Hyperuniform random samples

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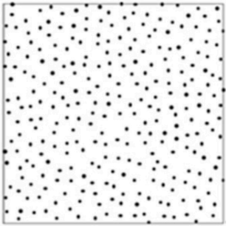
Mini-course for the 2025 Stochastic Geometry Days
Université Grenoble Alpes, June 23-27

Introduction

How to sample space properly?



Receptors in a chicken's retina Jiao et al. [2014]

Insect eyes

R. J. Ullichney, *Dithering with blue noise*, (1988)

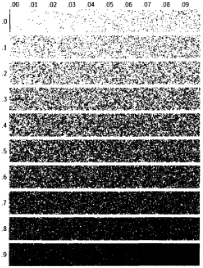


Fig. 6. Rectangular random dither of a gray-scale ramp.

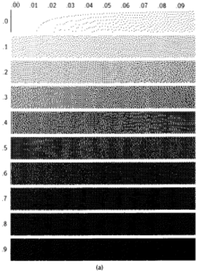


Fig. 7. Rectangular random dither of a scanned picture.

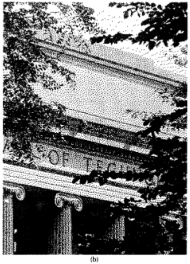


Fig. 16. Error diffusion with the Floyd and Steinberg filter. (a) Gray-scale ramp. (b) Scanned picture.

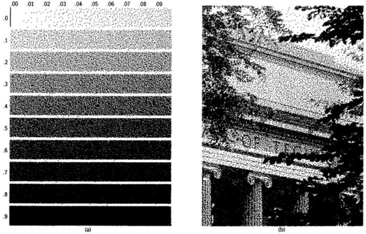
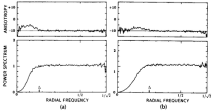


Fig. 24. Effect of one randomly positioned weight. (a) Gray-scale ramp. (b) Scanned picture.



Sample spectrum

Regular samples

There are many tasks where it is advantageous to sample points $\mathsf{P}_n := \{x_i; i = 1, \ldots, n\}$ in $\mathbb{R}^d$ in a "regular" way:

- **Numerical integration**: choose points to approximate the integral of a function
$$\int f \sim a_n \sum_{i=1}^{n} f(x_i) =: \mathsf{P}_n(f).$$
- **Choose hyperparameters** of a costly (learning) task, simulation.
- Produce patterns in graphical synthesis.
- **Diversity**: Kulesza and Taskar [2012] Negative correlation helps *"finding diverse sets of high-quality search results, building informative summaries by selecting diverse sentences from documents, modeling nonoverlapping human poses in images or video, and automatically building timelines of important news stories"*
- Estimate the distance between two measures through slices (TDA)

First definition of hyperuniformity

Several measures of regularity

- Low variance $\mathrm{Var}\left(\#\mathsf{P}_n \cap A\right)$ for $A \subset \mathbb{R}^d$
- Low discrepancy $\mathbf{E}|\#\mathsf{P}_n \cap A - (\mathbf{E}\#\mathsf{P}_n \cap A)|$.
- Low void/cluster probability $\mathbf{P}(\#\mathsf{P}_n \cap A > t), \ldots$

Hyperuniformity is about the **variance** of an infinite sample: for a homogeneous Poisson point process $\mathsf{P} \subset \mathbb{R}^d$, defined by

$$\mathsf{P}(A) := \#\mathsf{P} \cap A \sim \mathscr{P}(\mathbf{Leb}^d(A)), A \subset \mathbb{R}^d,$$

hence with B_R the ball centred in 0 with radius R

$$\mathrm{Var}\left(\mathsf{P}(B_R)\right) \sim \mathbf{Leb}^d(B_R) \sim R^d.$$

A "large" "homogeneous" random sample of points $\mathsf{P} \subset \mathbb{R}^d$ is **hyperuniform (HU)** if

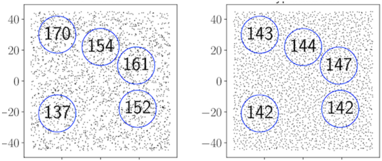
$$\lim_{R\to\infty} \frac{\mathrm{Var}\left(\mathsf{P}(B_R)\right)}{R^d} \to 0.$$

Hyperuniformity of finite samples

- For each $n \geqslant 1$, let P_n a sample with n points in the ball Λ_n with volume n, and $X^{(n)} := \{X_1, \dots, X_n\}$ i.i.d. points uniform in Λ_n.
- Then the family $\{\mathsf{P}_n; n \geqslant 1\}$ is hyperuniform if for $r > 0$

$$\sup_n \mathrm{Var}\left(\#\mathsf{P}_n \cap B_r\right) = o\left(\sup_n \mathrm{Var}\left(\#X^{(n)} \cap B_r\right)\right) = o(\mathbf{Leb}^d(B_r)).$$

- Exercise: $X^{(n)} \cap B_r \xrightarrow[n \to \infty]{} \mathrm{Poiss}(\mathbf{Leb}^d(B_r))$ (Alternative definition of the Poisson process)



Hyperuniformity (HU) is a good concept because ...

- It appears in many diverse models and phenomena (see next slide).
- It is a "simple" second order property but it has strong implications: rigidity, transport, Central Limit Theorem, ...
- It has a well defined perimeter with several equivalent definitions, for instance equiv. to low variance for MC estimators

$$R^{-d}\mathsf{P}(f_R) := R^{-d}\sum_{x\in\mathsf{P}} f(x/R) \xrightarrow[R\to\infty]{} \int f.$$

(For $f = 1_{B_1}$, $R^{-d}\sum_{x\in\mathsf{P}} f(x) = \frac{\#\mathsf{P}\cap B_R}{\mathbf{Leb}^d(B_R)}$)

For f smooth:

- If P is Poisson: Variance $\sim R^{-d}\int f^2$
- If P is hyperuniform: Variance $= o(R^{-d})$

Manifestations of hyperuniformity

- “Discovered” by **Torquato**’s team as a common phenomenon in **condensed matter physics**, seminal paper Torquato and Stillinger [2003]
  - Strictly jammed packings
  - Crystals and quasicrystals
  - “Self-organising systems at criticality”
  - Optical receptors of some birds Jiao et al. [2014]
  - Disposition of marine algae known as *Effrenium Voratum* Huang et al. [2021]
- In **Random matrix Theory**, many limiting distributions for eigenvalues were known to be hyperuniform (before this term existed). Systematic study by Lebowitz, Ghosh, Bufetov, etc...
  - 1D GUE (DPP)
  - 2D Ginibre (DPP)
  - GOE (Pfaffian)
- Particle systems: Riesz/Coulomb gases / OCPs / Jelliums
- Gaussian Analytic Functions

First example: perturbed lattices

Let $U \sim \mathscr{U}_{[0,1]^d}$, and the *shifted lattice*

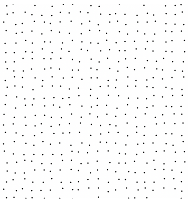
$$\mathbf{Z}^d = \{k + U; k \in \mathbb{Z}^d\},$$

U ensures stationarity:

$$\tau_x \mathbf{Z}^d \overset{(d)}{=} \mathbf{Z}^d \text{ where } \tau_x y = y + x; \quad x, y \in \mathbb{R}^d.$$

Let $U_k; k \in \mathbb{Z}^d$ i.i.d. with law μ, and the *independently perturbed lattice*

$$\mathbf{Z}^{d,\mu} := \{k + U + U_k\} \subset \mathbb{R}^d.$$



Reduced variance

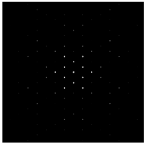
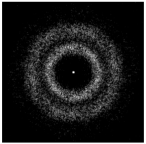
$$\mathrm{Var}\left(\#\mathbf{Z}^{\mu} \cap B_R\right) \sim \#\text{ points of } \mathbb{Z}^d \text{ close to } \partial B_R \sim R^{d-1}.$$

Compare with a homogeneous Poisson process P with intensity one

$$\mathrm{Var}\left(\#\mathsf{P} \cap B_R\right) = \mathrm{Var}\left(\mathsf{Poiss}(\mathbf{Leb}^d(B_R))\right) = \kappa_d R^d.$$

Problem of grids

- Periodicity / Spectral atoms

- In higher dimensions (except 24 Cohn et al. [2017]), densest packings are not periodic
- Irregular variance on rectangular windows: 1 or $\geqslant 2^d$ for $\mathbb{Z}^d$
- What is a disordered process?
  - No **Bragg peaks**, i.e. atoms in the spectrum
  - Ergodic theory **mixing**, i.e. for A, B bounded

$$\mathsf{P} \cap A | \mathsf{P} \cap \tau_x B \xrightarrow[x \to \infty]{\text{Law}} \mathsf{P} \cap A.$$

  - Isotropy: $\theta \mathsf{P} \overset{(d)}{=} \mathsf{P}$ for any rotation θ

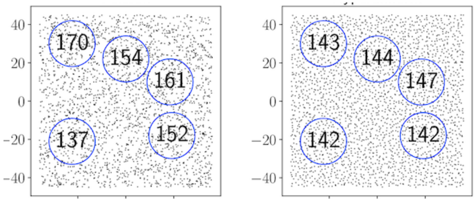
Shifted grid is not mixing

For $A = B = B_\varepsilon$, if one restricts $x \in \mathbb{Z}^d$,

$$\begin{aligned} &\mathbf{P}(\mathbf{Z} \cap B_\varepsilon \neq \emptyset \,|\, \mathbf{Z} \cap B(x, \varepsilon) \neq \emptyset) \\ &= \mathbf{P}(\tau_U \mathbb{Z}^d \cap B_\varepsilon \neq \emptyset | U - x \in \mathbb{N} \pm \varepsilon) \to 1. \end{aligned}$$

Rk: intersections of stationary lines process is mixing but not so much disordered.

Disordered HU "Chicken optical receptors"



Left: Poisson process. *Right:* Disordered HU process (Ginibre)

We present the following examples of disordered mathematically HU processes:

- Projector DPPs / Coulomb systems / OCPs (Sine kernel in 1D, Ginibre in 2D, ...)
- Zeros of planar GAFs in 2D
- Poisson-Coulomb allocation process in 3D +

Stationary point processes and structure factor

Stationary point processes

- A **Simple point configuration** is a measure $P = \sum_i \delta_{x_i}$ where the x_i are distinct, countable, isolated. Let $\mathcal{N} = \mathcal{N}(\mathbb{R}^d)$ the class of all such configurations.
- σ-algebra $\mathcal{B}(\mathcal{N})$ generated by mappings

$$\varphi_K : P \mapsto \# P \cap K, K \subset \mathbb{R}^d \text{ compact.}$$

- **Simple point process** P: Random element of $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$
- Can be seen as a **random set** "$\# P \cap A := P(A)$".
- P is **stationary** iff $\forall x \in \mathbb{R}^d$,

$$\tau_x \mathsf{P} := \mathsf{P} + x := \{y + x; y \in \mathsf{P}\} \stackrel{(d)}{=} \mathsf{P}.$$

Covariance measure

- Let P **stationary** and **locally square integrable** (LSI):
$$\mathbf{E}(\mathsf{P}(K)^2) < \infty, K \text{ compact.}$$

- The covariance measure $\mathscr{C}$ is defined by
$$\mathscr{C}(dx) = \mathrm{Cov}\left(\mathsf{P}(d0), \mathsf{P}(dx)\right)$$

- More formally, define for $f, g \in \mathcal{C}_c^b(\mathbb{R}^d)$ (bounded with compact support)
$$\mathsf{P}(f) = \sum_{x \in \mathsf{P}} f(x)$$
$$\mathrm{Cov}\left(\mathsf{P}(f), \mathsf{P}(g)\right) = \int f(x)\bar{g}(x+y)\mathscr{C}(dy)dx$$

Remark: Only well defined on bounded sets

Disintegration

$$\begin{aligned}\mathrm{Cov}\left(\mathsf{P}(f), \mathsf{P}(g)\right) &\text{ is linear in both } f, g \\ &= \int f(x)\bar{g}(x+y)\mu(dx, dy)\end{aligned}$$

μ invariant under x-translations:

$$\mu(dx, dy) = dx\mathscr{C}(dy)$$

SDP:

$$\int f(x)\bar{f}(x+y)dx\mathcal{C}(dy) = \sum_i f_i\bar{f}_j C_{j-i} \geqslant 0$$

implies $\hat{C} \geqslant 0$.
$C = \sum_j \hat{c}_j \omega_n^j J^j$ with $\hat{c}_j \geqslant 0$

Spectral measure

$\mathscr{C}$ is semi-definite positive:

$$\int \underbrace{f(x)\bar{f}(x+y)}_{f\tilde{\star}f(y)} dx\mathscr{C}(dy) = \mathrm{Var}\left(\mathsf{P}(f)\right) \geqslant 0.$$

- **Bochner's theorem**: There is a non-negative symmetric measure $\mathscr{S}$ such that $\mathscr{C} = \mathrm{Fourier}(\mathscr{S})$ in the sense of distributions:

$$\langle \mathscr{C}, f\rangle = c_d \langle \mathscr{S}, \hat{f}\rangle$$

  for f a Schwarz function ($\mathcal{C}^\infty$ with fast decaying derivatives).

- It gives the *phase-space variance formula*

$$\mathrm{Var}\left(\mathsf{P}(f)\right) = c_d \int |\hat{f}(u)|^2 \mathscr{S}(du), f \in \mathscr{C}_c^b(\mathbb{R}^d), \qquad (c_d = (2\pi)^{-d})$$

- $\mathscr{S}$ is called *Bartlett's spectral measure*,

Time series

- Let $\{X_k; k \in \mathbb{Z}^d\}$ a class of stationary L^2 (non-necessarily independent) random variables
- $\mathscr{C}(k) = \mathrm{Cov}\,(\mathsf{X}(0), \mathsf{X}(k)) \leqslant \mathrm{Var}\,(\mathsf{X}(0))$
- $\mathscr{S}(u) = \sum_k \mathscr{C}(k) e^{ik \cdot u}$ (if $\mathscr{C}(k)$ decreases fast enough) on the torus

Example: homogeneous Poisson process P with unit intensity

Variance of linear statistics: only the diagonal terms remain (replace $\sum_{x\in \mathsf{P}} f(x)$ with $\sim \sum_{i=1}^n f(X_i)$ and $A = \mathrm{supp}(f)$ is fixed)

$$\begin{aligned}\mathrm{Var}\,(\mathsf{P}(f)) =& \mathbf{E}\sum_{x,y} f(x)f(y) - \left(\mathbf{E}\sum_x f(x)\right)^2\\ =& \mathbf{E}\sum_{x\neq y} f(x)f(y) + \mathbf{E}\sum_x f(x)^2 - \left(\int f\right)^2\\ =& \int f(x)f(y)dxdy + \int f^2 - \int f(x)f(y)dxdy = \int f(x)f(x+y)\delta_0(\end{aligned}$$

Hence $\mathscr{C} = \delta_0$ (atomic nature of a point process), and

$$\mathscr{S} = \mathbf{Leb}^d.$$

For a "disordered process" P, one expects $\mathscr{S}(du) = (1 + o_{u\to\infty}(1))du$.

Example: shifted lattice $\mathsf{P} = \mathbf{Z} = \{k + U; k \in \mathbb{Z}^d\}$. Let f, g test functions

$$\mathbf{E}\mathbf{Z}(f)\mathbf{Z}(g) = \sum_{k,m} \int_{[0,1]^d} f(k+u)g(m+u)du$$
$$= \sum_k \int_{[0,1]^d} f(k+u) \underbrace{\sum_l g(k+u+l)\, du}_{\tilde{g}(k+u)} = \int_{\mathbb{R}^d} f(v) \sum_l g($$

$$\mathrm{Cov}\left(\mathbf{Z}(f), \mathbf{Z}(g)\right) = \int f(v)g(v+y) \sum_l \delta_l(y)dv - \int f \int g$$

$\Rightarrow \mathscr{C} = \sum_{\mathbf{m} \in \mathbb{Z}^d} \delta_{\mathbf{m}} - \mathbf{Leb}^d$ Then with **Poisson summation formula**

$$\langle \mathscr{S}, \varphi \rangle = \langle \mathscr{C}, \hat{\varphi} \rangle = \sum_{\mathbf{m} \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}) - \int \hat{\varphi} = \sum_{\mathbf{k} \in 2\pi\mathbb{Z}^d} \varphi(\mathbf{k}) - \varphi(0)$$

$$\mathscr{S} = \sum_{\mathbf{k} \in 2\pi\mathbb{Z}^d \setminus \{0\}} \delta_{\mathbf{k}}. \quad \text{Remark: } \mathscr{S}(B_{1/2}) = 0.$$

Equivalent definitions of hyperuniformity (HU)

Say $f \in L^1(\mathbb{R}^d)$ is regular if

$$|\hat{f}(u)| \leqslant c(1+\|u\|)^{-\frac{d+1}{2}} \text{ (implies } f, \hat{f} \in L^2(\mathbb{R}^d)).$$

Proposition

Let P *a LSI stationary point process, the three following conditions are equivalent*

(i) P *is HU, i.e.* $Var(\mathsf{P}(B_R)) = o(R^d)$

(ii) *for some* f *regular with* $\int f \neq 0$ $Var(\mathsf{P}(f_R)) = o(R^d)$
(Remark: (i) is (ii) with $f = 1_{B_1}$*)*

(iii) P *is "spectrally hyperuniform "* $\mathscr{S}(B_{1/R}) = o(R^{-d})$ *Björklund and Hartnick [2024]*

Lemma

- $f = 1_{B_1}$ *is regular and satisfies*

$$|\hat{f}(u)| = (1 + \|u\|)^{-\frac{d+1}{2}} (\sin(\|u\| + c_d) + o_{u\to\infty}(1))$$

- *The spectral measure of any stationary point process satisfies*

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-\frac{d+1}{2}} \mathscr{S}(du) < \infty.$$

(It implies that $\mathscr{S}$ is a tempered measure)

Remark: Most of the theoretical material of this mini-course is valid for wide-sense stationary measures i.e. M LSI such that

$$\mathrm{Var}\left(\mathsf{M}(\tau_x f)\right) = \mathrm{Var}\left(\mathsf{M}(f)\right)$$

proof I

(i) implies (ii) is obvious
(ii) implies (iii): $f \in L^1$ implies $\hat{f}$ continuous, hence $\hat{f} \geqslant \kappa > 0$ on some B_ε. Then

$$\kappa^2 \mathscr{S}(B_{\varepsilon/R}) \leqslant \int_{B_{\varepsilon/R}} |\hat{f}(Ru)|^2 \mathscr{S}(du) \leqslant \int |\hat{f}(Ru)|^2 \mathscr{S}(du)$$
$$= cR^{-2d} \mathrm{Var}\left(\mathsf{P}(f_R)\right)$$

QED.
(iii) implies (i) (or (ii)):

$$\mathrm{Var}\left(\mathsf{P}(B_r)\right) \leqslant R^{2d} \int (1 + R\|u\|)^{-d-1} \mathscr{S}(du)$$
$$\leqslant R^{2d} \underbrace{\int_{B_1} (1 + R\|u\|)^{-d-1} \mathscr{S}(du)}_{o(R^{-d}) \text{ if } \mathscr{S}(du) = o(1)du} + R^{2d} \int_{B_1^c} (R\|u\|)^{-d-1} \mathscr{S}(du)$$

proof II

Easy proof here if $\mathscr{S}(du) = s(u)du$ with $s(u) \to 0$. Otherwise...

$$\begin{aligned}
\leqslant & cR^{2d} \int_0^1 \mathscr{S}(\{u \in B_1 : (1 + R\|u\|)^{-d-1} > t\})dt \\
& + R^{-d-1} \int_{B_1^c} \|u\|^{-d-1} \mathscr{S}(du) \\
\leqslant & cR^{2d} \int_0^1 \underbrace{\mathscr{S}(\{u \in B_1 : \|u\| < \frac{t^{-\frac{1}{d+1}} - 1}{R}\})}_{o(R^{-d}(t^{-\frac{1}{d+1}} - 1)^d)} dt + O(R^{-d-1}) \\
\leqslant & cR^{2d} \int_0^{10^d R^{-d-1}} \mathscr{S}(B_1) + \int_{10^d R^{-d-1}}^1 \mathscr{S}(B_{t^{-\frac{1}{d+1}}/R}) + O(R^{-d-1}) \\
\leqslant & O(R^{-d-1}) + \int_{10^d R^{-d-1}} R^{-d} o(t^{-d/(d+1)})
\end{aligned}$$

Proof of lemma 2: take $f = 1_{B_1}$. Then as $u \to \infty$

$$\hat{f}(u) \sim \|u\|^{-\frac{d+1}{2}} \sin(\|u\| + c_d)$$

$$\exists z_1, \dots, z_{d+1} : \sum_{j=1}^{m} |\tau_{z_j} \hat{f}(u)|^2 \geqslant \|u\|^{-(d+1)} \sum_{j=1}^{d+1} \sin(\|u - z_i\|^2 + c_d)$$

$$\geqslant c\|u\|^{-(d+1)} \text{ (for large } u\text{)}$$

(1D: $\widehat{1_{[-1,1]}}(u) = \frac{\sin(u)}{u}, z_1 = 0, z_2 = \pi/2$) and

$$\infty > \sum_j \mathrm{Var}\left(\mathsf{P}(e^{iz_j \cdot} f)\right) \geqslant c' \int_{B_1^c} (1 + \|u\|)^{-d-1} \mathscr{S}(du)$$

Universal bounds (a.k.a. Rectangles do not fit well with hyperuniformity)

Theorem (Beck [1987])

For a stationary point process P,

$$\limsup_R \frac{Var(\mathsf{P}(B_R))}{R^{d-1}} > 0.$$

Still true for other "regular shapes" or for "disordered point processes" Kim and Torquato [2017].

(Still true for a wide sense stationary random measure)

Counter-examples

- Shifted lattice: $\mathrm{Var}\left(\mathbf{Z}^d([-n,n]^d)\right) = 0$
- Bylehn and Bjorklund [2024] $\liminf_R \frac{\mathrm{Var}(\mathbf{Z}(B_R))}{R^{d-1}} = 0$ iff $d \equiv 1 \mod 4$.
- $\limsup_R \frac{\mathbf{Z}'([-R,R]^d)}{R^{d-1}} = 0$ for some rotated "admissible" lattice $\mathbf{Z}'$
- There exists mixing M such that $\sup_R \mathrm{Var}\left(\mathsf{M}([-R,R]^d)\right) < \infty$

Beck

$$
\begin{aligned}
\int_0^{\mathbf{R}} \frac{\mathrm{Var}\,(\mathsf{P}(B_R))}{R^{d-1}} d\mathbf{R} &\geqslant \iint R^{d+1} |\hat{f}(Ru)|^2 dR \mathscr{S}(du) \\
&\geqslant \sim \iint_0^{\mathbf{R}} R^{d+1}(1+\|u\|R)^{-(d+1)} dR \mathscr{S}(du) \\
&= \iint_0^{\mathbf{R}} (1/R+\|u\|)^{-(d+1)} dR \mathscr{S}(du) \\
&\geqslant \iint_0^{\mathbf{R}} (1+\|u\|)^{-(d+1)} dR \mathscr{S}(du) > c\mathbf{R}
\end{aligned}
$$

Some consequences

- Proving the HU of some model is easier with a smooth linear statistic f than with 1_{B_1}
- The shifted lattice $\mathbf{Z}^d$ is HU because $\mathscr{S}(B(0,1/2)) = 0$
- **Exercise**: The structure factor of the perturbed lattice $\mathbf{Z}^{d,\mu}$ is

$$\mathscr{S}(du) = (1 - |\hat{\mu}(u)|^2)\mathbf{Leb}^d(du) + \sum_{\mathbf{k}\in 2\pi\mathbb{Z}^d\setminus\{0\}} \delta_{\mathbf{k}}|\hat{\mu}(\mathbf{k})|^2$$

where

$$\hat{\mu}(u) = \int e^{iu\cdot x}\mu(dx), u \in \mathbb{R}^d.$$

Hence it is HU: $\mathscr{S}(du) = o(u)du$ as $u \to 0$.

Factorial moment measures

- If P has **local moments** of order $k \in \mathbb{N}$, i.e. for K compact

$$\mathbf{E}\left[(\#\mathsf{P} \cap K)^k\right] < \infty,$$

its (symmetric) k-th **factorial moment measure** ρ_k^{P} is defined through symmetric test functions φ :

$$\mathbf{E} \sum_{\underbrace{\{x_1, \dots, x_k\}}_{\text{distinct}} \subset \mathsf{P}} \varphi(x_1, \dots, x_k) = \int \varphi(x_1, \dots, x_k) d\rho_k^{\mathsf{P}}(x_1, \dots, x_k)$$

- **Intensity**: for stationary P, $\rho_1(dx) = \lambda dx$ where $\lambda > 0$ ($\lambda = 1$ by default, *unit intensity*).
- HU, $\mathscr{C}$, $\mathscr{S}$ only involve ρ_1, ρ_2 (variance of linear statistics)

Characterisation by moments

- **Poisson:** Campbell formulas yield $\rho_m^{\mathsf{P}} \equiv \rho_1^{\mathsf{P}} \equiv \lambda$ (intensity)
- ρ_m^{P} determines ρ_k^{P} and $\mathbf{E}\mathsf{P}(A)^k$ for $k \leqslant m, A \subset \mathbb{R}^d$
- The $\mathbf{E}\mathsf{P}(A)^k$ characterise the law of $\mathsf{P}(A)$ if $\mathsf{P}(A)$ has some finite exponential moment

Convergence:

- Vague topology on $\mathcal{N}(\mathbb{R}^d) : P_n \to P$ if $P_n(f) \to P(f)$ for all $f \in \mathscr{C}_c^\infty(\mathbb{R}^d)$.
- For stationary P with exponential moments

$$\mathsf{P}_n \xrightarrow[n\to\infty]{\text{weak}} \mathsf{P} \text{ iff } \mathsf{P}_n(A) \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}(A)$$

for bounded A , iff for each $m \in \mathbb{N}$, $\rho_m^{\mathsf{P}_n} \to \rho_m^{\mathsf{P}}$ on each bounded set. This is a local convergence.

Remark: for $k < m$

$$\rho_k^{\mathsf{P}}(x_1, \dots, x_k) = \binom{k}{m}^{-1} \int \rho_m^{\mathsf{P}}(x_1, \dots, x_m) dx_{k+1} \dots \dots dx_m$$

Let $X^{(n)} = \{X_1, \dots, X_n\}$ uniform in $B_{1/\sqrt{\pi}}$ (volume 1). Then

$$n^{1/d} X^{(n)} \to \mathsf{P}^{\text{ Poisson}}$$

Proof:

$$\mathbf{P}(\#\{k : X_k \in n^{-1/d} A\}) \to \mathsf{Poiss}(\mathbf{Leb}^d(A))$$

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