# Realisability problems

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# Expected functionals

- $\xi$ is a random element in space $\mathbb{E}$
- Distribution of $\xi$ is uniquely specified by

$$\mathbf{E}g(\xi)$$

for all bounded continuous functions $g$

- $g \mapsto f(g) = \mathbf{E}g(\xi)$ is a linear functional
- What if $f(g)$ is given only for some functions $g$?

## Classical example (Kellerer 1964)

- Let $\mathcal{I}$ be a certain family of subsets $I \subset \{1, \dots, d\}$

- Given probability distributions $\mathbf{P}_I$ on $\mathbb{R}^{\mathrm{card}(I)}$, does there exist a probability distribution on $\mathbb{R}^d$ whose marginals are $\mathbf{P}_I$, $I \in \mathcal{I}$.

# Solution

- Need existence of a probability measure $\mathbf{P}$ on $\mathbb{R}^d$ such that

$$f(v_I) = \int v_I(x) d\mathbf{P}(x) = \int v_I(x) d\mathbf{P}_I(x)$$

for all functions $v_I$ that depend on the coordinates with numbers from $I$.

- Extend $f$ from functions of the type $v_I$ and their linear combinations to all functions $v$ on $\mathbb{R}^d$.

- In order to ensure that the extension is positive need

$$v^{\#}(x) = \sum_{I \in \mathcal{I}} a_i v_I(x) \geq 0 \text{ for all } x \quad \Rightarrow \quad f(v^{\#}) = \sum_{I \in \mathcal{I}} a_i \int v_I(x) d\mathbf{P}_I(x) \geq 0 \,.$$

# Random sets

- $X$ is a random closed set in space $\mathcal{X} = \mathbb{R}^d$

- The distribution is defined by the capacity functional

$$T(K) = \mathbf{P}\{X \cap K \neq \emptyset\}, \quad K \in \mathcal{K}.$$

- A functional $T$ on the family $\mathcal{K}$ of compact sets is the capacity functional of a random closed set $X$ if and only if $T$ is upper semicontinuous and completely alternating.

- Now assume that $T$ is defined only on some compact sets.

- If the family of test sets is sufficiently rich (separating), no problems.

# All finite sets

**Theorem.** *A completely alternating functional $T$ defined on the family of all finite sets is the capacity functional of a random closed set if and only if $T$ is upper semicontinuous on finite sets, i.e.*

$$\limsup T(K_n) \leq T(K)$$

*for each sequence $K_n$ of finite sets that converges to a finite set $K$ in the Hausdorff metric.*

# Singletons

One-point covering function

$$p_x = T(\{x\}) = \mathbf{P}\{x \in X\}, \quad x \in \mathcal{X}\,.$$

**Theorem.** *A function* $p_x$, $x \in \mathcal{X}$, *with values in* $[0,1]$ *is the one-point covering function of a random closed set if and only if* $p$ *is upper semicontinuous.*

*Proof.* Let $x_n \to x$ and let $\bar{U}_n$ be the closure of an open neighbourhood $U_n$ of $x$ that shrinks to $x$ as $n \to \infty$. The upper semicontinuity and monotonicity of $T$ yield that

$$\limsup p_{x_n} \le \limsup T(\bar{U}_n) \le T(\{x\}) = p_x \,,$$

that is $p_x$, $x \in \mathcal{X}$, is upper semicontinuous.

In the other direction, consider a random variable $\upsilon$ uniformly distributed on $[0, 1]$. Then

$$X = \{x : \ p_x \ge \upsilon\}$$

is closed by the upper semicontinuity of $p$ and $\mathbf{P}\{x \in X\} = \mathbf{P}\{\upsilon \le f(x)\} = p_x$ for all $x$. □

## Two-point covering function

- Two-point covering function of a random closed set $X$ is

$$p_{x_1,x_2} = \mathbf{P}\{x_1, x_2 \in X\}, \quad x_1, x_2 \in \mathcal{X} .$$

- If $X$ is stationary, then $p_{x_1,x_2}$ depends only on $x_1 - x_2$.

- Question: Characterise two-point covering functions of random closed sets, i.e. covariance functions of (stationary) upper semicontinuous indicator functions.

## Example (closedness condition)

- Let $p_{x,y} = \frac{1}{4}$ and let $p_x = \frac{1}{2}$ for all points $x, y \in \mathbb{R}$.

- While this two-point covering function corresponds, e.g., to the indicator field with independent values, it cannot be obtained as the covering function of a random closed set.

# Correlation measures of point processes

- $\xi$ is a point process in $\mathcal{X}$

- Which measures $\rho$ on $\mathcal{X} \times \mathcal{X}$ appear as the second-order factorial moment measure (correlation measure) of a point process.

- Kuna, Lebowitz and Speer (2011) characterised correlation measures for point processes with finite third moments or under hard-core exclusion condition.

# Plan

- Tool: extension of a positive linear functional, i.e. $f(g) \geq 0$ if $g \geq 0$ (w.r.t. the chosen order).

- Positivity condition: difficult computational problem.

- How to ensure that the extension is sufficiently regular and so the corresponding linear functional corresponds to a probability measure?

- Note: linear functionals on functions are not always representable as Lebesgue integral (Daniell's theory)!

# Riesz–Markov theorem

**Theorem.** *Let $\mathbb{E}$ be a compact space with its Borel $\sigma$-algebra.*

*Consider a vector lattice $G$ containing constants such that each $g \in G$ is continuous and a map $f : G \mapsto \mathbb{R}$ such that $f(1) = 1$.*

*Then there exists a random element $\xi$ in $\mathbb{E}$ such that $\mathbf{E}g(\xi) = f(g)$ for all $g \in G$ if and only if $f$ is a linear positive functional on $G$.*

- Note: vector lattice is a partially ordered vector space closed for maximum operation.

## Positive functionals and their extensions

- $E$ is a vector lattice, $G$ is a vector sub-space of $E$ (e.g. $E$ is generated by $\mathbb{I}_{x_1,\dots,x_k \in F}$ for all $k \geq 1$ and $G$ is a linear space generated by $\mathbb{I}_{x \in F}$)

- $G$ majorises $E$ if each $v \in E$ satisfies $v \leq g$ for some $g \in G$

**Theorem** (Kantorovich). *Assume that $G$ is a majorising vector subspace of a vector lattice $E$. Then each positive linear functional on $G$ admits a positive extension on the whole $E$.*

- Regularity (e.g. continuity) of the extension is not guaranteed!

## Positivity

**Theorem.** *Assume that* $G$ *can be represented as the direct sum of* $\mathbb{R}$ *(i.e. constant functions) and a vector space* $G'$. *Then a linear functional* $f : G \mapsto \mathbb{R}$ *with* $f(1) = 1$ *is positive if and only if*

$$f(g) \geq \inf_{x \in \mathbb{E}} g(x), \quad g \in G'.$$

# Another extension theorem

**Theorem.** *Let $f$ be a linear functional on a vector subspace $G$ of $E$. If $V$ is another vector subspace of $E$, then the existence of a positive extension of $f$ onto $G+V$ (including the positivity of $f$ on $G$) is equivalent to*

$$\sup_{g\in G,\ g\leq v} f(g) \leq q(v)$$

*for a linear functional $q$ on $V$.*

- Applied with $V=\{t\chi:\ t\in\mathbb{R}\}$ is the one-dimensional space generated by a non-negative function $\chi$ that does not belong to $G$. Condition

$$\sup_{g\in G,\ g\leq \chi} f(g) = r < \infty\,.$$

# Regularity modulus

- Assume that the vector space $G$ of functions on $\mathbb{E}$ contains constants and, for each $g_1, g_2 \in G$, there is $g \in G$ such that $(g_1 \vee g_2) \leq g$.

- A regularity modulus is a lower semicontinuous function $\chi : \mathbb{E} \mapsto [0, \infty]$ such that

$$H_g = \{x \in \mathbb{E} : \; \chi(x) \leq g\}$$

is relatively compact for each $g \in G$.

- Assume that each $g \in G$ is $\chi$-regular, i.e. $g$ is continuous on $H_{g'}$ for each $g'$ in $G$.

# Regular extension (main theorem)

**Theorem.** *If $f$ is a linear functional on $G$ with $f(1) = 1$, then there exists a Borel random element $\xi$ in $\mathbb{E}$ such that*

$$\begin{cases} \mathbf{E}g(\xi) = f(g) & \text{for all } g \in G\,, \\ \mathbf{E}\chi(\xi) \le r, \end{cases}$$

*for some real $r$ if and only if*

$$\sup_{g \in G,\; g \le \chi} f(g) \le r\,.$$

- Equivalent condition (if $G = \mathbb{R} + G'$)

$$\inf_{x \in \mathbb{E}} (\chi(x) - g(x)) + f(g) \le r\,, \quad g \in G'\,.$$

# Family of extensions

**Theorem.** *Assume that* $G$ *consists of continuous functions. Let* $f$ *be a linear positive functional on* $G$.

*Then the family* $\mathcal{M}$ *of all Borel random elements* $\xi$ *that satisfy* $\mathbf{E}g(\xi) = f(g)$ *for all* $g \in G$ *and* $\mathbf{E}\chi(\xi) \le r$ *is weak compact.*

**Theorem.** *Let* $f_n(g) \to f(g)$ *for all* $g \in G$ *for a sequence of linear positive functionals on* $G$ *such that*

$$\liminf_n \sup_{g \in G,\, g \le \chi} f_n(g) < \infty .$$

*Then* $f$ *is realisable as a random element* $\xi$ *satisfying* $\mathbf{E}g(\xi) = f(g)$ *for all* $g \in G$ *and* $\mathbf{E}\chi(\xi) < \infty$ *and such that* $\xi$ *is the weak limit of random elements realising* $f_{n_k}$ *for a subsequence* $\{n_k\}$.

# Invariant extension

- $\theta \in \Theta$ are transformations acting on $\mathbb{E}$
- $f$ is a $\Theta$-invariant functional and $\chi$ is regularity modulus

**Theorem.** *Assume that at least one of the following conditions holds*

*1) $G$ consists of continuous functions on $\mathbb{E}$ and $\chi$ is pointwisely approximated from below by a monotone sequence of functions $g_n \in G$.*

*2) $\chi$ is $\Theta$-invariant*

*Then $f$ is realisable by a $\Theta$-stationary random element $\xi$ with $\mathbf{E}\chi(\xi) \le r$ if and only if*

$$\sup_{g \in G,\ g \le \chi} f(g) \le r\,.$$

# Finite point processes

- $\mathcal{N}$ — locally finite counting measures with the vague topology
- $Y \in \mathcal{N}$ is a counting measure
- $\xi$ is a point process
- Functional

$$g_h(Y) = \sum_{x_i, x_j \in Y,\ i \neq j} h(x_i, x_j)\,, \quad Y \in \mathcal{N}\,,$$

$h \in \mathcal{C}_o$ — symmetric continuous with compact support

- Correlation measure $\rho$ of point process $\xi$

$$\int_{\mathcal{X} \times \mathcal{X}} h(x, y)\rho(dx, dy) = \mathbf{E} g_h(\xi)\,, \quad h \in \mathcal{C}_o\,.$$

# Positivity condition

- Functional $f$ on $G = \{g_h : \; h \in \mathcal{C}_\mathrm{o}\}$ is positive if

$$f(g_h) \geq \inf_{Y \in \mathbb{E}} g_h(Y)\,,$$

where $\mathbb{E} \subseteq \mathcal{N}$.

- Need to ensure that the extension of $f$ corresponds to a probability measure. Note that
  - each function $g_h$ is vague-continuous, and so is $\chi$-regular for any $\chi$;
  - $\mathcal{N}$ is locally compact.

## Bounded cardinality

- $\mathbb{E}_k$ consists of counting measures with total mass at most $k$ on a compact space $\mathcal{X}$.

- $\mathbb{E}_k$ is compact, $g_h$ is continuous and so Riesz–Markov is applicable

- Example. If $k = 2$, then $\inf_{Y \in \mathbb{E}_2} g_h(Y)$ is the minimum of zero (in case $Y$ is empty or consists of a single point) or the minimum of $h$.

A point process $\xi$ realising $\rho$ consists of coordinates of a random vector distributed according to the normalised $\rho$ with probability $\rho(\mathcal{X} \times \mathcal{X})$ and otherwise letting $\xi = \emptyset$.

## Moment condition

- For $\alpha > 2$

$$\chi_\alpha(Y) = Y(\mathcal{X})^\alpha, \quad Y \in \mathcal{N}.$$

is a regularity modulus, since

$$\{Y \in \mathcal{N} : \chi_\alpha(Y) \le c + g_h(Y)\} \subset \{Y \in \mathcal{N} : Y(\mathcal{X})^\alpha \le c + c' Y(\mathcal{X})^2\}$$

- The realisability condition

$$\sup_{g \in G,\ g \le \chi_\alpha} f(g) < \infty$$

ensures the existence of a point process with finite $\alpha$-moment.

# Non-finite point processes

- Let $\beta : \mathcal{X} \mapsto \mathbb{R}$ be a lower semi-continuous strictly positive function.

- Then

$$\chi_{\alpha,\beta}(Y) = \left( \sum_{x \in Y} \beta(x) \right)^{\alpha} , \quad Y \in \mathcal{N} ,$$

is a regularity modulus for $\alpha > 2$.

- Realisability condition becomes

$$\inf_{Y \in \mathbb{E}} (\chi_{\alpha,\beta}(Y) - g_h(Y)) + \int_{\mathcal{X} \times \mathcal{X}} h(x,y) \rho(dx,dy) \le r , \quad h \in \mathcal{C}_{\mathrm{o}} .$$

# Hard-core point processes with a fixed exclusion distance

- $\mathcal{X}$ is a compact metric space with metric d
- $\mathbb{E}^{\varepsilon}$ be the family of $\varepsilon$-hard-core point sets and so is a subset of $\mathcal{N}_s$ (simple counting measures)
- $\mathbb{E}^{\varepsilon}$ is compact, so Riesz–Markov applies. Only positivity condition is required

$$f(g_h) \geq \inf_{Y \in \mathbb{E}^{\varepsilon}} g_h(Y)$$

This is a stronger condition than

$$f(g_h) \geq \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

# Hard-core point process on a compact space

- Counting measures from $\cup_{\varepsilon>0}\mathbb{E}^{\varepsilon}$ (the exclusion distance is not specified)

- Define

$$\chi_{\psi}^{\mathrm{hc}}(Y) = \sum_{x_i, x_j \in Y,\ i \neq j} \psi(\mathsf{d}(x_i, x_j)), \quad Y \in \mathcal{N}_s\,,$$

where $\psi : (0,\infty) \mapsto [0,\infty]$ be a monotone decreasing right-continuous function, such that $\psi(t) \to \infty$ as $t \downarrow 0$.

- $\psi$ should grow at zero sufficiently fast to ensure that $\chi_{\psi}^{\mathrm{hc}}$ is a regularity modulus.

# A combinatorial lemma

- $P_t(\mathcal{X})$ denotes the packing number of $\mathcal{X}$ with metric d, i.e. the maximum number of points in the space $\mathcal{X}$ with pairwise distance exceeding $t$.

**Lemma 1.** *If $Y = \sum \delta_{x_i}$ is a counting measure of total mass $n$, then for all $t > 0$,*

$$\sum_{i \neq j} \mathbb{I}_{\mathsf{d}(x_i, x_j) \leq t} \geq n\Big(\frac{n}{P_t(\mathcal{X})} - 1\Big)\,.$$

*Proof.* Successive transformations of $Y$ that decrease the value of the left-hand sum and eventually bring $Y$ to a set of multiple points with equal multiplicities. □

## Asymptotics for the minimal number of close pairs

- $\gamma_t(n)$ is the minimal number of pairs $(x_i, x_j)$ with $\mathrm{d}(x_i, x_j) \leq t$ and $x_i, x_j \in Y$ over all counting measures of total mass $n$

- Then, for $t > 0$,

$$\lim_{n \to \infty} n^{-2} \gamma_t(n) = P_t(\mathcal{X})^{-1} \,,$$

## Regularity modulus $\chi_\psi^{\mathrm{hc}}$

**Lemma 2.** *Function* $\chi_\psi^{\mathrm{hc}}$ *is a regularity modulus on* $\mathcal{N}_s$ *if*

$$\psi(t)/P_t(\mathcal{X}) \to \infty \quad \text{as } t \downarrow 0\,.$$

*Proof.* Show that the total mass of any $Y$ from

$$\begin{aligned} H_\lambda &= \{Y \in \mathcal{N}_s : \ \chi_\psi^{\mathrm{hc}}(Y) \le \lambda Y(\mathcal{X})^2\} \\ &\subset \{Y : \ n^{-2}\gamma_t(n)\psi(t) \le \lambda\} \end{aligned}$$

is bounded from above. □

# Realisability condition

**Theorem.** *A locally finite measure $\rho$ on $\mathcal{X} \times \mathcal{X}$ is the correlation measure of a simple point process $\xi$ such that $\mathbf{E}\chi_\psi^{\mathrm{hc}}(\xi) \le r$ if and only if*

$$f(g_h) \ge \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

*and*

$$\int_{\mathcal{X}\times\mathcal{X}} \psi(\mathsf{d}(x,y))\rho(dx,dy) \le r\,.$$

- Note that $\chi_\psi^{\mathrm{hc}}$ can be approximated from below by functions $g_h$ and so this yields the existence of invariant extensions.

# A direct condition on $\rho$

**Theorem.** *A locally finite measure $\rho$ on $\mathcal{X} \times \mathcal{X}$ is a correlation measure of a simple point process $\xi$ if*

$$f(g_h) \geq \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

*and*

$$r = \int_{\mathcal{X} \times \mathcal{X}} P_{\mathsf{d}(x,y)}(\mathcal{X}) \rho(dx, dy) < \infty\,.$$

*In this case, for every $r' > r$, there exists $\xi$ such that*

$$\mathbf{E} \sum_{x_i, x_j \in \xi,\ i \neq j} P_{\mathsf{d}(x_i, x_j)}(\mathcal{X}) \leq r'\,.$$

*for every $r' > r$.*

*If furthermore $\Theta$ is a group of continuous transformations on $\mathcal{X}$ that leave $\rho$ invariant, then $\xi$ can be chosen $\Theta$-stationary.*

# Non-compact case ($\mathbb{R}^d$)

- Usual positivity condition

$$f(g_h) \geq \inf_{Y \in \mathcal{N}_s} g_h(Y)$$

- Regularity condition

$$\int_{B_n \times B_n} \|x - y\|^{-d} \rho(dx, dy) < \infty$$

for each ball $B_n$ of radius $n$.

## Random binary functions

- $\mathbb{E}$ is the family of all subsets of $\mathcal{X}$ with the topology of pointwise convergence

- Random indicator function $\xi$

- Let $G$ be a family of continuous (for pointwise convergence) functionals on $\mathbb{E}$.

- A linear functional $f$ on $G$ is realisable, i.e. $f(g) = \mathbf{E}g(\xi)$ if and only if $f$ is positive, i.e.

$$f(g) \geq \inf_{F \subset \mathcal{X}} g(F)\,, \quad g \in G\,.$$

## Example: one-point covering function

- $G$ generated by constants and $g_x(F) = \mathbb{I}_{x \in F}$
- $f : G \mapsto \mathbb{R}$ is determined by $f(g_x) = p_x$.
- Positivity condition $p_x \in [0, 1]$
- Note that $\xi$ is the indicator of a not necessarily closed random set.

# Example: two-point covering function (covariance)

- $G$ generated by constants and $g_{x,y}(F) = \mathbb{I}_{x,y\in F}$, $x, y \in \mathcal{X}$. $f : G \mapsto \mathbb{R}$ is determined by $f(g_{x,y}) = p_{x,y}$, i.e. two-point covering probabilities.

- $f$ is positive if and only if

$$\sum_{ij=1}^{n} a_{ij} p_{x_i,x_j} \geq \inf_{F\subset\mathcal{X}} \sum_{ij=1}^{n} a_{ij}\, \mathbb{I}_{x_i,x_j\in F}$$

- Equivalently, $\sum_{ij=1}^{n} a_{ij} p_{x_i,x_j} \geq 0$ for all corner-positive matrices $(a_{ij})$ (nonnegative sums of all main minors), McMillan (1955), Shepp (1967)

- If $a_{ij} = a_i a_j$, we recover the positive-definiteness of $p_{x,y}$ (only necessary condition).

# Covariances in the product form

**Theorem.** *Assume that $\mathcal{X}$ is a separable space. A function*

$$p_{x_1,x_2} = \begin{cases} p_{x_1} p_{x_2} & \text{if } x_1 \neq x_2\,, \\ p_{x_1} & \text{if } x_1 = x_2 \end{cases}$$

*is a two-point covering function of a random closed set if and only if $p_x$, $x \in \mathcal{X}$, is an upper semicontinuous function with values in $[0,1]$ such that each point $x$ with $p_x \in (0,1)$ has an open neighbourhood $U$ such that $p_y > 0$ only for at most a countable number of $y \in U$ and*

$$\sum_{y \in U} p_y < \infty\,.$$

# Functionals

- $\nu$ is a $\sigma$-finite reference measure on $\mathcal{X}$

- Define

$$g_h(F) = \int_{F \times F} h(x,y)\nu(dx)\nu(dy)$$

$$f(g_h) = \int_{\mathcal{X} \times \mathcal{X}} p_{x,y} h(x,y)\nu(dx)\nu(dy)\,.$$

- $p_{x,y}$ is weakly realisable if there exists random set $X$ such that $\mathbf{E} g_h(X) = f(g_h)$ for all $h$.

- $p_{x,y}$ is strongly realisable if there exists $X$ with $p_{x,y} = \mathbf{P}\{x, y \in X\}$.

# Difficulties

- Aim to realise a random closed set

- The family $\mathcal{F}$ of closed sets is compact in the Fell topology

- The functional $g_h$ is not continuous

- The finite point covering probabilities do not generate the Fell $\sigma$-algebra (only for the subfamily of regular closed sets).

## $\varepsilon$-neighbourhoods

- $\mathcal{F}^\varepsilon$ is the family of $\varepsilon$-neighbourhoods of closed sets in $\mathbb{R}^d$

- $G$ is generated by constants and the functions $g_h$, which are shown to be continuous on $\mathcal{F}^\varepsilon$.

**Theorem.** *A function $p_{x,y}$, $x, y \in \mathbb{R}^d$, is weakly realisable by a random closed set $X$ with realisations in $\mathcal{F}^\varepsilon$ for some $\varepsilon > 0$ if and only if*

$$f(g_h) \geq \inf_{F \in \mathcal{F}^\varepsilon} g_h(F), \quad h \in \mathcal{C}_\mathrm{o}.$$

## $\varepsilon$-neighbourhoods of varying $\varepsilon$

Regularity modulus

$$\chi(F) = \inf\{\varepsilon > 0 : \; F \in \mathcal{F}^{1/\varepsilon}\}$$

**Theorem.** *A function* $p_{x,y}$, $x, y \in \mathbb{R}^d$, *is weakly realisable by a random closed set* $X$ *such that* $\mathbf{E}\chi(X) \le r$ *if and only if*

$$\inf_{F \in \mathcal{F}^0} [\chi(F) - g_h(F)] + f(g_h) \le r \,, \qquad h \in \mathcal{C}_{\mathrm{o}} \,.$$

*If* $p_{x,y} = S_{x-y}$, $x, y \in \mathcal{X} = \mathbb{R}^d$, *with an even continuous function* $S$, *then* $p_{x,y}$ *is strongly realisable by a stationary random closed set* $X$.

## Convexity restriction

- $\mathcal{C}$ is the family for convex closed sets

- A functional $f(g_h)$ is realisable as the covariance of a convex random closed set if and only if

$$f(g_h) \geq \inf_{F\in\mathcal{C}} g_h(F)\,.$$

- For sets from the convex ring the regularity modulus is the smallest number of convex components.

## One-dimensional case

- Assume $\mathcal{X} = [0,1]$
- Let $\chi(F)$ be the number of convex components of $F$

**Theorem.** *If $p_{x,y}$ is a function of $x, y \in [0,1]$ such that*

$$\sup_{\varphi \in \mathcal{C}^1_o, 0 \le \varphi \le 1} \int_{\mathcal{X} \times \mathcal{X}} p_{x,y} \varphi'(x) \varphi'(y) dx dy = \infty ,$$

*then there is no random closed set $X$ satisfying $\mathbf{E}\chi(X)^2 < \infty$ having $p_{x,y}$ as its two-point covering function.*

## Contact distribution function

$$T_X(B_r(x)) = \mathbf{P}\{X \cap B_r(x) \neq \emptyset\}$$

is related to the spherical contact distribution function

$$H_X(r;x) = \mathbf{P}\{\mathsf{d}(x,X) \leq r | x \notin X\}, \quad r \geq 0.$$

# Realisability of capacity functionals on balls

**Theorem.** *A function* $\tau_x(r)$, $r \geq 0$, $x \in \mathbb{R}^d$, *is realisable as* $T_X(B_r(x))$ *for a random closed set* $X$ *if and only if*

$$f(g) = \sum_{i=1}^{q} a_i \tau_{x_i}(r_i) \geq 0$$

*for each non-negative function*

$$g(F) = \sum_{i=1}^{q} a_i \, \mathbb{1}_{B_{r_i}(x_i) \cap F \neq \emptyset} \geq 0 \,, \quad F \in \mathcal{F} \,.$$

## Example: two-points

**Theorem.** *Let* $x_1, x_2 \in \mathbb{R}^d$*, with* $l = \|x_1 - x_2\|$*, and let* $\tau_{x_1}(r)$ *and* $\tau_{x_2}(r)$ *be cumulative distribution functions of two sub-probability measures on* $\mathbb{R}_+$*. Then there exists a random closed set* $X$ *such that* $\tau_{x_i}(r) = T_X(B_r(x_i))$ *for* $r \geq 0$ *and* $i = 1, 2$ *if and only if for all* $r \geq 0$

$$\tau_{x_1}(\max(r - l, 0)) \leq \tau_{x_2}(r) \leq \tau_{x_1}(r + l).$$

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