Rigidity of point processes and spectral analysis

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The rigidity phenomenon (reverse plan of the talk)

- Let U uniform on $[0,1]^d$, and the shifted lattice

$$\mathsf{P} = \{k + U : k \in \mathbb{Z}^d\}.$$

This random point process is **maximally rigid** on any bounded $A \subset \mathbb{R}^d$:

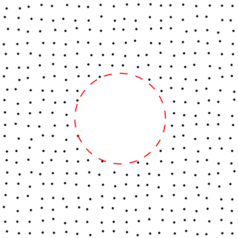
$$\mathsf{P} \cap A \in \sigma(\mathsf{P}_{A^c}).$$

- Let now a perturbed (shifted) lattice

$$\mathsf{P}' = \{k + U + \underbrace{\varepsilon_k}_{i.i.d.} : k \in \mathbb{Z}^d\}.$$

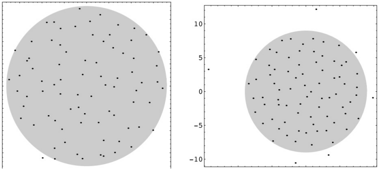
It is **number rigid** on A:

$$\#\mathsf{P}'_A \in \sigma(\mathsf{P}'_{A^c}).$$

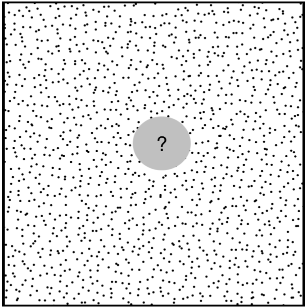


Disordered rigid point processes [Ghosh, Peres 2017]

- The infinite **complex** Ginibre eigenvalue DPP is number rigid
- The zero set of the **planar Gaussian analytic function** (GAF) is also number rigid
- For the GAF zeros, one can also guess the **barycenter of points**



[Forrester & Honner '99], n-order Ginibre set and GAF zeros (Weyl polynomials)



Number rigidity: One can infer the number of hidden points (Ginibre, GAF)
Barycenter rigidity: One can infer the barycenter of hidden points (GAF)

Rigidity bibliography

- Seminal paper **[Ghosh, Peres '17]**: The infinite Ginibre process and zero sets of the planar GAF are **number rigid**
- More general studies of the rigidity of **stationary DPPs** (Bufetov, Dabrowski, Qiu, ...)
- Rigidity of **Coulomb gases** (Dereudre, Leblé, Najnudel, Chaibi, Chatterjee, Thoma, ...)
- k-Rigidity for other GAFs **[Ghosh, Krishnapur '21]** where k-rigidity of P means that for $\sum_{i=1}^{d} |k_i| \leqslant k$

$$\int_A x_1^{k_1} \dots x_d^{k_d} d\mathsf{P}(x) \in \sigma(\mathsf{P}_{A^c})$$

- Rigidity of discrete DPPs by Lyons and Steif in the early '00s
- Older result of rigidity **[Aizenman, Martin '80]**
- Notion of tolerance **[Holroyd, Soo '13]**

Structure factor

- Let P a stationary point process on $\mathbb{R}^d$
- **Linear statistics:** $\mathsf{P}(f) = \sum_{x \in \mathsf{P}} f(x)$ (f bounded, compact support)
- **Running assumption:** $\mathbf{E}(\mathsf{P}(A)^2) < \infty$ for A bounded
- The **reduced covariance measure** C and its Fourier transform the **spectral measure** $\mathsf{S} = \text{Fourier}(\mathsf{C})$ are characterised on Schwartz functions by

$$\mathrm{Var}\,(\mathsf{P}(f)) = \int f(y) f(x+y) \mathsf{C}(dx) dy,$$

$$= (2\pi)^{-d} \int |\hat{f}|^2 d\mathsf{S} \qquad \textbf{(``Plancherel'')}$$

- Always well defined as tempered measures
- **Poisson process:** $\mathsf{C} = \delta_0, \ \mathsf{S}(d\xi) = d\xi$

Linear and non-linear rigidity

- In general, the rigidity of "disordered" processes is linear, i.e. in $L^2(\mathbf{P})$, $\exists h_n \in \mathsf{C}_c^\infty(\mathbb{R}^d)$:

$$\mathsf{P}(1_{B(0,1)}) = \#\mathsf{P} \cap B(0,1) = \lim_n \mathsf{P}(h_n 1_{B(0,1)^c})$$

Examples: DPPs, zeros of Gaussian functions, Coulomb gases, ...

- This can be formulated in the Hilbert space $L^2(\mathsf{S})$ by duality

$$\inf_{h \in \mathsf{C}_c^\infty(B(0,1))^c} \mathbf{E}\left[|\#\mathsf{P} \cap B(0,1) - \mathsf{P}(h)|^2\right] = \inf_h \int_{\mathbb{R}^d} \left|\widehat{1_{B(0,1)}} - \hat{h}\right|^2 \mathsf{S}(d\xi)$$

- For some lattice constructions, the rigidity can be proved directly and might not be linear

Example: **[Peres, Sly '14]** Gaussian perturbed lattice in dimension 3

Kolmogorov theorem on time series

Let $X_k, k \in \mathbb{Z}$ a **stationary process**, assume

$$\text{Covariance: } C(k) := \text{Cov}\,(X_0, X_k) \in \ell^1(\mathbb{R}),$$

$$\text{Spectral measure: } s(\xi) := \hat{C}(\xi) = \sum_k C(k) e^{-i\xi k}, \xi \in \mathbb{T} :\equiv [-\pi, \pi]$$

[Kolmogorov '41]

- X is **"number rigid":** $X_0 \in \sigma(X_k, k \neq 0)$ if s has a "weak zero"

$$\int_{\mathbb{T}} s(\xi)^{-1} d\xi = \infty.$$

- X is **predictable** (or **maximally rigid** on $\mathbb{N}$), or perfectly interpolable,

$$X_0 \in \sigma(X_k, k < 0)$$

if s has a **spectral gap**, i.e. $s \equiv 0$ on some interval $(u_0 - \varepsilon, u_0 + \varepsilon)$

Number rigidity for point processes

- Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ smooth with $f(0) = 1$. Then for f Schwartz

$$\#\mathsf{P} \cap B(0,1) \approx \sum_{x \in \mathsf{P} \cap B(0,1)} f_R(x) \text{ with } f_R(x) := f(x/R)$$

$$= \sum_{x \in \mathsf{P}} f_R(x) - \underbrace{\sum_{x \in \mathsf{P} \setminus B(0,1)} f_R(x)}_{\in \sigma(\mathsf{P} \cap B(0,1)^c)}$$

$$\approx \underbrace{\mathbf{E}\mathsf{P}(f_R)}_{\text{deterministic}} \pm \underbrace{\sqrt{\mathrm{Var}\left(\mathsf{P}(f_R)\right)}}_{\to 0?} - \underbrace{\sum_{x \in \mathsf{P} \setminus B(0,1)} f_R(x)}_{\in \sigma(\mathsf{P} \cap B(0,1)^c)}$$

- **Hyperuniformity**: $\Leftrightarrow \ \mathrm{Var}\left(\mathsf{P}(f_R)\right) = o(R^d)$ (sub-Poisson variance)
- **Rigidity**: $\Leftarrow \ \mathrm{Var}\left(\mathsf{P}(f_R)\right) \to 0$

Hyperuniformity and linear statistics

- Let f bounded with compact support. Monte-Carlo estimator:

$$\frac{1}{R^d}\int f(x)dx \approx_{R\to\infty} \frac{1}{R^d}\sum_{x\in \mathsf{P}} f(x/R) = \frac{1}{R^d}\mathsf{P}(f_R)$$

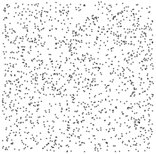
Variance for Poisson: $\mathrm{Var}\left(R^{-d}\mathsf{P}^{\mathrm{Pois}}(f_R)\right) \asymp R^{-d}$

Theorem (Torquato, Coste, Bjorklund & Hartnick, Lr, ...)

Let P *a stationary point process,* $\alpha > 0$*. (i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)*

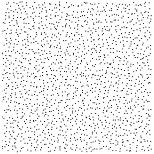
- *(i)* P *is hyperuniform for at least one* $f \geqslant 0$ *(smooth):* $Var\left(R^{-d}\mathsf{P}(f_R)\right) = o(R^{-d})$ $(= O(R^{-d-\alpha}))$
- *(ii)* P *is always better than Poisson for Monte-Carlo integration:* $Var\left(R^{-d}\mathsf{P}(f_R)\right) = o(R^{-d})$ $(= O(R^{-d-\alpha})$ if f is α-regular$)$
- *(iii)* $\frac{\mathsf{S}(B(0,\varepsilon))}{\mathbf{Leb}^d(B(0,\varepsilon))} \to 0$ $(= o(\varepsilon^\alpha)$, " α-hyperuniformity"$)$

Monte Carlo integration with hyperuniform samples



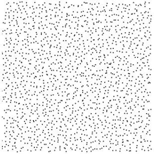
i.i.d. sample: Variance $\sim n^{-1}$

0-hyperuniform



Ginibre process: Variance $\sim n^{-2}$

1-hyperuniform, 0-rigid



Weyl zeros (GAF): Variance $\sim n^{-3}$

2-hyperuniform, 1-rigid

k-rigidity $\approx$ $(2k+d)$-hyperuniformity

$$\mathsf{S}(d\xi) = \underbrace{\mathsf{s}(\xi)}_{\text{spectral density}} d\xi + \underbrace{\mathsf{S}_s(d\xi)}_{\text{singular part}}$$

Theorem (Lr '24)

- *(i) A wide-sense stationary locally L^2 point process P is number rigid (or "0-rigid") if for all $\varepsilon > 0$*

$$\int_{B(0,\varepsilon)} \frac{1}{\mathsf{s}(\xi)} d\xi = \infty,$$

- *(ii) Under some additional assumptions (isotropy / separability / polynomial decay of s / ...),* **linear** *k-rigidity is equivalent to*

$$\int_{B(0,\varepsilon)} \frac{\|\xi\|^{2k}}{\mathsf{s}(\xi)} d\xi = \infty.$$

Paley-Wiener theorem

- Number rigidity means that $\widehat{1_{B(0,1)}}$ is in the $L^2(\mathsf{S})$-closure of

$$H = \{\hat{h} : h \in \mathsf{C}_c^\infty(B(0,1)^c)\}.$$

- The proof requires to understand $H^\perp = \overline{H}^\perp$, i.e. functions $\varphi \in L^2(\mathsf{S})$ such that $\hat{\varphi}$ is orthogonal to all $h \in \mathsf{C}_c^\infty(B(0,1)^c)$
- **Paley-Wiener theorem:** the functions φ with spectrum in $B(0,1)$ are the **analytic functions of type** 1 in each argument
- k-rigidity $\Leftrightarrow$ $\widehat{x^k 1_{B(0,1)}(x)} \in \overline{H}$, the same strategy works

Determinantal Point Processes (DPPs)

Theorem

Let P a stationary DPP with Hermitian kernel $K(x, y)$, L^2 in each variable. Then P is never k-rigid if $d \geqslant 3$ or if $k \neq 0$, and it is number rigid iff $d \in \{1, 2\}$ and

$$\int_{B(0,1)} \frac{1}{\mathsf{s}(\xi)} d\xi = \infty.$$

- **Examples:** Sine process (1D), Ginibre process (2D)
- Similar results for discrete DPPs (see **[Lyons, Steif '03]**).
- The theorem also applied to lattices perturbed by i.i.d. variables

Summary of first part

- A locally square integrable stationary field / measure M on $\mathbb{Z}^d/\mathbb{R}^d$ has a spectral measure characterised by

$$\mathrm{Var}\left(\mathsf{M}(f)\right) = (2\pi)^{-d} \int |\hat{f}|^2 d\mathsf{S}$$

- Quite often, k-rigidity is equivalent to α-hyperuniformity with

$$\alpha \geqslant 2k + d$$

- For most "disordered" hyperuniform processes,

$$\mathsf{S}(d\xi) = 0 + \kappa \|\xi\|^2 d\xi (1 + o(1))$$

  as $\xi \to 0$
- In this case we only have 0-rigidity (i.e. number rigidity for point processes) and only in dimension $d \in \{1, 2\}$.

Section 2

Spectral gaps

High order hyperuniformity / rigidity

- Higher order hyperuniformity $\Rightarrow$ Higher order rigidity
- High order hyperuniformity seems to be connected to optimal configurations
  - High order hyperuniformity occurs with some cancellations related to "sum rules" in physics
  - s-Riesz gases in $d = 1$ for $s \in (-4, -2]$ are expected to be 1-rigid
- Torquato conjectures that "jammed hard sphere models" are hyperuniform.

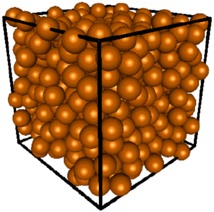


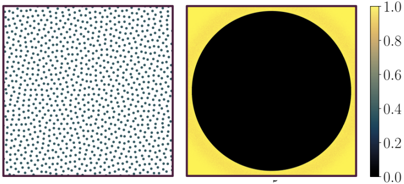
Figure: Jammed hard spheres

Spectral gaps and “stealthy point processes”

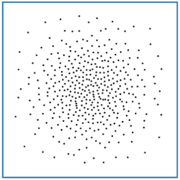
Definition

A stationary point process (or random measure) is **stealthy** if the spectral measure S has a gap around 0 (i.e. $\mathsf{S}(B(0,\varepsilon)) = 0$ for some $\varepsilon > 0$).

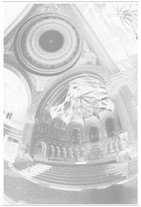
- Systematic study and simulations by **[Torquato, Zhang, Stillinger, ...]**
- Theoretical study by **[Ghosh & Lebowitz ’18]**
- Useful in image processing, quantization, optimal transport, ... (often under the terminology “blue noise”)



Stealthy process and spectrum **[Shih et al. ’23]**



Quantization



Dithering

Maximal rigidity

- $\mathsf{S}(B(0,\varepsilon)) = 0$ hence for all $r > 0$

$$\int_{B(0,r)} \frac{\|\xi\|^{2k}}{\mathsf{s}(\xi)} d\xi = \infty.$$

- Hence there is k-rigidity for each k on $B(0,1)$
- Since all the moments of the finite measure $\mathsf{P} \cap B(0,1)$ are determined by $\mathsf{P} \cap B(0,1)^c$, we have **maximal rigidity**

$$\mathsf{P} \cap B(0,1) \in \sigma(\mathsf{P} \cap B(0,1)^c)$$

(already found by **[Ghosh, Lebowitz '18]**)
- Unfortunately we do not really have examples of stealthy point processes except for (finite independant unions of) shifted lattices

$$\mathbb{Z}^d + U = \{k + U; k \in \mathbb{Z}^d\} \text{ with } U \sim \text{Unif}_{[0,1]^d}$$

General maximal rigidity and uniqueness pairs

- **Linear maximal rigidity** on A $\Leftrightarrow$ for any function $f \in \mathsf{C}_c^\infty(A)$, $\hat{f}$ is in the $L^2(\mathsf{S})$-closure of

$$H_A := \{\hat{h} : h \in \mathsf{C}^\infty(A^c)\}$$

- For A non-compact, we cannot use Paley-Wiener theorem anymore
- There is maximal rigidity if there cannot be $\varphi \in L^2(\mathsf{S})$ such that $\mathrm{Sp}(\varphi) \subset A$

Definition

$A, B \subset \mathbb{R}^d$ are a **uniqueness pair** if there is no $\varphi \neq 0$ such that

- $$\mathrm{supp}(\hat{\varphi}) \subset A, \qquad \mathrm{supp}(\varphi) \subset B$$

Theorem ([Lr '25])

If A and $\mathrm{supp}(\mathsf{S})$ form a uniqueness pair, there is maximal rigidity on A.

Uniqueness pairs and dense packings

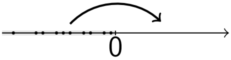
- **[Radchenko & Viazovska 2017]** proved that

$$A = \{\pm\sqrt{n}; n \in \mathbb{N}\} \subset \mathbb{R}$$

forms a uniqueness pair with itself
- With co-authors, they prove that it implies that "the $E8$ lattice" in $\mathbb{R}^8$ is the densest sphere packing because the squared lengths of lattice vectors are $2, 4, 6, \dots$
- It used to be called "mutually annihilating pair" **[Havin, Joricke '94]** (?)

Examples of uniqueness pairs

- $\mathbb{T} \setminus [-\varepsilon, \varepsilon]$ forms a uniqueness pair with $\mathbb{Z}_-$.

 1D gap		$\mathbb{R}$: **[Krein '45]** $\mathbb{Z}$: **[Kolmogorov '41**

Theorem (Runge's theorem on polynomial approximation)

A continuous periodic function φ is approximable by polynomials on $[-\pi, \pi] \setminus [-\varepsilon, \varepsilon]$

$$\inf_{m,(a_k)\in\mathbb{C}^m} \int_{[-\pi,\pi]\setminus[-\varepsilon,\varepsilon]} \Big|\varphi(\xi) - \underbrace{\sum_{k=1}^{m} a_k e^{ik\xi}}_{\text{positive spectrum}} \Big|^2 d\xi = 0.$$

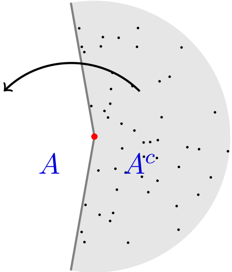
- Also true on the real line: $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ forms a uniqueness pair with $\mathbb{R}_-$

Levinson-Shapiro '73

If S has a gap $B(x,\varepsilon)$ in $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$ is a closed strictly convex cone, they form a uniqueness pair.

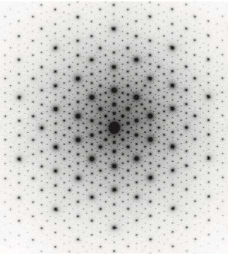
Corollary

A stealthy measure is maximally rigid on a strictly convex cone

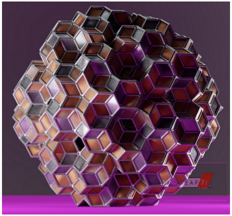
 S : Spectral hole		$\mathbb{R}^d$: Stealthy processes $\mathbb{Z}^d$: **[Helson & Lowdenslager '58]**

Quasicrystals, or *aperiodic order*

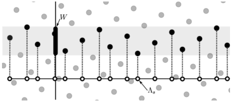
- A (mathematical) quasicrystal is a point configuration P
  - *uniformly discrete:* $\inf_{x \neq y \in \mathsf{P}} \|x - y\| > 0$
  - *relatively dense:* $\exists r > 0 : \cup_{x \in \mathsf{P}} B(x, r) = \mathbb{R}^d$
  - **pure point diffraction:** S *is purely atomic*



AiMnPd spectrum



3D quasicrystal



Cut-and-project

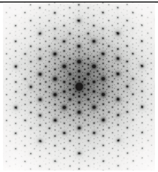
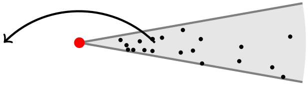
- **[Meyer '72]** cut-and-project models
- **[Bjorklund, Hartnick '24]** stationarised *cut-and-project* point processes: *Are they number rigid ?*

Atomic spectral measures and quasicrystals

Theorem (Maximal rigidity of quasicrystals)

Let C be a cone with non-empty interior.

- *Let* P *a stationary random measure on* $\mathbb{R}^d$ *with purely atomic spectrum. Then* P *is maximally rigid* *from* C.
- *Let* $\mathsf{X}_k, k \in \mathbb{Z}^d$ *a stationary random field* $\mathbb{Z}^d \to \mathbb{C}$ *with a purely atomic spectral measure, then* X *is maximally rigid from* $C \cap \mathbb{Z}^d$

 Purely atomic		$\mathbb{R}^d$:Quasicrystals (cut-and-project processes)

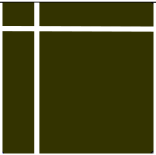
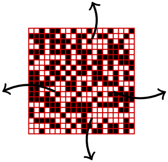
Periodic fields on $\mathbb{Z}^d$

Theorem (Borichev, Nishry, Sodin, Weiss '14,'17)

Let $X : \mathbb{Z} \to \mathbb{Z}$ **integer-valued** *time series with a spectral gap. Then X is a.s. periodic.*

Theorem ([Lr '25])

Let $X : \mathbb{Z}^d \to \mathbb{Z}$ **integer-valued** *stationary field with a simply connected spectrum in $\mathbb{T}^d$. Then X is a.s. periodic.*

 Gap in each coordinate $\Rightarrow$ Simply connected	 Rigid on $[-R, R]^c$ for R large enough

Random fields

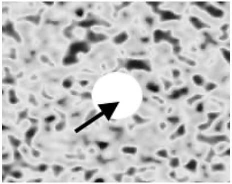
- Let the stationary random field $F(x)$ having covariance

$$\mathsf{C}(x) = \mathbf{E}(F(0)F(x)) = 1_{B(0,1)} \star 1_{B(0,1)}(x), x \in \mathbb{R}^d.$$

"completely standard"

  - Linear variance (not hyperuniform)
  - Continuous (can be made $\mathcal{C}^k$ for arbitrary k)
  - Small range: $F(0), F(x)$ are independent if $\|x\| \geqslant 2.$

Phase transition: Then there is maximal rigidity on $A = B(0, \rho)$ if $\rho < \frac{2}{\pi}$ (and there is independence for $\rho > 2$).

Concentric circles 		$\mathbb{R}^d$: Jensen's inequality and Paley-Wiener theorem

Thank you!