiener)}$$

Number rigidity of a point process

- We proved that **number rigidity** on A is equivalent to the fact that $\widehat{1_A}$ is orthogonal to analytic functions ψ of type 1 such that $\psi = \varphi \mathsf{s}$ for some $\varphi \in L^2(\mathsf{s})$.
- For such functions: $\widehat{\psi} \subset A$

$$\psi(0) = \int \widehat{\psi} = \int \widehat{\psi} 1_A = 0.$$

- Finally, number rigidity is equivalent to the fact that for all $\psi \in L^2(\mathsf{s}^{-1})$, $\psi(0) = 0$. This is the case if s^{-1} is not integrable around 0 (HU)
- $\varphi = \psi/\mathsf{s} \in L^2(\mathsf{s}) \;\Leftrightarrow\; \psi \in L^2(\mathsf{s}^{-1})$ and ψ is analytic of type 1.
- **Converse:** If $\int \frac{1}{\mathsf{s}(u)} du < \infty$, we can find $\psi \in L^2(\mathsf{s}^{-1})$ not vanishing in 0.

Maximal rigidity and stealthy processes

- For number rigidity, we had to investigate what are the functions of $L^2(\mathsf{s}^{-1})$ orthogonal to $\widehat{1_A}$.
- Maximal rigidity $\mathbf{P}_A \in \sigma(\mathbf{P}_{A^c})$ occurs when there are no analytic functions in $L^2(\mathsf{s}^{-1})$
- If s has a spectral gap (**Stealthy processes**), φ needs to vanish on a gap $\Rightarrow \ \varphi \equiv 0$, we have maximal rigidity. **[Ghosh, Lebowitz '18]**
- If the zero set of s has an accumulation point in dimension $d = 1$, or in higher dimensions if it has non-zero measure, then $\varphi \equiv 0$.
- A stealthy process (i.e. having a spectral gap) is maximally rigid on A non-bounded, more precisely for A a "minor cone"

Random fields

- Let a "completely standard" stationary random field $F(x)$ having covariance

$$\mathscr{C}(x) = \mathbf{E}(F(0)F(x)) = 1_{B(0,1)} \star 1_{B(0,1)}(x)$$

  - Linear variance (not HU)
  - Continuous (can be made $\mathcal{C}^k$ for arbitrary k)
  - Small range

  Phase transition: Then there is maximal rigidity on $A = B(0, \rho)$ if and only if $\rho < \rho_c$ (otherwise there is not even number rigidity).

- Relies on Jensen's identity: the zeros of a complex analytic function of exponential type 1 cannot have a density $> 1/\rho_c$.