Percolation of random fields excursions

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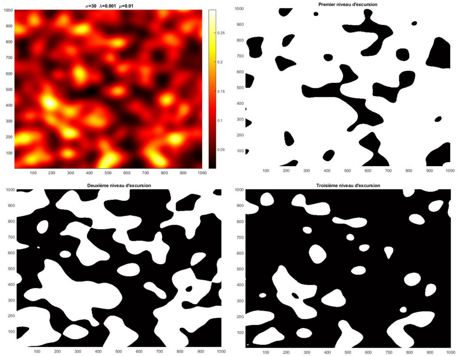
Excursions

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ a stationary random field (law invariant under translations).
- For $\ell \in \mathbb{R}$, define

$$\mathcal{E}_\ell = \mathcal{E}_\ell(f) = \{x \in \mathbb{R}^2 : \ f(x) \geqslant \ell\}$$

Figure – Excursions of a shot noise field (Credit : *PhD Thesis, Antoine Lerbet*)



Percolation

We are interested in the following questions :

1. Does $\mathcal{E}_\ell$ have (a unique) unbounded connected component(s) ?
2. Is there a critical value ℓ_c ?
3. Behaviour of

$$\mathbb{P}(\mathcal{E}_\ell \text{ crosses large rectangles})$$

for $\ell = \ell_c$ or $\ell \neq \ell_c$?

Poisson shot noise fields

- Let $\mathcal{P} = \{x_i;\ i \in \mathbb{N}\}$ be a homogeneous Poisson process on $\mathbb{R}^2$.
- Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ integrable
- **Poisson shot noise field** with kernel g :

$$f(x) := \sum_{i \in \mathbb{N}} g(x - x_i); x \in \mathbb{R}^2.$$

- Let $Y_i, i \in \mathbb{N}$ iid **symmetric** integrable variables with law μ.
- **Symmetric Poisson shot noise field** with kernel g and mark distribution μ :

$$f(x) = \sum_{i \in \mathbb{N}} Y_i g(x - x_i); x \in \mathbb{R}^2.$$

- Well defined in virtue of Campbell formula :

$$\mathbb{E}\left[\sum_{i \in \mathbb{N}} |Y_i g(x - x_i)|\right] = \int_{\mathbb{R}^2} |yg(x - t)| dt \mu(dy) = \mathbb{E}(|Y_1|)\|g\|_{L^1} < \infty$$

Gaussian Random Fields

The same questions have been thouroughly investigated for stationary continuous centred **Gaussian fields**, i.e. random functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

- $\forall x_1, \ldots, x_n \in \mathbb{R}^2, (f(x_1), \ldots, f(x_n))$ is a **centred Gaussian vector**
- a.s., $x \to f(x)$ is continuous
- Such a field is uniquely determined by its covariance function

$$\mathbb{E}(f(x)f(y)) =: C(x-y).$$

- Reciprocally, to each SDP function C, i.e. such that

$$\sum_{i=1}^{n} a_i a_j C(x_i - x_j) \geqslant 0$$

for all $x_1, \ldots, x_n \in \mathbb{R}^d, a_1, \ldots, a_n \in \mathbb{R}$, one can associate a unique centred stationary Gaussian field.

White noise construction

Most fields can actually be seen as the convolution of a kernel $g \in L^1(\mathbb{R}^d)$ with a **white noise** $\mathcal{W}$

$$f(x) = g \star \mathcal{W}(x) := \int g(x-y)d\mathcal{W}(y)$$

- $\mathcal{W}$: **random signed measure** satisfying for A, B disjoint
  - $\mathcal{W}(A)$ and $\mathcal{W}(B)$ are independent
  - $\mathcal{W}(A \cup B) = \mathcal{W}(A) + \mathcal{W}(B)$
  - $\mathrm{Var}(\mathcal{W}(A)) = \mathcal{L}^d(A)$
- **Poisson shot noise fields** : $\mathcal{W}_{\mathcal{P}}(A) := \#\mathcal{P} \cap A \sim \mathrm{Poiss}(\mathcal{L}^d(A))$
- **Gaussian fields** : $\mathcal{W}_{\mathcal{G}}(A) \sim \mathcal{N}(0, \mathcal{L}^d(A))$
- In **dimension** 1, the Gaussian white noise can be built from a **Brownian motion** $\{B_t; t \in \mathbb{R}\}$,

$$\mathcal{W}_{\mathcal{G}}([a,b]) := B_b - B_a.$$

- Similar constructions exist in all dimensions with Brownian sheets

Covariance property

- For A, B with finite measure,

$$\mathrm{Cov}(\mathcal{W}(A), \mathcal{W}(B)) = \mathcal{L}^d(A \cap B) = \langle \mathbf{1}_{\{A\}}, \mathbf{1}_{\{B\}} \rangle_{L^2(\mathbb{R}^d)}$$

- For all $g_1, g_2 \in L^2(\mathbb{R}^d)$

$$\mathrm{Cov}\left(\int g_1 d\mathcal{W}, \int g_2 d\mathcal{W}\right) = \langle g_1, g_2 \rangle = \int g_1 g_2.$$

- In particular, the **covariance function of** f satisfies

$$\begin{aligned} C(x-y) = \mathrm{Cov}(f(x), f(y)) =& \langle g(x-\cdot), g(y-\cdot), \rangle \\ =& \int g(x-y)g(x-y-z)dz \\ =& g \tilde{\star} g(x-y) \end{aligned}$$

- Some SDP functions with singular spectral measures cannot be built this way (e.g. Gaussian Random Planar Wave with C =Bessel Function)

Percolation of Gaussian excursions

Figure – Credit : D. Beliaev

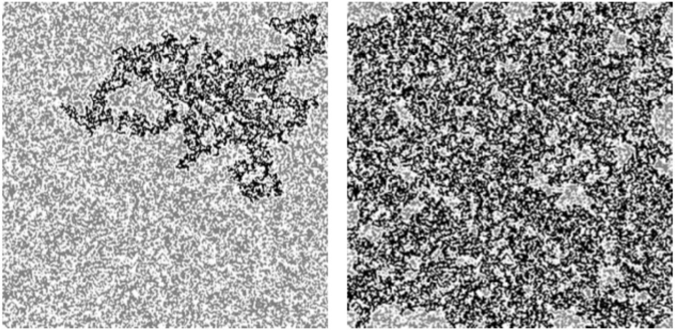


FIGURE 1. A simulation of the excursion set $\mathcal{E}_\ell$ of the *Bargmann-Fock* field restricted to a large square (in grey) at (i) the zero level $\ell = 0$ (left figure), at (ii) the level $\ell = 0.1$ (right figure), with the connected component of greatest area distinguished (in black). The Bargmann-Fock field is the stationary, centred Gaussian field with covariance kernel $\kappa(x) = e^{-|x|^2/2}$. Credit: Dmitry Beliaev.

Assumptions (Gaussian case)

Assumption	**Field** f	**Kernel** g
Regularity	$\mathcal{C}^3$	$\mathcal{C}^3$
Symmetry	D^4 $(Axis\ reflections, \frac{\pi}{2}-rotations)$	D^4
Positive Association	for A,B increasing events $\mathbb{P}(A\cap B)\geqslant \mathbb{P}(A)\mathbb{P}(B)$	$g\geqslant 0$
Asymptotic Independence	for A, B "far away" $\mathbb{P}(A\cap B)\approx \mathbb{P}(A)\mathbb{P}(B)$	for some $\beta{>}2$ $g(x)\leqslant c(1+\|x\|)^{-\beta}$

- **Increasing event** $A = A(f) : \mathbf{1}_{\{A(f)\}} \leqslant \mathbf{1}_{\{A(g)\}}$ for $f\leqslant g$

$$\text{Example : } A(f) = \{\mathcal{E}_\ell(f) \text{ crosses } Q\} \text{ for some } Q\subset \mathbb{R}^2$$

- **Symmetry** $f \overset{(d)}{=} -f$ entails **self-duality**

$$\mathcal{E}_0 \overset{(d)}{=} \mathcal{E}_0^c \qquad \text{(up to the boundary)}$$

$\Rightarrow$ It is natural to expect $\ell_c = 0$.

Bernoulli-like percolation (Gaussian case)

Theorem (Sharp phase transition (Beffara & Gayet, Vanneuville, Muirhead, Ribera))

Under the previous assumptions, $\{\mathcal{E}_\ell, \ell \in \mathbb{R}\}$ *behaves like* **Bernoulli percolation** *around the critical value : for* Q *a rectangle*

1. $\ell < 0$: $\mathcal{E}_\ell$ *has a unique unbounded component a.s. and*

$$\mathbb{P}(\mathcal{E}_\ell \textit{ crosses } rQ) > 1 - Ce^{-cr}, r > 0$$

2. $\ell > 0$: $\mathcal{E}_\ell$ *has bounded components a.s.*
3. $\ell = 0$: $\mathcal{E}_\ell$ *has bounded components and*

$$\mathbb{P}(\mathcal{E}_0 \textit{ crosses from } \partial B(0,r) \textit{ to } \partial B(0,R)) \leqslant c\left(\frac{r}{R}\right)^{\beta_{arm}}, r > 0.$$

$$0 < \inf_r \mathbb{P}(\mathcal{E}_0 \textit{ crosses } rQ) \leqslant \sup_r \mathbb{P}(\mathcal{E}_0 \textit{ crosses } rQ) < 1.$$

Early works

- **Molchanov and Stepanov '83** : give conditions for $\ell_c < \infty$ for some positive shot noise fields
- **Alexander '96** : For a stationary $\mathcal{C}^1$ random field on $\mathbb{R}^2$, ergodic and positively associated, the level lines are a.s. bounded.
- **Broman and Meester '17** : Conditions for $\ell_c < \infty$
- **Beffara Gayet '17** : Bounded components for ℓ sufficiently large
- **Ribera Vanneuville '19** : Bounded components for $\ell > 0$
- **Muirhead Vanneuville '19 :** Optimal condition $\beta > 2$ on decay of g, sharp phase transition
- **Muirhead, Rivera, Vanneuville '20** : Results without positive association and fast decay outside the critical level

Assumptions (Symmetric Poisson case)

Regularity	$\mathcal{C}^3$	$\mathcal{C}^3$
Symmetry	D^4 $(Axis\ reflections, \frac{\pi}{2}-rotations)$	D^4
Positive Association	for A,B increasing events $\mathbb{P}(A \cap B) \geqslant \mathbb{P}(A)\mathbb{P}(B)$	$g \geqslant 0$
Asymptotic Independence	for A, B "far away" $\mathbb{P}(A \cap B) \approx \mathbb{P}(A)\mathbb{P}(B)$	for some $\beta>3$, for $\lvert k\rvert \leqslant 3$ $\partial^k g(x) \leqslant c(1+\lVert x\rVert)^{-\beta-\lvert k\rvert}$
Self-Duality	$\mathcal{E}_0 \stackrel{(d)}{=} \overline{\mathcal{E}_0^c}$	$Y_i \stackrel{(d)}{=} -Y_i$
Density	$(f(0),\nabla f(0))$ has bounded joint density	$g(x)=c\exp(-\lVert x\rVert^\alpha)$, $\alpha\in(0,1)$ or $g(x)=c(1+\lVert x\rVert)^{-\beta}$, $\beta>d$
Concentration +	Use of OSSS inequality	Law of Y_i log-concave

Theorem (Lr,Muirhead 2022)

Under these assumptions, there is Bernoulli-like percolation for Poisson shot noise fields.

Non-symmetric case

- Let $\lambda > 0$, $\mathcal{P}_\lambda \stackrel{(d)}{=} \lambda^{-1/d}\mathcal{P}$ a Poisson homogeneous process with intensity λ. We consider

$$f_\lambda(x) = g \star \mathcal{W}_{\mathcal{P}_\lambda}(x) = \sum_{y \in \mathcal{P}_\lambda} g(y - x).$$

- Under mild assumptions, there is a finite **critical density**

$$\ell_c(f_\lambda) = \sup\{\ell : \mathbb{P}(\mathcal{E}_\ell \text{ has unbounded component}) > 0\} < \infty$$

- **Asymptotic regime** $\lambda \to \infty$? Elementary Central Limit Theorem

$$\tilde{f}_\lambda(x) := \frac{f_\lambda(x) - \mathbb{E}(f_\lambda(x))}{\sqrt{\mathrm{Var}(f_\lambda(x))}} \to G(x) \text{ with } \begin{cases} \mathbb{E}(f_\lambda(x)) = \lambda \int g, \\ \mathrm{Var}(f_\lambda(x)) = \lambda \int g^2 \end{cases}$$

- **Multivariate CLT** (Heinrich, Schmidt '85) : Convergence of FDD
- $G(x)$ is Gaussian centred with **same covariance** $g \tilde{\star} g$
- Question :

$$\ell_c(\tilde{f}_\lambda) \to \ell_c(G) = 0?$$

Assumptions (Non-symmetric Poisson case)

Assumption	**Field** f	**Kernel** g
Regularity	$\mathcal{C}^4$	$\mathcal{C}^4$
Symmetry	Isotropy (invariance to rotations)	Isotropy
Positive Association	for A,B increasing events $\mathbb{P}(A \cap B) \geqslant \mathbb{P}(A)\mathbb{P}(B)$	$g \geqslant 0$
Asymptotic Independence	for A, B "far away" $\mathbb{P}(A \cap B) \approx \mathbb{P}(A)\mathbb{P}(B)$	for some $\beta > 2$, for $\lvert k \rvert \leqslant 3$ $\partial^k g(x) \leqslant c(1 + \lVert x \rVert)^{-\beta}$
Density	$(f_\lambda(0), \nabla f_\lambda(0))$ has bounded density	$g(x) = c \exp(-\lVert x \rVert^\alpha)$, $\alpha \in (0,1)$ or $g(x) = c(1 + \lVert x \rVert)^{-\beta}$, $\beta > d$

Critical value approximation

Theorem (Lr,Muirhead 21+)

Recall

$$\ell_c(\tilde{f}_\lambda) = c\lambda^{-1/2}(\ell_c(f_\lambda) - \lambda \int g)$$

Assume the previous hypotheses, except positive association. Then

- *without positive association,*

$$\ell_c(\tilde{f}_\lambda) \to 0$$

- *with positive association ($g \geqslant 0$),*

$$\ell_c(\tilde{f}_\lambda) = O(\lambda^{-1/2} \log(\lambda)^{3/2})$$

Strong Invariance principles

- Proof based on the **construction of a coupling** (f_λ, g), for each $\lambda > 0$.
- **Historical result :** Komlos, Major, Tusnady 85', coupling of X_i, i.i.d

Rademacher variables with i.i.d Gaussian variables $G_1, \ldots, G_n$ such that

$$\mathbb{P}(\sup_{0 \leqslant k \leqslant n} |\sum_{i=1}^{k} X_i - \sum_{i=1}^{k} G_i| \geqslant c \ln(n) + t) \leqslant Ce^{-ct}$$

and the order $\ln(n)$ is optimal.

- **"Random measure"** point of view

$$\sum_{i=1}^{k} X_i =(\sum_{i=1}^{n} \delta_{X_i})(\mathbf{1}_{[1,\ldots,k]}), 1 \leqslant k \leqslant n$$

$$\text{Similarly } f_\lambda(\mathbf{k}) = \mathcal{W}_{\mathcal{P}_\lambda}(g(\mathbf{k} - \cdot)), \mathbf{k} \in \mathbb{Z}^d$$

$$G(\mathbf{k}) = \mathcal{W}_{\mathcal{G}}(g(\mathbf{k} - \cdot)), \mathbf{k} \in \mathbb{Z}^d$$

Strong invariance principle for shot noise fields

Theorem (Lr,Muirhead 21+)

$$\mathbb{P}\left(\sup_{x\in B(0,R)} |\tilde{f}_\lambda(x) - G(x)| > \lambda^{-1/2}\ln(\lambda)^{1/2}t\right) < CR^d\lambda^c\exp(-ct)$$

- Optimal up to the power of $\ln(\lambda)$ (see also Berry-Esseen inequality)
- Based on Koltchinski 94' : There is a coupling of $\mathcal{P}_\lambda$ and $\mathcal{W}_\mathcal{G}$ such that for any $\mathbf{k}\in\mathbb{Z}^d$,

$$\mathbb{P}(|\tilde{f}_\lambda(\mathbf{k}) - G(\mathbf{k})| \geqslant t\lambda^{-1/2}\ln(\lambda)) \leqslant Ce^{-ct}$$

- For $x\in\mathbb{R}^d\setminus\mathbb{Z}^d$, approximate $f(x)$ by $f([x]) + \nabla f(\xi)\cdot(x-[x])$.
- There is a coupling of $N\sim\text{Pois}(\lambda)$ and $Z\sim\mathcal{N}(0,1)$ such that

$$\mathbb{P}(|N-\lambda-\sqrt{\lambda}Z| > t) \leqslant Ce^{-ct}$$

.

Elements of proof for the symmetric case

- Box crossing estimates (RSW) stem from the work of Tassion '16 because we have :
  - Positive association of the discretised field (FKG inequality on a finite space)
  - $\mathcal{E}_0$ is invariant in law under reflections and rotation by $\pi/2$
  - Spatial asymptotic independence (of f, hence of $\mathcal{E}_\ell$)
- One arm decay stems from
  - Positive association of the discretised field (FKG inequality on a finite space)
  - Asymptotic independence
  - Box crossing estimates (RSW)

Proof of sharp phase transition (bounded Mills ratio case)

1. First prove that $\mathbb{P}(\mathrm{Cross}_\ell(2R, R)) \to 0$ and then use bootstraping argument
2. Proof based on a differential inequality of

$$\theta : h \to \mathbb{P}(f_r^{\varepsilon,h} \in \mathrm{Cross}_\ell(2R, R))$$

   where $f_r^{\varepsilon,h}$ is obtained from f_r^ε by adding h to all the marks. We prove

$$\frac{\partial}{\partial h}\theta(h) \geqslant c \frac{\theta(h)(1-\theta(h))}{\inf_{2r<\rho<R/2}\{2\rho/R + \mathbb{P}(f_r^\varepsilon \in \mathrm{Arm}_\ell(2r, \rho))\}}$$

3. Use of the OSSS inequality applied to randomized algorithms ; after the ideas of Duminil-Copin, Tassion, Raoufi.

Sharp phase transition

Theorem (Lr & Muirhead 19+)

For $\ell > 0$ there is $c > 0$ such that

$$\mathbb{P}(\mathrm{Cross}_\ell(2R, R)) \leqslant 1 - \exp(-cR), R > 0$$

It implies the main result :

- For $\ell \geqslant 0$, $\mathcal{E}_\ell$ has only bounded connected components a.s..
- For $\ell < 0$, $\mathcal{E}_\ell$ has a unique unbounded component a.s..

Proof : $\bullet \ell \geqslant 0$: $\mathbb{P}(\mathrm{Arm}_0(1, R)) \to 0$.

$\bullet \ell < 0$: Borel-Cantelli lemma with

$$\sum_{k \geqslant 1} (1 - \mathbb{P}(\mathrm{Cross}_\ell(2^{k+1}, 2^k))) < \infty \Rightarrow (\mathrm{Cross}_\ell(2^{k+1}, 2^k)) \text{ occurs for } k > k_0$$

and arrange the rectangles so that the connected components overlap.

OSSS inequality (O'Donnell, Saks, Schramm, Servedio '05)

For an event A on a product probability space (E^n, μ^n) and a random algorithm determining A, $(\theta := \mathbb{P}(A))$

$$\mathrm{Var}(\mathbf{1}_{\{A\}}) = \theta(1-\theta) \leqslant \sum_{i=1}^{n} \delta_i^\mu(\mathcal{A}) I_i^\mu(A)$$

where

- $\delta_i^\mu(\mathcal{A})$: Probability that coordinate i is revealed by the algorithm
- Influence of coordinate i : $I_i^\mu(A) = \mathbb{P}(\mathbf{1}_{\{A\}} \neq \mathbf{1}_{\{A^i\}})$ where A^i is obtained by resampling coordinate i

For percolation events, typically :

- $\mathcal{A}$ is a progressive uncovering of all the connected components touching a random crossing line (in a rectangle) / circle (in a disc)
- $\delta_i^\mu(\mathcal{A})$ is the probability that a point i is "close" to one of these connected components (one-arm decay is useful here)
- $I_i^\mu(A)$ is related to $\partial_h \theta(h)$ for $h \sim 0$

Key point

- First remark that crossing events are monotonous in the marks (higher mark = more chances to percolate). Hence for each i there is a.s. a random level y_i such that there is percolation for $Y_i \geqslant y_i$.
- Assume for f^ε that mark Y_i is replaced by $Y_i + h_i$ for some parameter $h_i \in \mathbb{R}$. Then

$$\begin{aligned}
\frac{\partial}{\partial h_i}\mathbb{P}(\mathrm{Cross}_\ell(2R,R)) =& \frac{\partial}{\partial h_i}\mathbb{P}(Y_i + h_i \geqslant y_i) \geqslant u_{\mu_{ac}}(y_i - h_i) \\
I_i = \mathbb{P}(\mathbf{1}_{\{A\}} \neq \mathbf{1}_{\{A_i\}}) =& \mathbb{P}(Y_i + h_i \geqslant y_i, Y_i' + h_i < Y_i') \\
& + \mathbb{P}(Y_i + h_i < y_i, Y_i' + h_i \geqslant Y_i') \\
\leqslant & 2\mathbb{P}(Y_i \geqslant y_i - h_i)
\end{aligned}$$

$$\overset{\text{Mills}}{\leqslant} c u_{\mu_{ac}}(y_i - h_i)$$

- We end up with

$$\frac{\partial}{\partial h}\theta(h) = \sum_i \frac{\partial}{\partial h_i}\theta(h) \geqslant c \sum_i I_i \geqslant c\frac{\sum_i I_i \delta_i}{\sup_i \delta_i} \overset{OSSS}{=} \frac{\theta(1-\theta)}{\sup_i \delta_i}$$