Percolation of random fields excursions

Raphaël Lachièze-Rey, Univ. Paris Cité, MAP5 UMR CNRS
Joint work with Stephen Muirhead, University of Melbourne
Excursions

- Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) a stationary random field (law invariant under translations).
- For \( \ell \in \mathbb{R} \), define

\[
\mathcal{E}_\ell = \mathcal{E}_\ell(f) = \{ x \in \mathbb{R}^2 : f(x) \geq \ell \}
\]

Figure – Excursions of a shot noise field (Credit: PhD Thesis, Antoine Lerbet)
We are interested in the following questions:

1. Does $\mathcal{E}_\ell$ have (a unique) unbounded connected component(s)?
2. Is there a critical value $\ell_c$?
3. Behaviour of

$$\mathbb{P}(\mathcal{E}_\ell \text{ crosses large rectangles})$$

for $\ell = \ell_c$ or $\ell \neq \ell_c$?
Poisson shot noise fields

- Let $\mathcal{P} = \{x_i; \ i \in \mathbb{N}\}$ be a homogeneous Poisson process on $\mathbb{R}^2$.
- Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ integrable
- **Poisson shot noise field** with kernel $g$:

  $$f(x) := \sum_{i \in \mathbb{N}} g(x - x_i); x \in \mathbb{R}^2.$$ 

- Let $Y_i, i \in \mathbb{N}$ iid **symmetric** integrable variables with law $\mu$.
- **Symmetric Poisson shot noise field** with kernel $g$ and mark distribution $\mu$:

  $$f(x) = \sum_{i \in \mathbb{N}} Y_i g(x - x_i); x \in \mathbb{R}^2.$$ 

- Well defined in virtue of Campbell formula:

  $$\mathbb{E} \left[ \sum_{i \in \mathbb{N}} |Y_i g(x - x_i)| \right] = \int_{\mathbb{R}^2} |yg(x - t)| dt \mu(dy) = \mathbb{E}(|Y_1|)\|g\|_{L^1} < \infty$$
Gaussian Random Fields

The same questions have been thoroughly investigated for stationary continuous centred Gaussian fields, i.e. random functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

- $\forall x_1, \ldots, x_n \in \mathbb{R}^2, (f(x_1), \ldots, f(x_n))$ is a centred Gaussian vector
- a.s., $x \to f(x)$ is continuous
- Such a field is uniquely determined by its covariance function $\mathbb{E}(f(x)f(y)) =: C(x - y)$.

- Reciprocally, to each SDP function $C$, i.e. such that
  \[ \sum_{i=1}^{n} a_i a_j C(x_i - x_j) \geq 0 \]
  for all $x_1, \ldots, x_n \in \mathbb{R}^d, a_1, \ldots, a_n \in \mathbb{R}$, one can associate a unique centred stationary Gaussian field.
White noise construction

Most fields can actually be seen as the convolution of a kernel $g \in L^1(\mathbb{R}^d)$ with a **white noise** $\mathcal{W}$

\[
f(x) = g \ast \mathcal{W}(x) := \int g(x - y)d\mathcal{W}(y)
\]

- $\mathcal{W}$: **random signed measure** satisfying for $A, B$ disjoint
  - $\mathcal{W}(A)$ and $\mathcal{W}(B)$ are independent
  - $\mathcal{W}(A \cup B) = \mathcal{W}(A) + \mathcal{W}(B)$
  - $\text{Var}(\mathcal{W}(A)) = \mathcal{L}^d(A)$

- **Poisson shot noise fields**: $\mathcal{W}_P(A) := \# \mathcal{P} \cap A \sim \text{Poiss}(\mathcal{L}^d(A))$

- **Gaussian fields**: $\mathcal{W}_G(A) \sim \mathcal{N}(0, \mathcal{L}^d(A))$

- In **dimension 1**, the Gaussian white noise can be built from a **Brownian motion** $\{B_t; t \in \mathbb{R}\}$,

\[
\mathcal{W}_G([a, b]) := B_b - B_a.
\]

- Similar constructions exist in all dimensions with Brownian sheets
Covariance property

• For $A, B$ with finite measure,

\[ \text{Cov}(\mathcal{W}(A), \mathcal{W}(B)) = \mathcal{L}^d(A \cap B) = \langle 1_A, 1_B \rangle_{L^2(\mathbb{R}^d)} \]

• For all $g_1, g_2 \in L^2(\mathbb{R}^d)$

\[ \text{Cov}\left( \int g_1 d\mathcal{W}, \int g_2 d\mathcal{W} \right) = \langle g_1, g_2 \rangle = \int g_1 g_2. \]

• In particular, the covariance function of $f$ satisfies

\[ \text{C}(x - y) = \text{Cov}(f(x), f(y)) = \langle g(x - \cdot), g(y - \cdot), \rangle \]

\[ = \int g(x - y)g(x - y - z)dz \]

\[ = \tilde{g} \ast g(x - y) \]

• Some SDP functions with singular spectral measures cannot be built this way (e.g. Gaussian Random Planar Wave with $C = \text{Bessel Function}$)
Figure 1. A simulation of the excursion set $\mathcal{E}_\ell$ of the Bargmann-Fock field restricted to a large square (in grey) at (i) the zero level $\ell = 0$ (left figure), at (ii) the level $\ell = 0.1$ (right figure), with the connected component of greatest area distinguished (in black). The Bargmann-Fock field is the stationary, centred Gaussian field with covariance kernel $\kappa(x) = e^{-|x|^2/2}$. Credit: Dmitry Beliaev.
### Assumptions (Gaussian case)

<table>
<thead>
<tr>
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<th>Field $f$</th>
<th>Kernel $g$</th>
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<td>Regularity</td>
<td>$C^3$</td>
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<td>Symmetry</td>
<td>$D^4$ (Axis reflections, $\frac{\pi}{2}$−rotations)</td>
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- **Increasing event** $A = A(f) : 1\{A(f)\} \leq 1\{A(g)\}$ for $f \leq g$

  Example: $A(f) = \{\mathcal{E}_\ell(f) \text{ crosses } Q\}$ for some $Q \subset \mathbb{R}^2$

- **Symmetry** $f \overset{(d)}{=} -f$ entails self-duality

\[
\mathcal{E}_0 \overset{(d)}{=} \mathcal{E}_0^C \quad \text{(up to the boundary)}
\]

$\Rightarrow$ It is natural to expect $\ell_c = 0$. 
Bernoulli-like percolation (Gaussian case)

Theorem (Sharp phase transition (Beffara & Gayet, Vanneuville, Muirhead, Ribera))

Under the previous assumptions, \( \{ \mathcal{E}_\ell, \ell \in \mathbb{R} \} \) behaves like Bernoulli percolation around the critical value: for \( Q \) a rectangle

1. \( \ell < 0 \): \( \mathcal{E}_\ell \) has a unique unbounded component a.s. and
   \[
   \mathbb{P}(\mathcal{E}_\ell \text{ crosses } rQ) > 1 - C e^{-cr}, \, r > 0
   \]

2. \( \ell > 0 \): \( \mathcal{E}_\ell \) has bounded components a.s.

3. \( \ell = 0 \): \( \mathcal{E}_\ell \) has bounded components and
   \[
   \mathbb{P}(\mathcal{E}_0 \text{ crosses from } \partial B(0, r) \text{ to } \partial B(0, R)) \leq c \left( \frac{r}{R} \right)^{\beta_{\text{arm}}}, \, r > 0.
   \]

\[ 0 < \inf_r \mathbb{P}(\mathcal{E}_0 \text{ crosses } rQ) \leq \sup_r \mathbb{P}(\mathcal{E}_0 \text{ crosses } rQ) < 1. \]
Early works

- **Molchanov and Stepanov ’83**: give conditions for $\ell_c < \infty$ for some positive shot noise fields
- **Alexander ’96**: For a stationary $C^1$ random field on $\mathbb{R}^2$, ergodic and positively associated, the level lines are a.s. bounded.
- **Broman and Meester ’17**: Conditions for $\ell_c < \infty$
- **Beffara Gayet ’17**: Bounded components for $\ell$ sufficiently large
- **Ribera Vanneuville ’19**: Bounded components for $\ell > 0$
- **Muirhead Vanneuville ’19**: Optimal condition $\beta > 2$ on decay of $g$, sharp phase transition
- **Muirhead, Rivera, Vanneuville ’20**: Results without positive association and fast decay outside the critical level
### Assumptions (Symmetric Poisson case)

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<td><strong>Self-Duality</strong></td>
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<td><strong>Density</strong></td>
<td>$Y_i^{(d)} = \overline{Y_i^{(d)}}$</td>
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<td><strong>Concentration +</strong></td>
<td>Use of OSSS inequality</td>
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**Theorem (Lr,Muirhead 2022)**

*Under these assumptions, there is Bernoulli-like percolation for Poisson shot noise fields.*
Non-symmetric case

• Let $\lambda > 0$, $\mathcal{P}_\lambda \overset{(d)}{=} \lambda^{-1/d}\mathcal{P}$ a Poisson homogeneous process with intensity $\lambda$. We consider

$$f_\lambda(x) = g \star \mathcal{W}_{\mathcal{P}_\lambda}(x) = \sum_{y \in \mathcal{P}_\lambda} g(y - x).$$

• Under mild assumptions, there is a finite critical density

$$\ell_c(f_\lambda) = \sup\{\ell : \mathbb{P}(E_\ell \text{ has unbounded component}) > 0\} < \infty$$

• Asymptotic regime $\lambda \to \infty$? Elementary Central Limit Theorem

$$\tilde{f}_\lambda(x) := \frac{f_\lambda(x) - \mathbb{E}(f_\lambda(x))}{\sqrt{\text{Var}(f_\lambda(x))}} \to G(x) \text{ with } \begin{cases} \mathbb{E}(f_\lambda(x)) = \lambda \int g, \\ \text{Var}(f_\lambda(x)) = \lambda \int g^2 \end{cases}$$

• Multivariate CLT (Heinrich, Schmidt ’85): Convergence of FDD

• $G(x)$ is Gaussian centred with same covariance $g\tilde{x}g$

• Question:

$$\ell_c(\tilde{f}_\lambda) \to \ell_c(G) = 0?$$
## Assumptions (Non-symmetric Poisson case)

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<td>$g(x) = c \exp(-|x|^\alpha)$, $\alpha \in (0, 1)$ or $g(x) = c(1 + |x|)^{-\beta}$, $\beta &gt; d$</td>
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R. Lachieze-Rey

Percolation of random fields excursions
Critical value approximation

**Theorem (Lr, Muirhead 21+)**

Recall

\[ \ell_c(\tilde{f}_\lambda) = c \lambda^{-1/2}(\ell_c(f_\lambda) - \lambda \int g) \]

Assume the previous hypotheses, except positive association. Then

- **without positive association,**
  \[ \ell_c(\tilde{f}_\lambda) \to 0 \]

- **with positive association** \((g \geq 0),\)
  \[ \ell_c(\tilde{f}_\lambda) = O(\lambda^{-1/2} \log(\lambda)^{3/2}) \]
Strong Invariance principles

- Proof based on the **construction of a coupling** \((f_\lambda, g)\), for each \(\lambda > 0\).
- **Historical result**: Komlos, Major, Tusnady 85’, coupling of \(X_i\), i.i.d Rademacher variables with i.i.d Gaussian variables \(G_1, \ldots, G_n\) such that

\[
\mathbb{P}\left( \sup_{0 \leq k \leq n} \left| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} G_i \right| \geq c \ln(n) + t \right) \leq C e^{-ct}
\]

and the order \(\ln(n)\) is optimal.

- “**Random measure**” point of view

\[
\sum_{i=1}^{k} X_i = (\sum_{i=1}^{n} \delta_{X_i})(1_{[1, \ldots, k]}), 1 \leq k \leq n
\]

Similarly \(f_\lambda(k) = \mathcal{W}_{\mathcal{P}_\lambda}(g(k - \cdot)), k \in \mathbb{Z}^d\)

\(G(k) = \mathcal{W}_{\mathcal{G}}(g(k - \cdot)), k \in \mathbb{Z}^d\)
Strong invariance principle for shot noise fields

**Theorem (Lr, Muirhead 21+)**

\[
P \left( \sup_{x \in B(0,R)} |\tilde{f}_\lambda(x) - G(x)| > \lambda^{-1/2} \ln(\lambda)^{1/2} t \right) < CR^d \lambda^c \exp(-ct)
\]

- Optimal up to the power of \(\ln(\lambda)\) (see also Berry-Esseen inequality)
- Based on Koltchinski 94’: There is a coupling of \(\mathcal{P}_\lambda\) and \(\mathcal{W}_G\) such that for any \(k \in \mathbb{Z}^d\),

\[
P(|\tilde{f}_\lambda(k) - G(k)| \geq t\lambda^{-1/2} \ln(\lambda)) \leq Ce^{-ct}
\]

- For \(x \in \mathbb{R}^d \setminus \mathbb{Z}^d\), approximate \(f(x)\) by \(f([x]) + \nabla f(\xi) \cdot (x - [x])\).
- There is a coupling of \(N \sim \text{Pois}(\lambda)\) and \(Z \sim \mathcal{N}(0, 1)\) such that

\[
P(|N - \lambda - \sqrt{\lambda}Z| > t) \leq Ce^{-ct}
\]
Elements of proof for the symmetric case
• Box crossing estimates (RSW) stem from the work of Tassion '16 because we have:
  ▶ Positive association of the discretised field (FKG inequality on a finite space)
  ▶ $\mathcal{E}_0$ is invariant in law under reflections and rotation by $\pi/2$
  ▶ Spatial asymptotic independence (of $f$, hence of $\mathcal{E}_\ell$)

• One arm decay stems from
  ▶ Positive association of the discretised field (FKG inequality on a finite space)
  ▶ Asymptotic independence
  ▶ Box crossing estimates (RSW)
Proof of sharp phase transition (bounded Mills ratio case)

1. First prove that $\mathbb{P}(\text{Cross}_\ell(2R, R)) \to 0$ and then use bootstrapping argument.

2. Proof based on a differential inequality of

$$
\theta : h \to \mathbb{P}(f^\varepsilon_{r, h} \in \text{Cross}_\ell(2R, R))
$$

where $f^\varepsilon_{r, h}$ is obtained from $f^\varepsilon_r$ by adding $h$ to all the marks. We prove

$$
\frac{\partial}{\partial h} \theta(h) \geq c \frac{\theta(h)(1 - \theta(h))}{\inf_{2r < \rho < R/2} \{2\rho/R + \mathbb{P}(f^\varepsilon_r \in \text{Arm}_\ell(2r, \rho))\}}
$$

3. Use of the OSSS inequality applied to randomized algorithms; after the ideas of Duminil-Copin, Tassion, Raoufi.
Sharp phase transition

**Theorem (Lr & Muirhead 19+)**

For $\ell > 0$ there is $c > 0$ such that

$$
\mathbb{P}(\text{Cross}_\ell(2R, R)) \leq 1 - \exp(-cR), R > 0
$$

It implies the main result:

- For $\ell \geq 0$, $\mathcal{E}_\ell$ has only bounded connected components a.s..
- For $\ell < 0$, $\mathcal{E}_\ell$ has a unique unbounded component a.s..

**Proof:**
- $\ell \geq 0$: $\mathbb{P}(\text{Arm}_0(1, R)) \to 0$.
- $\ell < 0$: Borel-Cantelli lemma with

$$
\sum_{k \geq 1} (1 - \mathbb{P}(\text{Cross}_\ell(2^{k+1}, 2^k))) < \infty \Rightarrow (\text{Cross}_\ell(2^{k+1}, 2^k)) \text{ occurs for } k > k_0
$$

and arrange the rectangles so that the connected components overlap.
OSSS inequality (O’Donnell, Saks, Schramm, Servedio ’05)

For an event $A$ on a product probability space $(E^n, \mu^n)$ and a random algorithm determining $A$, $(\theta := \mathbb{P}(A))$

\[
\text{Var}(1_{\{A\}}) = \theta(1 - \theta) \leq \sum_{i=1}^{n} \delta_{i}^{\mu}(A) I_{i}^{\mu}(A)
\]

where

- $\delta_{i}^{\mu}(A)$ : Probability that coordinate $i$ is revealed by the algorithm
- Influence of coordinate $i$ : $I_{i}^{\mu}(A) = \mathbb{P}(1_{\{A\}} \neq 1_{\{A_i\}})$ where $A_i$ is obtained by resampling coordinate $i$

For percolation events, typically :

- $A$ is a progressive uncovering of all the connected components touching a random crossing line (in a rectangle) / circle (in a disc)
- $\delta_{i}^{\mu}(A)$ is the probability that a point $i$ is “close” to one of these connected components (one-arm decay is useful here)
- $I_{i}^{\mu}(A)$ is related to $\partial_{h} \theta(h)$ for $h \sim 0$
Key point

- First remark that crossing events are monotonous in the marks (higher mark = more chances to percolate). Hence for each $i$ there is a.s. a random level $y_i$ such that there is percolation for $Y_i \geq y_i$.
- Assume for $f^c$ that mark $Y_i$ is replaced by $Y_i + h_i$ for some parameter $h_i \in \mathbb{R}$. Then

$$\frac{\partial}{\partial h_i} \mathbb{P}(\text{Cross}_\ell(2R, R)) = \frac{\partial}{\partial h_i} \mathbb{P}(Y_i + h_i \geq y_i) \geq u_{\mu_{ac}}(y_i - h_i)$$

$$l_i = \mathbb{P}(1_{\{A\}} \neq 1\{A_i\}) = \mathbb{P}(Y_i + h_i \geq y_i, Y'_i + h_i < Y'_i)$$

$$+ \mathbb{P}(Y_i + h_i < y_i, Y'_i + h_i \geq Y'_i)$$

$$\leq 2\mathbb{P}(Y_i \geq y_i - h_i)$$

Mills

$$\leq cu_{\mu_{ac}}(y_i - h_i)$$

- We end up with

$$\frac{\partial}{\partial h} \theta(h) = \sum_i \frac{\partial}{\partial h_i} \theta(h) \geq c \sum_i l_i \geq c \frac{\sum_i l_i \delta_i}{\sup_i \delta_i} \overset{\text{ossss}}{=} \frac{\theta(1 - \theta)}{\sup_i \delta_i}$$