Hyperuniform random samples

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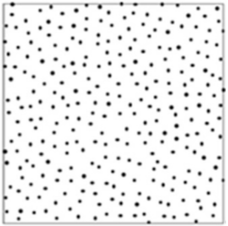
Introduction

How to sample space properly?



Receptors in a chicken's retina Jiao et al. [2014]

Insect eyes

R. J. Ullichney, *Dithering with blue noise*, (1988)

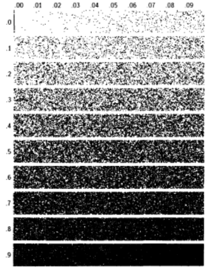
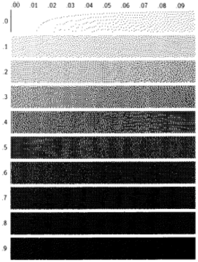
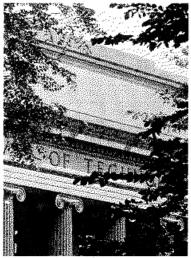


Fig. 6. Rectangular random dither of a gray-scale ramp.

Fig. 7. Rectangular random dither of a scanned picture.

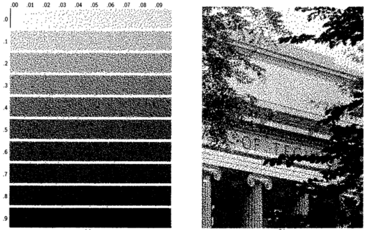


(a)



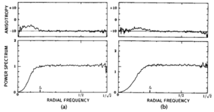
(b)

Fig. 16. Error diffusion with the Floyd and Steinberg filter. (a) Gray-scale ramp. (b) Scanned picture.



(a) (b)

Fig. 24. Effect of one randomly positioned weight. (a) Gray-scale ramp. (b) Scanned picture.



Sample spectrum

Regular samples

There are many tasks where it is advantageous to sample points $\mathsf{P}_n := \{x_i; i = 1, \ldots, n\}$ in $\mathbb{R}^d$ in a "regular" way:

- **Numerical integration**: choose points to approximate the integral of a function

$$\int f \sim a_n \sum_{i=1}^{n} f(x_i) =: \mathsf{P}_n(f).$$

- **Choose hyperparameters** of a costly (learning) task, simulation.
- Produce patterns in graphical synthesis.
- **Diversity**: Kulesza and Taskar [2012] Negative correlation helps *"finding diverse sets of high-quality search results, building informative summaries by selecting diverse sentences from documents, modeling nonoverlapping human poses in images or video, and automatically building timelines of important news stories"*
- Estimate the distance between two measures through slices (TDA)

First definition of hyperuniformity

Several measures of regularity

- Low variance $\mathrm{Var}\left(\#\mathsf{P}_n \cap A\right)$ for $A \subset \mathbb{R}^d$
- Low discrepancy $\mathbf{E}|\#\mathsf{P}_n \cap A - (\mathbf{E}\#\mathsf{P}_n \cap A)|$.
- Low void/cluster probability $\mathbf{P}(\#\mathsf{P}_n \cap A > t), \dots$

Hyperuniformity is about the **variance** of an infinite sample: for a homogeneous Poisson point process $\mathsf{P} \subset \mathbb{R}^d$, defined by

$$\mathsf{P}(A) := \#\mathsf{P} \cap A \sim \mathscr{P}(\mathbf{Leb}^d(A)), A \subset \mathbb{R}^d,$$

hence with B_R the ball centred in 0 with radius R

$$\mathrm{Var}\left(\mathsf{P}(B_R)\right) \sim \mathbf{Leb}^d(B_R) \sim R^d.$$

A "large" "homogeneous" random sample of points $\mathsf{P} \subset \mathbb{R}^d$ is **hyperuniform (HU)** if

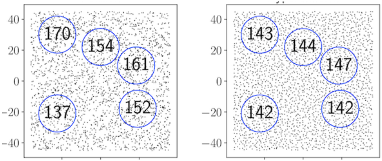
$$\lim_{R \to \infty} \frac{\mathrm{Var}\left(\mathsf{P}(B_R)\right)}{R^d} \to 0.$$

Hyperuniformity of finite samples

- For each $n \geqslant 1$, let P_n a sample with n points in the ball Λ_n with volume n, and $X^{(n)} := \{X_1, \dots, X_n\}$ i.i.d. points uniform in Λ_n.
- Then the family $\{\mathsf{P}_n; n \geqslant 1\}$ is hyperuniform if for $r > 0$

$$\sup_n \mathrm{Var}\left(\#\mathsf{P}_n \cap B_r\right) = o\left(\sup_n \mathrm{Var}\left(\#X^{(n)} \cap B_r\right)\right) = o(\mathbf{Leb}^d(B_r)).$$

- Exercise: $X^{(n)} \cap B_r \xrightarrow[n\to\infty]{} \mathrm{Poiss}(\mathbf{Leb}^d(B_r))$ (Alternative definition of the Poisson process)



Hyperuniformity (HU) is a good concept because ...

- It appears in many diverse models and phenomena (see next slide).
- It is a "simple" second order property but it has strong implications: rigidity, transport, Central Limit Theorem, ...
- It has a well defined perimeter with several equivalent definitions, for instance equiv. to low variance for MC estimators

$$R^{-d}\mathsf{P}(f_R) := R^{-d}\sum_{x\in\mathsf{P}} f(x/R) \xrightarrow[R\to\infty]{} \int f.$$

(For $f = 1_{B_1}$, $R^{-d}\sum_{x\in\mathsf{P}} f(x) = \frac{\#\mathsf{P}\cap B_R}{\mathbf{Leb}^d(B_R)}$)

For f smooth:

- If P is Poisson: Variance $\sim R^{-d}\int f^2$
- If P is hyperuniform: Variance $= o(R^{-d})$

Manifestations of hyperuniformity

- “Discovered” by **Torquato**’s team as a common phenomenon in **condensed matter physics**, seminal paper Torquato and Stillinger [2003]
  - Strictly jammed packings
  - Crystals and quasicrystals
  - “Self-organising systems at criticality”
  - Optical receptors of some birds Jiao et al. [2014]
  - Disposition of marine algae known as *Effrenium Voratum* Huang et al. [2021]
- In **Random matrix Theory**, many limiting distributions for eigenvalues were known to be hyperuniform (before this term existed). Systematic study by Lebowitz, Ghosh, Bufetov, etc...
  - 1D GUE (DPP)
  - 2D Ginibre (DPP)
  - GOE (Pfaffian)
- Particle systems: Riesz/Coulomb gases / OCPs / Jelliums
- Gaussian Analytic Functions

First example: perturbed lattices

Let $U \sim \mathscr{U}_{[0,1]^d}$, and the *shifted lattice*

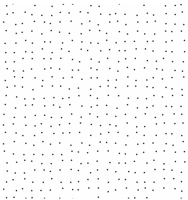
$$\mathbf{Z}^d = \{k + U; k \in \mathbb{Z}^d\},$$

U ensures stationarity:

$$\tau_x \mathbf{Z}^d \stackrel{(d)}{=} \mathbf{Z}^d \text{ where } \tau_x y = y + x; \quad x, y \in \mathbb{R}^d.$$

Let $U_k; k \in \mathbb{Z}^d$ i.i.d. with law μ, and the *independently perturbed lattice*

$$\mathbf{Z}^{d,\mu} := \{k + U + U_k\} \subset \mathbb{R}^d.$$



Reduced variance

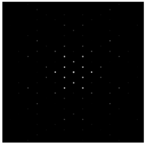
$$\mathrm{Var}\left(\#\mathbf{Z}^\mu \cap B_R\right) \sim \#\text{ points of } \mathbb{Z}^d \text{ close to } \partial B_R \sim R^{d-1}.$$

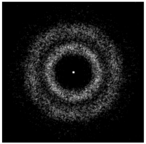
Compare with a homogeneous Poisson process P with intensity one

$$\mathrm{Var}\left(\#\mathsf{P} \cap B_R\right) = \mathrm{Var}\left(\mathsf{Poiss}(\mathbf{Leb}^d(B_R))\right) = \kappa_d R^d.$$

Problem of grids

- Periodicity / Spectral atoms





- In higher dimensions (except 24 Cohn et al. [2017]), densest packings are not periodic
- Irregular variance on rectangular windows: 1 or $\geqslant 2^d$ for $\mathbb{Z}^d$
- What is a disordered process?
  - No **Bragg peaks**, i.e. atoms in the spectrum
  - Ergodic theory **mixing**, i.e. for A, B bounded

$$\mathsf{P} \cap A | \mathsf{P} \cap \tau_x B \xrightarrow[x \to \infty]{\text{Law}} \mathsf{P} \cap A.$$

  - Isotropy: $\theta \mathsf{P} \stackrel{(d)}{=} \mathsf{P}$ for any rotation θ

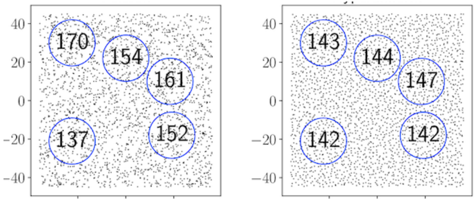
Shifted grid is not mixing

For $A = B = B_\varepsilon$, if one restricts $x \in \mathbb{Z}^d$,

$$\mathbf{P}(\mathbf{Z} \cap B_\varepsilon \neq \emptyset \,|\, \mathbf{Z} \cap B(x,\varepsilon) \neq \emptyset)$$
$$= \mathbf{P}(\tau_U \mathbb{Z}^d \cap B_\varepsilon \neq \emptyset | U - x \in \mathbb{N} \pm \varepsilon) \to 1.$$

Rk: intersections of stationary lines process is mixing but not so much disordered.

Disordered HU "Chicken optical receptors"



Left: Poisson process. *Right:* Disordered HU process (Ginibre)

We present the following examples of disordered mathematically HU processes:

- Projector DPPs / Coulomb systems / OCPs (Sine kernel in 1D, Ginibre in 2D, ...)
- Zeros of planar GAFs in 2D
- Poisson-Coulomb allocation process in 3D +

Stationary point processes and structure factor

Stationary point processes

- A **Simple point configuration** is a measure $P = \sum_i \delta_{x_i}$ where the x_i are distinct, countable, isolated. Let $\mathcal{N} = \mathcal{N}(\mathbb{R}^d)$ the class of all such configurations.
- σ-algebra $\mathcal{B}(\mathcal{N})$ generated by mappings

$$\varphi_K : P \mapsto \# P \cap K, K \subset \mathbb{R}^d \text{ compact.}$$

- **Simple point process** P: Random element of $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$
- Can be seen as a **random set** "$\# P \cap A := P(A)$".
- P is **stationary** iff $\forall x \in \mathbb{R}^d$,

$$\tau_x \mathsf{P} := \mathsf{P} + x := \{y + x; y \in \mathsf{P}\} \overset{(d)}{=} \mathsf{P}.$$

Covariance measure

- Let P **stationary** and **locally square integrable** (LSI):

$$\mathbf{E}(\mathsf{P}(K)^2) < \infty, K \text{ compact}.$$

- The covariance measure $\mathscr{C}$ is defined by

$$\mathscr{C}(dx) = \mathrm{Cov}\left(\mathsf{P}(d0), \mathsf{P}(dx)\right)$$

- More formally, define for $f, g \in \mathcal{C}_c^b(\mathbb{R}^d)$ (bounded with compact support)

$$\mathsf{P}(f) = \sum_{x \in \mathsf{P}} f(x)$$

$$\mathrm{Cov}\left(\mathsf{P}(f), \mathsf{P}(g)\right) = \int f(x) g(x+y) \mathscr{C}(dy) dx$$

Remark: Only well defined on bounded sets

Disintegration

$$\begin{aligned}\mathrm{Cov}\left(\mathsf{P}(f), \mathsf{P}(g)\right) &\text{ is linear in both } f, g\\ &= \int f(x)\bar{g}(x+y)\mu(dx, dy)\end{aligned}$$

μ invariant under x-translations:

$$\mu(dx, dy) = dx\mathscr{C}(dy)$$

SDP:

$$\int f(x)\bar{f}(x+y)dx\mathcal{C}(dy) = \sum_i f_i \bar{f}_j C_{j-i} \geqslant 0$$

implies $\hat{C} \geqslant 0$.
$C = \sum_j \hat{c}_j \omega_n^j J^j$ with $\hat{c}_j \geqslant 0$

Spectral measure

$\mathscr{C}$ is semi-definite positive:

$$\int \underbrace{f(x)f(x+y)}_{f\tilde{\star}f(y)} dx\mathscr{C}(dy) = \mathrm{Var}\,(\mathsf{P}(f)) \geqslant 0.$$

- **Bochner's theorem**: There is a non-negative symmetric measure $\mathscr{S}$ such that $\mathscr{C} = \mathrm{Fourier}(\mathscr{S})$ in the sense of distributions:

$$\langle \mathscr{C}, f\rangle = c_d \langle \mathscr{S}, \hat{f}\rangle$$

  for f a Schwarz function ($\mathcal{C}^\infty$ with fast decaying derivatives).
- It gives the *phase-space variance formula*

$$\mathrm{Var}\,(\mathsf{P}(f)) = c_d \int |\hat{f}(u)|^2 \mathscr{S}(du), f \in \mathscr{C}_c^b(\mathbb{R}^d), \qquad (c_d = (2\pi)^{-d})$$

- $\mathscr{S}$ is called *Bartlett's spectral measure*,

Time series

- Let $\{X_k; k \in \mathbb{Z}^d\}$ a class of stationary L^2 (non-necessarily independent) random variables
- $\mathscr{C}(k) = \mathrm{Cov}\,(\mathsf{X}(0), \mathsf{X}(k)) \leqslant \mathrm{Var}\,(\mathsf{X}(0))$
- $\mathscr{S}(u) = \sum_k \mathscr{C}(k) e^{ik \cdot u}$ (if $\mathscr{C}(k)$ decreases fast enough) on the torus

Example: homogeneous Poisson process P with unit intensity

Variance of linear statistics: only the diagonal terms remain (replace $\sum_{x\in \mathsf{P}} f(x)$ with $\sim \sum_{i=1}^n f(X_i)$ and $A = \mathrm{supp}(f)$ is fixed)

$$\begin{aligned}
\mathrm{Var}\,(\mathsf{P}(f)) =& \mathbf{E}\sum_{x,y} f(x)f(y) - \left(\mathbf{E}\sum_{x} f(x)\right)^2 \\
=& \mathbf{E}\sum_{x\neq y} f(x)f(y) + \mathbf{E}\sum_{x} f(x)^2 - \left(\int f\right)^2 \\
=& \int f(x)f(y)dxdy + \int f^2 - \int f(x)f(y)dxdy = \int f(x)f(x+y)\delta_0
\end{aligned}$$

Hence $\mathscr{C} = \delta_0$ (atomic nature of a point process), and

$$\mathscr{S} = \textbf{Leb}^d.$$

For a "disordered process" P, one expects $\mathscr{S}(du) = (1 + o_{u\to\infty}(1))du$.

Example: shifted lattice $\mathsf{P} = \mathbf{Z} = \{k + U; k \in \mathbb{Z}^d\}$. Let f, g test functions

$$\mathbf{E}\mathbf{Z}(f)\mathbf{Z}(g) = \sum_{k,m} \int_{[0,1]^d} f(k+u)g(m+u)du$$

$$= \sum_k \int_{[0,1]^d} f(k+u) \underbrace{\sum_l g(k+u+l)}_{\tilde{g}(k+u)} du = \int_{\mathbb{R}^d} f(v) \sum_l g(v+l) dv$$

$$\mathrm{Cov}\left(\mathbf{Z}(f), \mathbf{Z}(g)\right) = \int f(v) g(v+y) \sum_l \delta_l(y) dv - \int f \int g$$

$\Rightarrow$ $\mathscr{C} = \sum_{\mathbf{m} \in \mathbb{Z}^d} \delta_{\mathbf{m}} - \mathbf{Leb}^d$ Then with **Poisson summation formula**

$$\langle \mathscr{S}, \varphi \rangle = \langle \mathscr{C}, \hat{\varphi} \rangle = \sum_{\mathbf{m} \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}) - \int \hat{\varphi} = \sum_{\mathbf{k} \in 2\pi\mathbb{Z}^d} \varphi(\mathbf{k}) - \varphi(0)$$

$$\mathscr{S} = \sum_{\mathbf{k} \in 2\pi\mathbb{Z}^d \setminus \{0\}} \delta_{\mathbf{k}}. \quad \text{Remark: } \mathscr{S}(B_{1/2}) = 0.$$

Equivalent definitions of hyperuniformity (HU)

Say $f \in L^1(\mathbb{R}^d)$ is regular if

$$|\hat{f}(u)| \leqslant c(1 + \|u\|)^{-\frac{d+1}{2}} \text{ (implies } f, \hat{f} \in L^2(\mathbb{R}^d)).$$

Proposition

Let P *a LSI stationary point process, the three following conditions are equivalent*

(i) P *is HU, i.e.* $Var(\mathsf{P}(B_R)) = o(R^d)$

(ii) *for some* f *regular with* $\int f \neq 0$ $Var(\mathsf{P}(f_R)) = o(R^d)$
(Remark: (i) is (ii) with $f = 1_{B_1}$*)*

(iii) P *is "spectrally hyperuniform "* $\mathscr{S}(B_{1/R}) = o(R^{-d})$ *Björklund and Hartnick [2024]*

Lemma

- $f = 1_{B_1}$ *is regular and satisfies*

$$|\hat{f}(u)| = (1 + \|u\|)^{-\frac{d+1}{2}}(\sin(\|u\| + c_d) + o_{u\to\infty}(1))$$

- *The spectral measure of any stationary point process satisfies*

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-\frac{d+1}{2}} \mathscr{S}(du) < \infty.$$

(It implies that $\mathscr{S}$ *is a tempered measure)*

Remark: Most of the theoretical material of this mini-course is valid for wide-sense stationary measures i.e. M LSI such that

$$\mathrm{Var}\left(\mathsf{M}(\tau_x f)\right) = \mathrm{Var}\left(\mathsf{M}(f)\right)$$

proof I

(i) implies (ii) is obvious
(ii) implies (iii): $f \in L^1$ implies $\hat{f}$ continuous, hence $\hat{f} \geqslant \kappa > 0$ on some B_ε. Then

$$\kappa^2 \mathscr{S}(B_{\varepsilon/R}) \leqslant \int_{B_{\varepsilon/R}} |\hat{f}(Ru)|^2 \mathscr{S}(du) \leqslant \int |\hat{f}(Ru)|^2 \mathscr{S}(du)$$
$$= cR^{-2d} \mathrm{Var}\left(\mathsf{P}(f_R)\right)$$

QED.
(iii) implies (i) (or (ii)):

$$\mathrm{Var}\left(\mathsf{P}(B_r)\right) \leqslant R^{2d} \int (1 + R\|u\|)^{-d-1} \mathscr{S}(du)$$
$$\leqslant R^{2d} \underbrace{\int_{B_1} (1 + R\|u\|)^{-d-1} \mathscr{S}(du)}_{o(R^{-d}) \text{ if } \mathscr{S}(du) = o(1)du} + R^{2d} \int_{B_1^c} (R\|u\|)^{-d-1} \mathscr{S}(du)$$

proof II

Easy proof here if $\mathscr{S}(du) = s(u)du$ with $s(u) \to 0$. Otherwise...

$$\begin{aligned}
\leqslant & cR^{2d} \int_0^1 \mathscr{S}(\{u \in B_1 : (1 + R\|u\|)^{-d-1} > t\})dt \\
& + R^{-d-1} \int_{B_1^c} \|u\|^{-d-1} \mathscr{S}(du) \\
\leqslant & cR^{2d} \int_0^1 \underbrace{\mathscr{S}(\{u \in B_1 : \|u\| < \frac{t^{-\frac{1}{d+1}} - 1}{R}\})}_{o(R^{-d}(t^{-\frac{1}{d+1}}-1)^d)} dt + O(R^{-d-1}) \\
\leqslant & cR^{2d} \int_0^{10^d R^{-d-1}} \mathscr{S}(B_1) + \int_{10^d R^{-d-1}}^1 \mathscr{S}(B_{t^{-\frac{1}{d+1}}/R}) + O(R^{-d-1}) \\
\leqslant & O(R^{-d-1}) + \int_{10^d R^{-d-1}} R^{-d} o(t^{-d/(d+1)})
\end{aligned}$$

Proof of lemma 2: take $f = 1_{B_1}$. Then as $u \to \infty$

$$\hat{f}(u) \sim \|u\|^{-\frac{d+1}{2}} \sin(\|u\| + c_d)$$

$$\exists z_1, \dots, z_{d+1} : \sum_{j=1}^{m} |\tau_{z_j} \hat{f}(u)|^2 \geqslant \|u\|^{-(d+1)} \sum_{j=1}^{d+1} \sin(\|u - z_i\|^2 + c_d)$$

$$\geqslant c\|u\|^{-(d+1)} \text{ (for large } u\text{)}$$

(1D: $\widehat{1_{[-1,1]}}(u) = \frac{\sin(u)}{u}, z_1 = 0, z_2 = \pi/2$) and

$$\infty > \sum_j \mathrm{Var}\left(\mathsf{P}(e^{iz_j \cdot} f)\right) \geqslant c' \int_{B_1^c} (1 + \|u\|)^{-d-1} \mathscr{S}(du)$$

Universal bounds (a.k.a. Rectangles do not fit well with hyperuniformity)

Theorem (Beck [1987])

For a stationary point process P,

$$\limsup_R \frac{Var(\mathsf{P}(B_R))}{R^{d-1}} > 0.$$

Still true for other "regular shapes" or for "disordered point processes" Kim and Torquato [2017].

(Still true for a wide sense stationary random measure)

Counter-examples

- Shifted lattice: $\mathrm{Var}\left(\mathbf{Z}^d([-n,n]^d)\right) = 0$
- Bylehn and Bjorklund [2024] $\liminf_R \frac{\mathrm{Var}(\mathbf{Z}(B_R))}{R^{d-1}} = 0$ iff $d \equiv 1 \mod 4$.
- $\limsup_R \frac{\mathbf{Z}'([-R,R]^d)}{R^{d-1}} = 0$ for some rotated "admissible" lattice $\mathbf{Z}'$
- There exists mixing M such that $\sup_R \mathrm{Var}\left(\mathsf{M}([-R,R]^d)\right) < \infty$

Beck

$$\int_0^{\mathbf{R}} \frac{\mathrm{Var}\left(\mathsf{P}(B_R)\right)}{R^{d-1}} d\mathbf{R} \geqslant \int\int R^{d+1} |\hat{f}(Ru)|^2 dR \mathscr{S}(du)$$
$$\geqslant \sim \int\int_0^{\mathbf{R}} R^{d+1}(1+\|u\|R)^{-(d+1)} dR \mathscr{S}(du)$$
$$= \int\int_0^{\mathbf{R}} (1/R+\|u\|)^{-(d+1)} dR \mathscr{S}(du)$$
$$\geqslant \int\int_0^{\mathbf{R}} (1+\|u\|)^{-(d+1)} dR \mathscr{S}(du) > c\mathbf{R}$$

Some consequences

- Proving the HU of some model is easier with a smooth linear statistic f than with 1_{B_1}
- The shifted lattice $\mathbf{Z}^d$ is HU because $\mathscr{S}(B(0,1/2)) = 0$
- **Exercise**: The structure factor of the perturbed lattice $\mathbf{Z}^{d,\mu}$ is

$$\mathscr{S}(du) = (1 - |\hat{\mu}(u)|^2)\textbf{Leb}^d(du) + \sum_{\mathbf{k}\in 2\pi\mathbb{Z}^d \setminus \{0\}} \delta_{\mathbf{k}} |\hat{\mu}(\mathbf{k})|^2$$

where

$$\hat{\mu}(u) = \int e^{iu\cdot x}\mu(dx), u \in \mathbb{R}^d.$$

Hence it is HU: $\mathscr{S}(du) = o(1)du$ as $u \to 0$.

Factorial moment measures

- If P has **local moments** of order $k \in \mathbb{N}$, i.e. for K compact

$$\mathbf{E}\left[(\#\mathsf{P} \cap K)^k\right] < \infty,$$

its (symmetric) k-th **factorial moment measure** ρ_k^{P} is defined through symmetric test functions φ :

$$\mathbf{E} \sum_{\underbrace{\{x_1, \dots, x_k\}}_{\text{distinct}} \subset \mathsf{P}} \varphi(x_1, \dots, x_k) = \int \varphi(x_1, \dots, x_k) d\rho_k^{\mathsf{P}}(x_1, \dots, x_k)$$

- **Intensity**: for stationary P, $\rho_1(dx) = \lambda dx$ where $\lambda > 0$ ($\lambda = 1$ by default, *unit intensity*).
- HU, $\mathscr{C}$, $\mathscr{S}$ only involve ρ_1, ρ_2 (variance of linear statistics)

Characterisation by moments

- **Poisson:** Campbell formulas yield $\rho_m^{\mathsf{P}} \equiv \rho_1^{\mathsf{P}} \equiv \lambda$ (intensity)
- ρ_m^{P} determines ρ_k^{P} and $\mathbf{E}\mathsf{P}(A)^k$ for $k \leqslant m, A \subset \mathbb{R}^d$
- The $\mathbf{E}\mathsf{P}(A)^k$ characterise the law of $\mathsf{P}(A)$ if $\mathsf{P}(A)$ has some finite exponential moment

Convergence:

- Vague topology on $\mathcal{N}(\mathbb{R}^d) : P_n \to P$ if $P_n(f) \to P(f)$ for all $f \in \mathscr{C}_c^\infty(\mathbb{R}^d)$.
- For stationary P with exponential moments

$$\mathsf{P}_n \xrightarrow[n\to\infty]{\text{weak}} \mathsf{P} \text{ iff } \mathsf{P}_n(A) \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}(A)$$

for bounded A , iff for each $m \in \mathbb{N}$, $\rho_m^{\mathsf{P}_n} \to \rho_m^{\mathsf{P}}$ on each bounded set. This is a <u>local</u> convergence.

Remark: for $k < m$

$$\rho_k^{\mathsf{P}}(x_1,\dots,x_k) = \binom{k}{m}^{-1} \int \rho_m^{\mathsf{P}}(x_1,\dots,x_m)dx_{k+1}\dots\dots dx_m$$

Let $X^{(n)} = \{X_1,\dots,X_n\}$ uniform in $B_{1/\sqrt{\pi}}$ (volume 1). Then

$$n^{1/d}X^{(n)} \to \mathsf{P}^{\text{ Poisson}}$$

Proof:

$$\mathbf{P}(\#\{k : X_k \in n^{-1/d}A\}) \to \mathsf{Poiss}(\textbf{Leb}^d(A))$$

Disordered examples: DPPs, GAFs, OCPs

Biblio

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- Hough, Krishnapur, Peres, and Viràg [2009]: Zeros of Gaussian Analytic Functions and Determinantal Point Processes, University Lecture Series. Institute of Mathematical Statistics, 2009.

Random matrix I: Gaussian Orthogonal Ensemble - GOE (1D)

Let $n \geqslant 1$ and

- $X_{i,j} \sim \mathcal{N}(0,1), 1 \leqslant i < j \leqslant n$ i.i.d.
- $X_{i,i} \sim \mathcal{N}(0,2), 1 \leqslant i \leqslant n$ i.i.d.
- $X_{j,i} := X_{i,j}$ for $j > i$,
- $\mathrm{GOE}_n := (X_{i,j})_{1 \leqslant i,j \leqslant n} \in \mathcal{S}_n(\mathbb{R})$ random symmetric matrices
- For $S = (S_{i,j}) \in \mathcal{S}_n(\mathbb{R})$,

$$d\mathbf{P}_{\mathrm{GOE}_n}(dS) \propto \exp(-\underbrace{\sum_{i,j} S_{i,j}^2}_{\mathrm{Tr}(SS^T)}/4)dS$$

- Invariant under the action of an orthogonal matrix O, i.e.

$$O \times \mathrm{GOE}_n \times O^T \overset{(d)}{=} \mathrm{GOE}_n.$$

Eigenvalues

- Let $\mathsf{P}_n^{\mathrm{GOE}} \in \mathcal{N}(\mathbb{R})$ the set of a.s. n distinct eigenvalues. It has density, for $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$,

$$\rho_n^{\mathrm{GOE}}(\Lambda) \propto \prod_{i<j} \underbrace{|\lambda_i - \lambda_j|}_{\text{repulsion}} \underbrace{\exp(-\sum_i \lambda_i^2/4)}_{\text{confinement}} \, d\Lambda$$

The n-th factorial moment measure can be interpreted as a density for a point process with n points a.s.

$$\mathbf{E}\varphi(\mathsf{P}_n^{\mathrm{GOE}}) = \int_{\mathbb{R}^n} \rho_n^{\mathrm{GOE}}(\Lambda)\varphi(\Lambda)d\Lambda.$$

Theorem

There exists a simple stationnary hyperuniform point process $\mathsf{P}^{\mathrm{GOE}} \subset \mathbb{R}$ *such that* $\sqrt{n}\mathsf{P}_n^{\mathrm{GOE}} \xrightarrow[n\to\infty]{\text{weak}} \mathsf{P}^{\mathrm{GOE}}$.

tableau

Decomposition $\mathrm{GOE}_n = O\Lambda O^T$ unique up to signs and order
We want to show the change of variables

$$\int \varphi(\Lambda(S)) \exp(-\mathrm{Tr}(SS^T))dS = \int\int \varphi(\Lambda) \exp(-|\Lambda|^2) \underbrace{J(\Lambda, O)}_{\text{Jacobian}} dSd\Lambda$$
$$= \int_{\mathbb{R}^n} \varphi(\Lambda)\exp(-|\Lambda|^2)| \prod_{i<j} |\lambda_i - \lambda_j| d\Lambda?$$

Rotation invariance: O determined by first eigenvector is uniform on $\mathbb{S}^{d-1}$, hence $J(\Lambda, O) = J_1(\Lambda)dO$. It suffices to understand $J_1(\Lambda)$ around $O = I_n$, with $dO^T = -dO$.

Tableau II

$$S + dS = (O + dO)(\Lambda + d\Lambda)(O + dO)^T$$
$$= O\Lambda O^T + \underbrace{dO\Lambda + \Lambda dO^T}_{dO\Lambda - \Lambda dO \text{ antidiagonal}} + \underbrace{d\Lambda}_{diagonal}$$

It follows that in $O = I_n$

$$J(\Lambda, O) = \begin{array}{c|cc} & \overbrace{}^{diagonal(n)(dO=0)} & \overbrace{}^{antidiagonal(n(n-1)/2)(d\Lambda=0)} \\ d\lambda_i \{ & 1 & 0 \\ dO_{i,j}, i < j \{ & ? & (\lambda_i - \lambda_j) \end{array}$$

rescaling

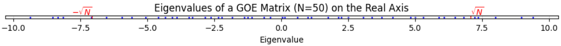
$$\mathbf{E}\sum_{i=1}^{n}\lambda_i^2 = \mathbf{E}\sum_{i,j}X_{i,j}^2 \sim n^2$$

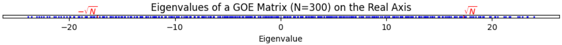
Compared to n i.i.d. variables in $[-n,n]$

$$\mathbf{E}\sum_{i=1}^{n}X_i^2 \sim n^3$$

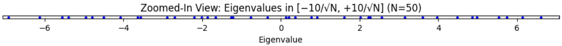
Simus GOE/GUE

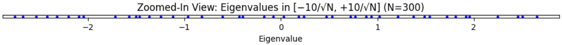
GOE:



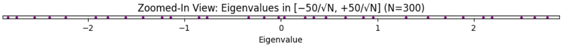


rescaled GOE:





rescaled GUE:



Random matrix II: Gaussian Unitary Ensemble - GUE

- $Z = X + iY \sim \mathcal{N}_{\mathbb{C}}(0,1)$ if and only if X, Y are i.i.d. $\mathcal{N}(0,1/2)$, i.e. Z has density

$$f_Z(z) = \frac{1}{\pi} e^{-|z|^2}, z \in \mathbb{C}$$

Let

- $Z_{i,j}, i < j$ i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$,
- $Z_{i,i} \sim \mathcal{N}_{\mathbb{C}}(0,2)$
- $Z_{j,i} := \bar{Z}_{i,j}$ and the **GUE** model

$$\mathrm{GUE}_n := (Z_{i,j})_{1 \leqslant i,j \leqslant n}.$$

It is a random Hermitian matrix model

$$\mathrm{GUE}_n^* := \overline{\mathrm{GUE}}_n^T = \mathrm{GUE}_n$$

GUE (Cont'd)

GUE Model: for $H \in \mathcal{H}_n(\mathbb{C})$ (Hermitian matrices)

$$\frac{d\mathbf{P}_{\mathrm{GUE}_n}}{dH} \propto \exp(-\sum_{i,j} |H_{i,j}|^2/2)$$

- Invariant under conjugation with unitary matrix U (i.e. $U\bar{U} = I_n$)

Theorem (Wigner)

The eigenvalues $\mathsf{P}_n^{\mathrm{GUE}} = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ *have the density*

$$\rho_n^{\mathrm{GUE}_n}(\lambda_1, \dots, \lambda_n) \propto \prod_{i<j} \underbrace{(\lambda_i - \lambda_j)^2}_{\text{stronger repulsion}} \underbrace{\exp(-\sum_i \lambda_i^2/2)}_{\text{same confining term}}$$

Furthermore, $\sqrt{n}\mathsf{P}_n^{\mathrm{GUE}} \to \mathsf{P}$, *a hyperuniform stationary point process.*

Random matrix III: 2D Complex Ginibre

- Let $Z_{i,j}, 1 \leqslant i, j \leqslant n$ i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1)$, and

$$\mathrm{Gin}_n = (Z_{i,j}),$$

random complex non-Hermitian matrix, density

$$\frac{d\mathbf{P}_{\mathrm{Gin}_n}}{dM} \propto \exp(-\sum_{i,j} |M_{i,j}|^2)$$

- Eigenvalues $\mathsf{P}_n^{\mathrm{Gin}} = \{z_1, \ldots, z_n\} \subset \mathbb{C}$ distributed as a 2D Coulomb system

$$\rho_n^{\mathrm{Gin}}(z) \propto \prod_{i<j} |z_i - z_j|^2 \exp(-\sum_{i,j} |z_i|^2)$$

Interlude: Determinantal Point Processes - DPPs

- A point process $\mathsf{P} \subset \mathbb{R}^d$ is determinantal (DPP) iff there is a kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that for each $m \in \mathbb{N}, x_1, \ldots, x_m \in \mathbb{R}^d$,

$$\rho_m^{\mathsf{P}}(x_1, \ldots, x_m) = \det(K(x_i, x_j)_{1 \leqslant i,j \leqslant m}) \qquad \text{(DPP)}$$

  Furthermore assume K **Hermitian:** $K(y, x) = \overline{K(x, y)}$
- **Poisson homogeneous:** $K(x, y) = \mathbf{1}\{x = y\} \times \text{intensity}$.
- **Canonical kernels**: of the form

$$K(x, y) = \sum_k a_k \varphi_k(x) \bar{\varphi}_k(y)$$

  where $a_k \in [0, 1]$ the φ_k form an orthonormal basis in $L^2(\mathbb{R}^d)$. Then if $a_k \equiv 1$ it has the **replicating** property

$$\int K(x, y) K(y, z) dy = K(x, z).$$

Projection DPPs

Proposition (Dyson)

If K is a replicating kernel, and if (DPP) is true for ρ_n^{P} it is also true for $\rho_m^{\mathsf{P}}, m \leqslant n$: for $x_1, \dots, x_m \in \mathbb{R}^d$,

$$\int_{(\mathbb{R}^d)^{n-m}} \det(K(x_i, x_j)) dx_{m+1} \dots dx_n \propto \det(K(x_i, x_j)_{1 \leqslant i,j \leqslant m})$$

K is also said to be a "projection" kernel in $L^2(\mathbb{R}^d)$: With

$$Kf(x) := \int K(x,y) f(y) dy, f \in L^2(\mathbb{R}^d)$$

we have $K(Kf)(x) = Kf(x)$. If $K = \sum \varphi_k \otimes \bar{\varphi}_k$, K is the projection on the space spanned by the φ_k.

for all m $\rho_m^{\mathsf{P}^{\mathrm{GOE}}} \to \rho_m^{\mathsf{P}}$ for the DPP P with kernel K. and K is invariant under translations: for some function φ,

$$K(z+y, w+y) = K(z, w)$$

proves that ρ_m^{P} too.
Assume $\rho_1 = K(0,0) = 1$ without loss of generality

$$\begin{aligned}
\mathrm{Var}\,(\mathsf{P}(B_R)) &= \int_{B_R^2} K(x,y)K(y,x)dxdy - \int_{B_R} K(x,x) \\
&= \int_{B_R}\int_{B_R} K(x,y)K(y,x)dydx - K(0,0)\textbf{Leb}^d(B_r) \\
&\sim \int_{B_R}\int_{\mathbb{R}^d} K(x,y)K(y,x)dydx - K(0,0)\textbf{Leb}^d(B_r) \\
&\sim \int_{B_R} K(x,x)dx - K(0,0)\textbf{Leb}^d(B_r) = 0.
\end{aligned}$$

Theorem (Ginibre)

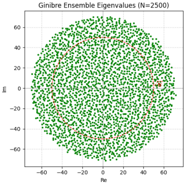
- $\mathsf{P}_n^{\mathrm{Gin}}$ *is a DPP with projection kernel*

$$K_n(z,w) = \sum_{k=0}^{n} \frac{1}{k!} z^k \bar{w}^k e^{-\frac{|z|^2+|w|^2}{2}},$$

- $\mathsf{P}_n^{\mathrm{Gin}}$ *converges to the stationary hyperuniform DPP* $\mathsf{P}^{\mathrm{Gin}}$ *with projection kernel*

$$K(z,w) = e^{z\bar{w}} e^{-\frac{|z|+|w|^2}{2}}.$$

No rescaling needed! $\mathbf{E}\sum_{i=1}^{n}|z_i|^2 = \mathbf{E}\sum_i |Z_{i,j}|^2 \sim n^2$



Ginibre as determinantal

- Where is the determinant? Van Der Monde:

$$\left(\prod_{i<j}|z_i - z_j|\right)^2 = |\det((z_i^{k-1})_{1\leqslant i,k\leqslant n})|^2 = \det|M(z)|^2 = \det(K(z_i, z_j$$

where

$$M(z) = (z_i^{k-1})_{1\leqslant i,k\leqslant n}$$
$$K(z,w) = M(z)M(w)^* = \sum_k z_i^{k-1}\bar{w}_j^{k-1}$$

$$\prod_{i<j}|z_i - z_j|^2 e^{-\sum_i |z_i|^2} = \det((K(z_i, z_j)e^{-(|z_i|^2+|z_j|^2)/2}))$$

DPP proved for ρ_n! ρ_k also if projector kernel.

- Changes nothing if instead

$$M(z) = (\alpha_k z_i^{k-1})_{i,k}$$

Ginibre as DPP I

- Canonical expression of kernels of DPPs:

$$K(z,w) = \sum_k \varphi_k(z)\bar{\varphi}_k(w)$$

where the φ_k form an ONB. Here

$$\int z^k \bar{z}^{k'} e^{-|z|^2} dz = \delta_{k=k'} 2\pi \underbrace{\int \rho^{2k} \rho e^{-\rho^2} d\rho}_{\Gamma(k+1)=k!}$$

hence we choose $\alpha_k = \sqrt{k!}^{-1}$ and we have

$$K_n(z,w) \propto \sum_{k=1}^{n} \frac{1}{k!} z^k \bar{w}^k$$

Ginibre stationarity I

$K(z,w)=e^{z\bar w}e^{-\frac{|z|^2+|w|^2}{2}}$ is not invariant under shifts:

$$
\begin{aligned}
K(z+v,w+v) =& K(z,w)e^{v\bar w+z\bar v}e^{-z\bar v/2-v\bar z/2}e^{-w\bar v/2-v\bar w/2}\\
=& K(z,w)\exp\Bigg(\underbrace{z\bar v/2-v\bar z/2}_{\varphi(z,v)}\Bigg)\exp\Bigg(\underbrace{v\bar w/2-w\bar v/2}_{\bar\varphi(w,v)}\Bigg)
\end{aligned}
$$

and φ has modulus 1

$$\bar\varphi(z,v)=-\varphi(z,v)$$

Ginibre stationarity II

hence with $\varphi_i = \varphi(z_i, v)$

$$\begin{aligned}\rho_m^{\mathsf{P}^{\mathrm{Gin}}}(z_1+v,\dots,z_m+v) &= \det((K(z_i+v,z_j+v))_{1\leqslant i,j\leqslant n}) \\ &= \det((K(z_i,z_j)\varphi_i\bar{\varphi}_j)_{i,j}) \\ &= \rho_m^{\mathsf{P}^{\mathrm{Gin}}}(z_1,\dots,z_q)\underbrace{|\varphi_1|^2\dots|\varphi_q|^2}_{=1}\end{aligned}$$

hence $\tau_v \mathsf{P}^{\mathrm{Gin}} \stackrel{(d)}{=} \mathsf{P}^{\mathrm{Gin}}$.
A similar reasoning holds for rotations, but isotropy of $\mathsf{P}^{\mathrm{Gin}}$ is also a consequence of that of $\mathsf{P}_n^{\mathrm{Gin}}$.

GUE

The GUE eigenvalues form a projection DPP: for some projection kernel K_n,

$$\prod_{i<j}(\lambda_i - \lambda_j)^2 \exp(-\sum_{i,j}\lambda_i^2/2) \propto \det(K_n(\lambda_i, \lambda_j); 1 \leqslant i, j \leqslant n)$$

Furthermore, on each compact of $\mathbb{R} \times \mathbb{R}$

$$K_n\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}\right) \to K(x, y) \propto \frac{\sin(c(x-y))}{x-y} \text{ “sine kernel”}.$$

Theorem

- *The DPP with the sine kernel is stationary projection and* $\sqrt{n}\mathsf{P}^{\mathrm{GUE}_n} \to \mathsf{P}^{\mathrm{GUE}}$
- *Stationary DPPs with L^2 projection kernels are hyperuniform.*

GUE as DPP I

- Back to

$$\prod_{i<j}(x_i - x_j)^2 \propto \det(M(x))^2$$

with

$$M(x)_{i,k} = (\alpha_k x_i^{k-1}).$$

- The determinant does not change if a linear combination of lower degree columns are added:

$$\tilde{M}(x)_{i,k} = (\underbrace{\alpha_k x_i^{k-1} + \sum_{l<k-1} a_l x_i^l}_{P_{k-1}(x)})$$

GUE as DPP II

- Choose as $P_k(x)$ the Hermite polynomials, defined as orthonormal basies for $e^{-x^2/2}$

$$\int P_k(x)P_l(x)e^{-x^2/2}dx = \delta_{k=l}$$

- Then $\mathsf{P}_n^{\mathrm{GUE}}$ is a DPP with kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} P_k(x)P_k(y)e^{-\frac{x^2+y^2}{4}}.$$

- Computing the limit of K_n is more delicate.

Gaussian Analytic Functions - GAFs (Hough, Krishnapur, Peres, and Viràg [2009])

- A Gaussian Analytic Function on a domain $D \subset \mathbb{C}$ is a random field

$$F(z) = \sum_k Z_k \psi_k(z), D \mapsto \mathbb{C}$$

  where the $Z_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are i.i.d. and the ψ_k are holomorphic.
- Alternatively, it is characterised by the fact that FIDI can be written

$$(F(z_1), \ldots, F(z_q)) \stackrel{(d)}{=} M_{z_1,\ldots,z_q} \times (Z_1, \ldots, Z_q)$$

  for some matrix $M_{z_1,\ldots,z_q}$ and i.i.d. $Z_i \sim \mathcal{N}_{\mathbb{C}}(0,1)$.
- It is not a random complex analytic function which is Gaussian .
- A centered GAF $F(z)$ is uniquely determined (in law) by its *complex covariance function*

$$C(z,w) = \mathbf{E}[F(z)\bar{F}(w)] = \sum_k \psi_k(z)\bar{\psi}_k(w)$$

Bonus slide: field is Analytic and Gaussian but not GAF

$F(z) = A\cos(z) + B\sin(z)$ where A, B are correlated. $F(0) = (A, B)$ is not $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ as A, B are not independent.

The planar GAF

- The **planar GAF** is the unique GAF F on $\mathbb{C}$ with complex covariance

$$C(z,w) = \mathbf{E}F(z)\bar{F}(w) = e^{z\bar{w}} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \bar{w}^k$$

or equivalently, for some i.i.d. $Z_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$,

$$F(z) = \sum_k Z_k \frac{z^k}{\sqrt{k!}}$$

Theorem

The zero set $\mathsf{P}^{\mathrm{GAF}} = \{z : F(z) = 0\}$ *is stationary, isotropic and hyperuniform.*

GAF stationarity

Stationarity: there is φ such that $\tau_w F(z) \overset{(d)}{=} e^{\varphi(z,w)} F(z)$, i.e.

$$\mathbf{E}F(z+w)\overline{F(z'+w)} = e^{\varphi(z,w)} e^{z\bar{w}} e^{\overline{\varphi(z',w)}}$$

.

Remember Ginibre

$$C(z+v, w+v) = C(z,w) e^{\varphi(z,v)} e^{\overline{\varphi(w,v)}}$$

F and $\tilde{F}$ have the same zeros.
Similar for rotations.

Forrester and Honner [1999]

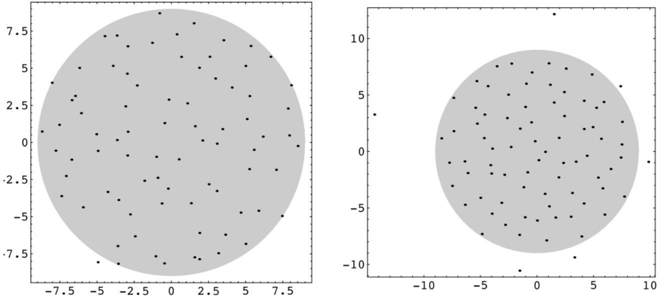


Figure 1. A typical realization of the eigenvalues of a 81×81 complex Gaussian random matrix (leftmost plot) and the zeros of a complex Gaussian random polynomial of degree 81 with variances given by (1.6). The shaded region represents the disc $|z| < 9$ which is the leading order support of the density in both cases. Outside the disc the density has a $1/r^2$ tail in the case of the zeros, whereas it falls off as a Gaussian for the eigenvalues, in keeping with the realizations in the figure.

Particles systems

Consider for $\beta > 0, n \in \mathbb{N}$ the P_n^β "β-ensembles" on $\mathbb{R}$ with density

$$\rho_n^\beta(\lambda_1, \dots, \lambda_n)) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\beta \sum_i \lambda_i^2/4)$$

(eigenvalues of some random matrix $H \in \mathcal{H}_{n,\beta}$)

- $\beta = 1$: $\mathsf{P}_n^{\mathrm{sine}_1} = \mathsf{P}_n^{\mathrm{GOE}}$
- $\beta = 2$: $\mathsf{P}_n^{\mathrm{sine}_2} = \mathsf{P}_n^{\mathrm{GUE}}$
- $\beta = 4$: Gaussian Symplectic Ensemble (GSE)

Theorem (Valkó and Viràg [2009])

There is a stationary hyperuniform process $\mathsf{P}^{\mathrm{Sine}_\beta}, \beta > 0$ *such that*

$$\sqrt{n}\mathsf{P}_n^{\mathrm{Sine}_\beta} \to \mathsf{P}^{\mathrm{Sine}_\beta}$$

Higher dimensional OCPs / Coulomb gases

Let the 2D OCP/β-Coulomb gas P_n^β

$$\rho_n^\beta(x_1, \dots, x_n) \propto \prod_{i<j} \|x_i - x_j\|^\beta \exp(-\beta \sum_i \|x_i\|^2/2)$$

We see that $\mathsf{P}_n^2 \overset{(d)}{=} \mathsf{P}_n^{\mathrm{Gin}}$, but it is the only tractable example. In higher dimensions, let the Coulomb (electrostatic/gravitationnal/...) potential

$$\mathrm{Coul}_d(x) = \begin{cases} -\ln(\|x\|) \text{ if } d = 2 \\ \frac{1}{\|x\|^{d-2}} \text{ if } d \geqslant 3. \end{cases}$$

Fundamental property: $\Delta \mathrm{Coul}_d = -\kappa_d \delta_0$ for some (explicit) $\kappa_d > 0$.
Define the n-particles Coulomb Gas $\mathsf{P}_n^{d,\beta}$ as the one with density

$$\rho_n^{d,\beta}(x_1, \dots, x_n) \propto \prod_{i<j} \exp(-\beta \mathrm{Coul}_d(x_i - x_j) - \beta \sum_i \|x_i\|^2/2)$$

Hyperuniformity of OCPs

- In dimension $d \geqslant 2$, the existence of an infinite stationary limit $\mathsf{P}_n^{d,\beta} \to \mathsf{P}^{d,\beta}$ is not clear, but there exist candidate stationary systems satisfying the corresponding DLR equations see the survey Lewin [2022]
- Leblé proved uniform hyperuniformity for finite systems:

Theorem (Leblé [2023], Hyperuniformity of the "bulk" 2D OCP)

$$\sup_n \frac{Var\left(\#\mathsf{P}_n^{2,\beta} \cap B_R\right)}{R^2} \leqslant \frac{c}{\ln(R)^{0.6}}.$$

- Hyperuniformity is "expected" for all Coulomb gases, and even the more general Riesz gases Lewin [2022]

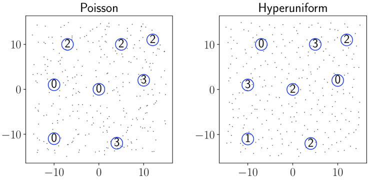
Summary

- $d = 1$, β-ensembles $\xrightarrow[n\to\infty]{}$ Sine_β stationary process, hyperuniform
  - $\beta = 1 :$ GOE, "Pfaffian" process
  - $\beta = 2 :$ GUE, Determinantal
  - $\beta = 4 :$ GSE, Pfaffian
- $d = 2$, OCPs, $\xrightarrow[\to\infty]{}?$, uniformly hyperuniform
  - $\beta = 2, \xrightarrow[n\to\infty]{}$ $\mathrm{Ginibre}$, Determinantal
- $d > 2$, OCPs, conjectured hyperuniform with stationary infinite limit

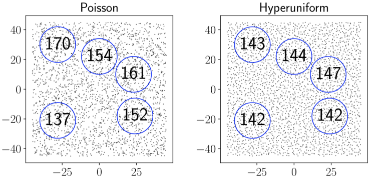
[Lewin, Leblé, Serfaty, ...]

Simulation section: we will see other ways to simulate disordered HU processes.

Global order and local disorder



Local disorder



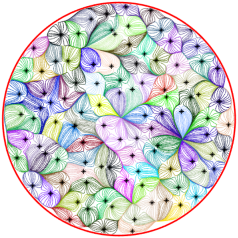
Global order

Two manifestations

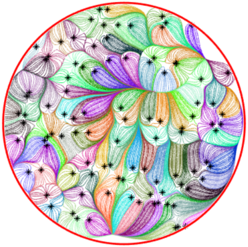
- Optimal transport / allocation / matching
- Rigidity

Efficient allocation

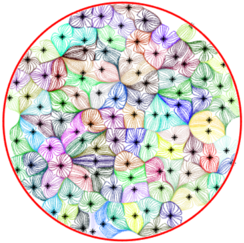
Sample regularity through allocation



The allocation of the disk to 100 Ginibre points.



The allocation of the disk to 100 Uniform points.



The allocation of the disk to 100 zeros of the GAF.

Credit: D. Hawat

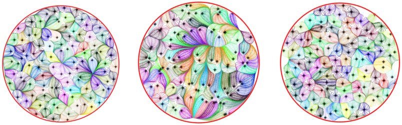
Wasserstein distance

- Let n points $\mathsf{P}_n = \{X_1, \dots, X_n\}$ in a ball Λ_n with volume n.
- **Allocation:** mapping $T : \Lambda_n \to \mathsf{P}_n$ such that for each i, the cell of points allocated to x_i has volume 1:

$$C_i := \mathbf{Leb}^d(T^{-1}(\{x_i\})) \text{ has volume } 1,$$

- Denote the p-transport cost, by

$$\begin{aligned} W_p^p(\mathsf{P}_n, \mathbf{Leb}^d_{\Lambda_n}) &= \inf_T \int \|T(x) - x\|^p \\ &= \inf_T \sum_i \int_{C_i} \|X_i - x\|^p dx. \end{aligned}$$



AKT Theorem

Mean expected p-transport cost per particle:

$$\frac{1}{n}\mathbf{E}W_p^p(\mathsf{P}_n, \mathbf{Leb}_n^d) \sim \mathbf{E}\int_{C_1} \|X_1 - x\|^p dx.$$

Theorem (Ajtai, Komlos, and Tusnady [1984])

Let P_n made up of n i.i.d. points X_i uniform in Λ_n.
The mean expected p-transport cost is finite...

- *if and only if $p < 1/2$ in dimension 1*
- *if and only if $p < 1$ in dimension 2*
- *for all p if $d \geqslant 3$ (there are actually finite exponential moments)*

Dimensions 1 and 2 are “special”.

Allocation for HU systems in dimension $1, 2$

- As we saw, for i.i.d. points $\frac{1}{n}\mathbf{E}W_1(X^{(n)}, \mathbf{Leb}_n^d) \to \infty$
- L^p-perturbed lattice: let $U_k \sim \mu$ i.i.d. with $\mathbf{E}\|U_k\|^p < \infty$,

$$\mathbf{Z}^\mu := \{\underbrace{k + U + U_k}_{X_k}; k \in \mathbb{Z}^d\}$$

Then $T : (k + [-1/2, 1/2)^d) \to X_k$, is a "good allocation":

$$\mathbf{E}\int_{T^{-1}(X_0)} \|T(x) - x\|^p \sim \underbrace{\mathrm{Vol}(T^{-1}(X_0))}_{=1} \mathbf{E}\|U_0\|^p < \infty.$$

- "Global order": Let P_n a hyperuniform sample,

$$\frac{1}{n}\mathbf{E}W_p^p(\mathsf{P}_n, \mathbf{Z}_n) < \infty?$$

Spoiler:

- "Almost" in dimension 2 for $p = 2$
- $d \geqslant 3$: true even for "Poisson-like" systems (AKT Theorem)

Let P a hyperuniform stationary point process on $\mathbb{R}^2$.

- Lr and Yogeshwaran [2024] : Assume $\mathscr{C}$ is integrable on $\mathbb{R}^d$ and as $x \to \infty$,

$$\int_{\mathbb{R}^d \setminus B_1} \ln(\|x\|)|\mathscr{C}|(dx) < \infty$$

Then with $N = \mathsf{P}(\Lambda_n) \sim n$,

$$\mathbf{E}W_2^2\left(\mathsf{P}1_{\Lambda_n}, \frac{N}{n}\mathbf{Leb}_n^d\right) \leqslant cn.$$

- Butez et al. [2024] Assume for some $p \geqslant 1$

$$\mathbf{E}|\underbrace{\mathsf{P}(\Lambda_n) - \mathbf{E}\mathsf{P}(\Lambda_n)}_{N-n}|^p \leqslant cn^{p/2}\ln(r)^{-p-\varepsilon}$$

then

$$\mathbf{E}W_p^p(\mathsf{P}1_{\Lambda_n}, \frac{N}{n}\mathbf{Leb}_n^d) \leqslant cn.$$

Infinite version

- Assume either P is HU with (Lr and Yogeshwaran [2024])

$$\int_{\mathbb{R}^d \setminus B_1} \ln(\|x\|)|\mathscr{C}|(dx) < \infty$$

or for some $\varepsilon > 0$ (Huesmann and Leblé [2024])

$$\frac{\mathrm{Var}\,(\mathsf{P}(B_r))}{\mathbf{Leb}^d(B_r)} \leqslant c\ln(r)^{-1-\varepsilon}$$

then there exist an invariant infinite L^2 allocation $T : \mathbb{R}^2 \to \mathsf{P}$, i.e. such that $\tau_x T \overset{(d)}{=} T$ for $x \in \mathbb{R}^2$, such that

$$\mathbf{E}\|T(0)\|^2 < \infty.$$

2D Applications

- 2D DPPs $\mathsf{P}^{\mathrm{Gin}}, \ldots$
- GAF zeros $\mathsf{P}^{\mathrm{GAF}}$ (Nazarov, Sodin, and Volberg [2007])
- Coulomb systems?

Heart of the proof (Lr and Yogeshwaran [2024])

Bobkov-Ledoux + inverse Sobolev: for μ with mass n on Λ_n

$$W_2^2(\mu, \mathbf{Leb}_n^d) \leqslant c \sum_{m \in \mathbb{Z}^d, 0 < \|m\| \leqslant n^{1/d}} \|m\|^{-2} \underbrace{\|\hat{\mu}(mn^{-1/d})\|^2}_{\mu = \mathsf{P}_n:\ \text{HU} \Leftrightarrow n \times o_{u \to 0}(1)} + cn$$

Approx structure factor: for $u \neq 0$

$$\mathscr{S}(u) \sim \hat{\mathscr{S}}_n(u) := \frac{1}{n} |\hat{\mathsf{P}}_n(u)|^2 = \frac{1}{n} \mathbf{E} \left| \sum_{x \in \mathsf{P}_n} e^{iu \cdot x} \right|^2, u \in \mathbb{R}^d,$$

Huesmann and Leblé [2024]: Interpretation in terms of finite Coulomb energy

SF approx

indeed, for a nice $\mathscr{C}$, since $1_{\Lambda_n}(u) = 1_{\Lambda_1}(u/n^{1/d})$

$$\begin{aligned}
\mathbf{E}\left|\sum_{x\in \mathsf{P}_n} e^{iu\cdot x}\right|^2 =& \mathrm{Var}\left(\mathsf{P}(\underbrace{1_{\Lambda_n} e^{iu\cdot}}_{f_{u,n}})\right) + (\mathbf{E}\mathsf{P}(f_{u,n}))^2 \\
=& \int f_{u,n} \star \bar{f}_{u,n}(x) d\mathscr{C}(x) + \left(\widehat{1_{\Lambda_n}}(u)\right)^2 \\
=& \int e^{iu\cdot x} \underbrace{1_{\Lambda_n} \star 1_{\Lambda_n}}_{\sim n}(x) \mathscr{C}(dx) + \left(n^{1/d} \widehat{1_{\Lambda_1}(un^{1/d})}\right)^2 \\
=& n\mathscr{S}(u) + o(n) + n^2(1 + n^{1/d}\|u\|)^{-d-1}(1 + o(1))
\end{aligned}$$

for $u \neq 0$

Summary: Allocation and perturbed lattices

- There are good allocations from $\mathbf{Leb}_n^d$ to (perturbed) grids $\mathbf{Z}^{d,\mu} \cap \Lambda_n$
- There are good allocations from $\mathbf{Leb}_n^2$ to HU(+) point samples P_n
- **Question:** Is there a "good matching (i.e. bijection)" between P_n and $\mathbf{Z}^2 \cap \Lambda_n$?
- **Answer:** Yes! (Triangle inequality in optimal transport, not constructive) $\Leftrightarrow$ A HU(+) point process P is a perturbed lattice:

$$\mathsf{P} = \{k + U + T_k\}$$

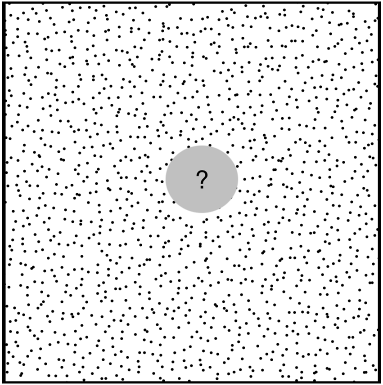
where $\{T_k; k \in \mathbb{Z}^d\}$ is a dependent field invariant under translations: $\tau_k T \stackrel{(d)}{=} T$.

- Dereudre et al. [2024] Converse statement in dimension $d = 1, 2$: a point process P of this form for T a L^2 dependent field is HU.

Matching and allocations

- Imagine each Dirac mass δ_{X_i} is "spread out" as a small disjoint circular shape $\tilde{\mathsf{P}}_n = \cup_{i \in \mathsf{P}_n} B(X_i, \varepsilon)$
- An allocation is a bijection $T : \mathbf{Leb}_n^d \to \tilde{\mathsf{P}}_n$ where $\mathbf{Leb}_n^d(B(X_i, \varepsilon)) = 1$.
- Good allocation $T_1 : \Lambda_n \to \tilde{\mathsf{P}}_n$
- Good allocation $T_2 : \Lambda_n \to \tilde{\mathbf{Z}}^d \cap \Lambda_n$
- $T_1 \circ T_2^{-1}$ bijection between $\tilde{\mathsf{P}}_n$ and $\tilde{\mathbf{Z}}^d \cap \Lambda_n$
- The best allocation between two sets of n (quasi-)points is obtained through a bijection $\sigma : T^{\text{optimal}}(B(x_i, \varepsilon)) = B(y_{\sigma(i)}, \varepsilon)$
  - Proof: Matrix $T_{i,j} =$ mass from i to j, $C_{i,j}$: cost
  - Admissible set is convex: $T_{i,j} \geqslant 0, \forall i, \sum_j T_{i,j} = 1, \forall j, \sum_i T_{i,j} = 1$
  - Cost is linear: $T \mapsto \sum_{i,j} C_{i,j} T_{i,j}$
  - Minima is on extreme point, of the form $T_{i,j} = \delta_{i,\sigma(i)}$

Rigidity and hyperuniform exponent



Rigidity of lattices

- Let U uniform on $[0,1]^d$, and the shifted lattice

$$\mathbf{Z}^d = \{k + U : k \in \mathbb{Z}^d\}.$$

This random point process is **maximally rigid** on any bounded $A \subset \mathbb{R}^d$:

$$\mathbf{Z}^d_A := \mathbf{Z}^d \cap A \in \sigma(\mathbf{Z}^d_{A^c}).$$

- Let now a perturbed (shifted) lattice

$$\mathbf{Z}^{d,\mu} = \{k + U + \underbrace{U_k}_{i.i.d.} : k \in \mathbb{Z}^d\}.$$

Is it **number rigid** on A:

$$\#\mathbf{Z}^{d,\mu}_A \in \sigma(\mathbf{Z}^{d,\mu}_{A^c})?$$

Rigidity of perturbed lattices I

- Are perturbed lattices number rigid?
  - In dimension $d = 1, 2$, the answer is YES if $\mathbf{E}U_k^2 < \infty$ (not optimal)
  - In dimension $d \geqslant 3$ with $U_k \sim \mathcal{N}(0, \sigma^2)$, YES iff $\sigma < \sigma_c$ (NO in dimension $d \geqslant 4$) Peres and Sly [2014]
  - General case?

Rigidity of perturbed lattices II

- k-rigidity: rigidity for higher order moments (on A bounded)

$$\text{0-rigidity} : \#\mathsf{P} \cap A = \int_A d\mathsf{P} \in \sigma(\mathsf{P}1_{A^c})$$

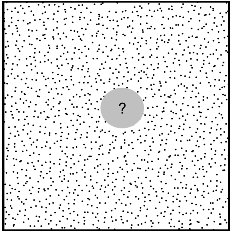
$$\text{1-rigidity} : \sum_{x \in \mathsf{P} \cap A} x_i = \int_A x_i d\mathsf{P}(x) \in \sigma(\mathsf{P}1_{A^c}) \text{ for } 1 \leqslant i \leqslant d$$

$$\Leftrightarrow \quad \text{“Barycenter rigidity”} \sum_{x \in \mathsf{P} \cap A} x \in \sigma(\mathsf{P}1_{A^c})$$

$$\text{k-rigidity} : \int_A x_1^{k_1} \dots x_d^{k_d} d\mathsf{P}(x) \in \sigma(\mathsf{P}1_{A^c}) \text{ for } k_i \geqslant 0, \sum_i k_i = k$$

$$\text{max-rigidity: } \mathsf{P}1_A \in \sigma(\mathsf{P}1_{A^c}) \Leftrightarrow k - \text{rigidity for all } k$$

Rigidity of disordered samples Ghosh and Peres [2017]



Rigidity: can you guess what is inside?

- **Poisson**: $\mathsf{P} \cap A$ and $\mathsf{P} \cap A^c$ are independent: impossible to guess
- **Ginibre** (2D infinite): 0-rigid: one can guess the number of points
- **GAF zeros**: 1-rigid: one can guess the number of points and the sum of the points (or equivalently the barycenter)

Rigidity bibliography

- Notion of tolerance Holroyd and Soo [2013]
- More general studies of the rigidity of DPPs (Bufetov, Dabrowski, Qiu, ...)
- Rigidity of Coulomb gases (Dereudre et al. [2020], Chhaibi and Najnudel [2018], Chatterjee [2019])
- k-Rigidity for zeros of non-stationary GAFs Ghosh and Krishnapur [2021]
- Rigidity of discrete DPPs in the early '00s Lyons and Steif [2003]
- Older result of rigidity Aizenman and Martin [1980]

Uses of rigity

- Use in continuous percolation by Ghosh et al. [2016]
- Used in signal theory and signal reconstruction, related to the completeness question Ghosh [2015]
- Relation with diffusive dynamics of particle systems Osada [2024]
- Used it to prove phase uniqueness for some discrete models from statistical physics Lyons and Steif [2003]

Linear rigidity for hyperuniform systems

- Let f smooth with $f(0)=1$, $f_R(x) := f(x/R), R > 0$

$$
\begin{aligned}
\#\mathsf{P} \cap B_1 &\approx \sum_{x \in \mathsf{P} \cap B_1} f_R(x) \\
&\approx \mathbf{E}(\mathsf{P}(f)) - \sum_{x \in \mathsf{P} \cap B_1^c} f_R(x) + \text{“}O\left(\mathrm{Var}\left(\mathsf{P}(f_R)\right)\right)\text{”}
\end{aligned}
$$

$$
\begin{aligned}
\int_A x^k d\mathsf{P} &\approx \sum_{x \in P \cap A} (x^k f)_R(x) \\
&\approx \mathbf{E}(\mathsf{P}(x^k f)) - \sum_{x \in \mathsf{P} \cap A^c} (x^k f)_R(x) + O\left(\sqrt{\mathrm{Var}\left(\mathsf{P}(x^k f)_R\right)}\right)
\end{aligned}
$$

- Low variance for large linear statistics should imply rigidity.

Linear and non-linear rigidity

- Previous disordered examples are *linearly number rigid*: a.s. and in L^2,

$$\#\mathsf{P} \cap A = \lim_{R\to\infty} \sum_{x\in \mathsf{P}\cap A^c} h_R^A(x) \text{ for some } h_R^A \in L^2(\mathbb{R}^d).$$

- Some perturbed lattice examples are number rigid but not linearly number rigid:
  - An independently perturbed lattice is linearly k-rigid if and only if $d \in \{1,2\}, k = 0$.Lr [2024]
  - if $\|U_k\| < 1/4$ in dimension $d \geqslant 3$
  - Gaussian case in dimension 3 if $\sigma < \sigma_c$ Peres and Sly [2014]
- Examples of rigid but non-linearly rigid disordered point processes?

Radon-Nikodym decomposition:

$$\mathscr{S}(du) = \underbrace{\mathsf{s}(u)}_{\text{spectral density}} du + \underbrace{\mathscr{S}_s(du)}_{\text{singular part}}$$

Theorem (Lr [2024])

A stationary point process P *is linearly* k*-rigid if it has a zero of order* k *"in some complicated algebraic sense".*
In the following simple cases: $k = 0$*, or* $d = 1$*, or* P *is isotropic, or* P *is a DPP,* s *has a zero of order* k *iff*

$$\int_{B_\varepsilon} \frac{\|u\|^{2k}}{\mathsf{s}(u)} du = \infty.$$

Corollary: Number rigidity $(k = 0)$ iff $\int \mathsf{s}^{-1}(u) du = \infty$

- $d = 1$: $\mathsf{s}(u) \leqslant c|u|$ (Lipschitz in 0) OK Ghosh and Lebowitz [2018], Bufetov, Dabrowski, and Qiu [2018]
- $d = 2$: $\mathsf{s}(u) = O(\|u\|^2)$ OK Ghosh and Lebowitz [2018]

Rigidity for DPPs

Theorem

Let P *a stationary DPP with locally* L^2 *kernel* K *and*

$$\kappa(x) := |K(0,x)| \qquad \textbf{Ginibre: } \kappa(z) = e^{-|z|^2/2}.$$

The spectral measure is

$$\mathscr{S}(du) = \underbrace{(1 - \widehat{\kappa^2}(u))}_{\geqslant \sigma u^2} du \qquad \textbf{Ginibre (rescaled): } \mathscr{S}(du) = (1 - e^{-|u|^2})du.$$

for some $\sigma > 0$*. Then* P *is never* k*-rigid for* $k \geqslant 1$ *and* P *is number rigid iff* $d \in \{1,2\}$ *and*

$$\int_{B_\varepsilon} \mathsf{s}(u)^{-1} du = \infty$$

The peculiarity of the GAF zeros

Theorem (Forrester and Honner [1999], Nazarov and Sodin [2011])

Let $\mathsf{P}^{\mathrm{GAF}}$ *the zeros of the planar GAF, then* $\mathscr{S}$ *has a density* s *such that as* $u \to 0$.

$$\mathsf{s}(u) \sim \sigma \|u\|^4 \text{ hence } \int \frac{\|u\|^2}{\mathsf{s}(u)} du = \infty$$

and $\mathsf{P}^{\mathrm{GAF}}$ *is* 1*-rigid (and not further rigid) Ghosh and Peres [2017].*

Based on the relation

$$\begin{aligned}
\mathrm{Var}\left(\mathsf{P}(f)\right) &\leqslant C \|\Delta f\|^2_{L^2(\mathbb{R}^d)} \text{ (Complex analysis)} \\
\mathrm{Var}\left(\mathsf{P}(f_R)\right) &\leqslant C R^{-2} \|(\Delta f)_R\|^2_{L^2(\mathbb{R}^d)}
\end{aligned}$$

Why??

Remember the harmonicity of the 2D Coulomb potential

$$\Delta \ln(\|x - x_i\|) = c_d \delta_{x_i}.$$

For a polynomial F (or for a GAF F on some compact K)

$$|F(z)| = \exp(\underbrace{\psi_K(z)}_{\text{harmonic}}) \prod_i |z - z_i| = c \exp(\psi_K(z) + \sum_i \ln(|z - z_i|)).$$

Linear statistics:

$$c_d \sum_i f(z_i) = \int \Delta \ln |F| f = \int \ln |F(z)| \Delta f \leqslant c_F \|\Delta f\|^2$$

Hyperuniformity exponent

Definition

In general, say $\alpha > 0$ is a *hyperuniformity exponent* of P if as $u \to 0$

$$\mathsf{s}(du) = O(\|u\|^{\alpha})du.$$

It implies k-rigidity for $k \leqslant \frac{\alpha-d}{2}$, and decay of linear statistics

$$\mathrm{Var}\,(\mathsf{P}(f_R)) \sim R^{d-\alpha} \text{ as } R \to \infty \text{ for } f \in \mathcal{C}_c^{\infty}(\mathbb{R}^d).$$

- For independently perturbed lattices, $\alpha = 2$ (0-rigid in $d = 1, 2$),
- For $\mathsf{P}^{\mathrm{Gin}}, \mathsf{P}^{\mathrm{GUE}}$, other HU DPPs, $\alpha \leqslant 2$
- For $\mathsf{P}^{\mathrm{GAF}}$, $\alpha = 4$
- For the shifted lattice, "$\alpha = \infty$" (more HU than a perturbed lattice)
- **Non-lattice process with $\alpha > 4$?**
- Coulomb gas in $d = 1, 2$ (not 1-rigid Dereudre et al. [2020]): $\alpha < 3$
- Coulomb gas in $d \geqslant 3$ (not 0-rigid Thoma [2023]): $\alpha < d$

Decay

Assume $\hat{f}(u) \leqslant c(1+\|u\|)^{-1000}$

$$\begin{aligned}\mathrm{Var}\,(\mathsf{P}(f_R)) \leqslant & \int |\hat{f}_R(u)|^2 \mathscr{S}(du) \\ \leqslant & R^{2d} \int_{B_\varepsilon} |\hat{f}(u/R)|^2 \|u\|^\alpha du + \int_{B_\varepsilon^c} R^{2d}(1+\|u\|/R)^{-2000} \mathscr{S}(du) \\ \leqslant & R^{d-\alpha}(\|\hat{f}\|_2^2(1+o(1))) + O(R^{-100})\end{aligned}$$

A p-rigid process Lr [2024]

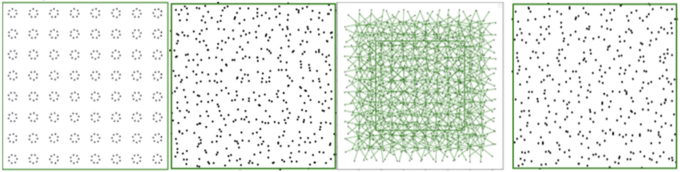
- Let $\mathbf{U}_p \subset \mathbb{C}$ the set of p-th roots of unity ($p \in \mathbb{N}^*$)
- Let $r > 0$ and θ_k i.i.d. uniform rotations in $\mathbb{S}^1$

$$\mathsf{P}_p = \{k + U + r\theta_k \mathbf{U}_p, k \in \mathbb{Z}^2\}$$

- If p is a prime number, the structure factor satisfies

$$\mathsf{s}(u) \sim c\|u\|^{2p}$$

$\Rightarrow$ P_p is $(p-1)$-rigid but not p-rigid on $A = B_{r+1}$.



$r = 0.2, p = 7$ | $r = 1.3, p = 7$ | Making of | $r = 1.3, p = 6$

Maximal rigidity and stealthy processes

Definition

A point process P is **stealthy hyperuniform** if $\mathscr{S}(B_\varepsilon) = 0$ for some $\varepsilon > 0$.

Example: (independent union of) shifted lattices (with pairwise irrational periods): $(\mathbb{Z} + U_1) \cup (\sqrt{2}(\mathbb{Z} + U_2)), \dots$

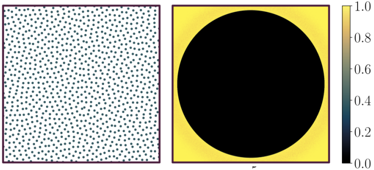


Figure: *Left:* Simulated stealthy HU system. *Right:* Spectral measure. *Credit:* Shih, Casiulis, and Martiniani [2024]

Physical properties: Transparent to certain wavelengths.

Image analysis: “Blue noise”

Theoretical properties of stealthy processes

Let P a stationary stealthy HU point process:

- **Bounded holes:** There is $\rho_-, \rho_+ > 0$ deterministic such that a.s.

$$\forall x \in \mathsf{P}, \mathsf{P}(B(x, \rho_-)) = 1, \forall y \in \mathbb{R}^d, \mathsf{P}(B(y, \rho_+)) > 0.$$

  (Conjectured by Zhang et al. [2017], proved by Ghosh and Lebowitz [2018])

- **Conjecture:** There is a bounded matching, i.e. $T : \mathbb{Z}^d \to \mathsf{P}, c > 0$ invariant such that a.s. for all $k \in \mathbb{Z}^d$, $\|T(k) - k\| \leqslant c$.
- A hyperuniform point process is **max rigid** on a strictly convex cone C (Lr [2025+]), i.e.

$$\mathsf{P}1_C \in \sigma(\mathsf{P}1_{C^c})$$

  (proved on a ball B by Ghosh and Lebowitz [2018])

- **Question:** Do “disordered” (e.g. mixing) stationary stealthy HU systems exist mathematically?? (NO for $d = 1$)

Numerical aspects

Estimation and detection

Estimation of Structure factor

Let P stationary and ergodic. **Empirical structure factor / scattering intensity:**

$$\hat{\mathscr{S}}_n(u) = \frac{1}{n}\left|\hat{\mathsf{P}}_n(u)\right|^2 = \frac{1}{n}\left|\sum_{x\in \mathsf{P}_n} e^{iu\cdot x}\right|^2 \to \mathscr{S}(u) \text{ for } u \neq 0.$$

- Coste [2021] $\hat{\mathscr{S}}_n - \delta_0 \to \mathscr{S}$ in the sense of distributions.
- Hawat, Gautier, Bardenet, and Lr [2023] Estimation on $[-n, n]^d$: If $\mathscr{S} - \textbf{Leb}^d$ is integrable,

$$\sup_{k : k_j \in 2\pi\mathbb{Z}/n \text{ for some } j} |\mathbf{E}\mathscr{S}_n(k) - \mathscr{S}(k)| \to 0.$$

  $+$ test of hyperuniformity from i.i.d. samples.
- Klatt, Last, and Henze [2023+]: Test of hyperuniformity from one sample.

Tapered variants

$$\text{Tapered:}\quad \mathscr{S}_n^{t_n}(u) = \frac{1}{n}\left|\sum_{x\in \mathsf{P}_n} t_n(x)e^{ix\cdot u}\right|^2$$

$$\text{Undirectly debiased tapered:}\quad \left|\mathscr{S}_n^{t_n} - m_{u,n}\right|$$

$$\text{Multi-tapered:}\quad \frac{1}{P}\sum_{p=1}^{P}\mathscr{S}_n^{t_{n,p}}$$

Grainger, Rajala, Murrell, and Olhede [2024]: Quantitative assessments (variance, etc...)
Mastrilli, Blaszczyszyn, and Lavancier [2024]: Estimation of the hyperuniformity index based on multi-tapered estimators with CLTs
Mastrilli [2025+]: Minimax bounds for the speed of convergence $\hat{\mathscr{S}}_n \to \mathscr{S}$

Package structure-factor

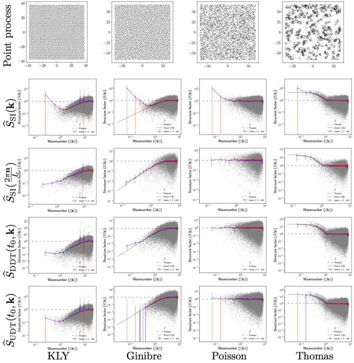


Fig. 5: Variants of the scattering intensity estimator applied to four point processes. The computation and visualization are done using structure_factor

Figure: Hawat, Gautier, Bardenet, and Lr [2023]

https://pypi.org/project/structure-factor/
https://github.com/For-a-few-DPPs-more/structure-factor

Simulation

Simulating 2D HU processes (on a personnal computer)

- Ginibre "naive" (with `np.linalg.eigvals`):
  - 3000pts in 17s
  - 4000pts in 35s
  - 5000pts in 62s
- 2D β-Ginibre ensembles: Lavancier and Rubak [2023]

Algo. 4/5	$\rho = 100$	$\rho = 200$	$\rho = 400$	$\rho = 800$
$\beta = \beta_{\max}/3$	0.58/**0.09**	2.99/**0.36**	19.21/**6.63**	130.14/**96.44**
$\beta = \beta_{\max}/2$	0.17/**0.07**	0.59/**0.27**	**3.11**/4.58	**18.43**/68.83
$\beta = \beta_{\max}$	0.09/**0.07**	0.28/**0.23**	**1.32**/3.43	**6.45**/50.09

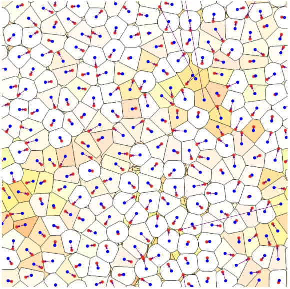
Table 2: Average computation time (in seconds) for a single realization of a β-Ginibre process on $\mathbf{B}_R$, $R = 1/\sqrt{\pi}$, with the given parameters, where $\beta_{\max} = 1/(\rho\pi)$, using Algorithm 4 (left value) and Algorithm 5 (right value). The fastest method is boldfaced.

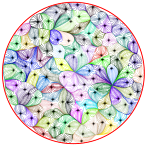
- 2D Jacobi β-ensembles with `DPPy` (Gautier et al. [2021]):
  - 1000pts: 2s
  - 2000pts: 14s
  - 2500pts: 30s

Fair partitions

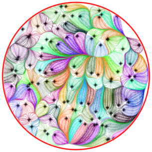
Theorem (Klatt, Last, Lotz, and Yogeshwaran [2025])

Let $C = \{C_i; i \geqslant 1\}$ *a random stationary fair partition (i.e.* $\mathbf{Leb}^d(C_i) = 1$ *a.s.). Let* Y_i *cond. i.i.d. with* $Y_i \sim \mathscr{U}_{C_i}$.
Then $\mathsf{P} = \{Y_i\}$ *is hyperuniform.*

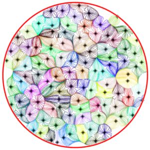




The allocation of the disk to 100 Ginibre points.



The allocation of the disk to 100 Uniform points.



The allocation of the disk to 100 zeros of the GAF.

Gravitationnal allocation

- Let $\mathsf{P} = \{X_i\}$ a stationary point process such that for all x, the Coulomb force field is well defined

$$F_P(x) = \sum_{\|X_i - x\|\uparrow} \underbrace{\frac{x - X_i}{\|x - X_i\|^d}}_{\text{Coulomb force}} \text{ is well defined}$$

- Example: $\mathsf{P}^{\mathrm{Gin}}$ or $\mathsf{P}^{\mathrm{GAF}}$ in $d = 2$, Poisson <u>only</u> in $d \geqslant 3$.
- Let $C_i = \{x \to X_i\}$ (points attracted to X_i)

Theorem (Sodin and Tsirelson [2006])

The C_i form a fair partition.

(Also based on the harmonicity of the Coulomb potential)

- Let Y_i uniform in C_i. $\mathsf{P}' := \{Y_i\}$ is hyperuniform.

Poisson Coulomb Allocation Process simulation in dimension $d \geqslant 3$

- Start from an "easy to simulate" point process $\mathsf{P} = \{X_i\}$ for which the gravitationnal force converges : e.g. Poisson dimension $d \geqslant 3$
- Draw i.i.d. θ_i uniform in $\mathbb{S}^{d-1}$ and i.i.d. $T_i \sim \mathscr{E}(1)$.
- Let Y_i a particle following the force field $-F_P$ (repulsive) starting from X_i in direction θ_i, for a time T_i.
- Then, $\mathsf{P}' := \{Y_i\}$ is hyperuniform.

Approximate simulation: blue noise in image analysis

Relies on hard sphere models.

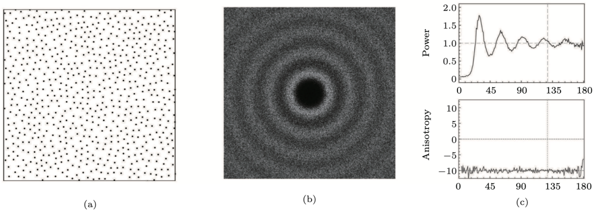


Fig. 1. Example of Poisson-disk sampling and its spectral analysis. (a) A sampled point set. (b) Power spectrum from this point set. (c) Radial means and normal anisotropy.

Figure: Yan et al. [2015]

Approximate simulation

Team of Torquato: *Collective coordinate energy minimisation scheme*
Method: Given fixed K Minimise for N particles the energy of particles interacting through a force F which radial Fourier transform is non-negative and supported by B_1 (in practice they take 1_{B_1}) the energy in the Fourier domain

$$\Phi(x_1, \dots, x_n) = \sum_{k \in \varepsilon\mathbb{Z}^d \cap B_1} \hat{F}(\|k\|) |\hat{\mathscr{S}}(\|k\|)|^2 + \underbrace{\kappa_F}_{\text{ignored}}$$

Parameter χ

Morse et al. [2023] $N = 10^6$ particles with $\mathscr{S}(u) \sim 10^{-50}$, cost $O(N^2)$ (parallelisable). Such precisions required for "understand key properties of the ground-state manifold of the stealthy hyperuniform potential, namely its connectivity and dimension, especially near critical points"

In the simulation, they can choose to move only a fraction χ of the available degrees of freedom

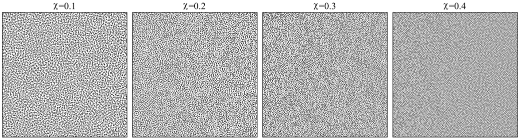


FIG. 1. Stealthy hyperuniform point patterns of $N = 2 \times 10^6$ particles with $\chi = 0.1, 0.2, 0.3$, and 0.4. Each image only shows 1/16th of all the data for better visualizations. Full configuration data are deposited through Princeton Data Commons [35].

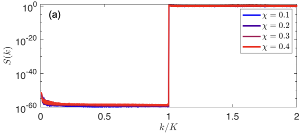
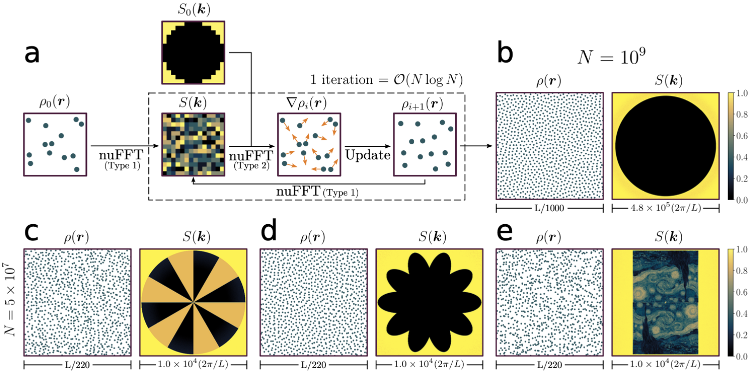


Figure: Morse, Kim, Steinhardt, and Torquato [2023]

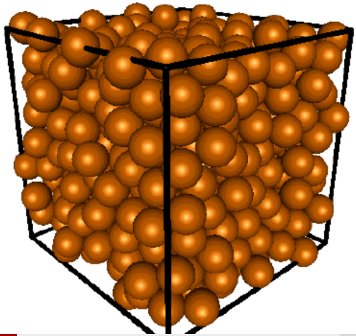
Shih, Casiulis, and Martiniani [2024] generate the "largest ever stealthy hyperuniform configuration"

- 2d: 10^9 particles
- 3d: 10^7 particles



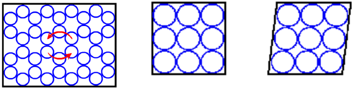
[Physical simulation] Strictly jammed packing Torquato [2018]

- Take n balls of volume $\rho > 0$ in a large volume, i.e. *hard spheres*
- Make a *strictly jammed packing*: "Any collectively jammed configuration that disallows all uniform volume-decreasing strains of the system boundary is strictly jammed, implying that their bulk and shear moduli are infinitely large"



Conjecture

Related to the conjecture that in higer dimensions ($\neq 2, 3, 8, 24$), the densest sphere packings are not lattices.



non-strict packings Torquato [2018]

Conjecture: Any infinite Strictly Jammed Packing is hyperuniform.
Tentative mathematical formulation: Say that a ρ-hard-sphere stationary point process P is *strictly jammed* if there is no family of ρ-hard-sphere stationary point processes $\mathsf{P}_t, t \in [0,1]$ such that $\mathsf{P}_0 = 0$, $t \mapsto \mathsf{P}_t$ is continuous in $\mathcal{N}(\mathbb{R}^d)$, and the intensity λ_t of P_t is non-decreasing with $\lambda_1 > \lambda_0$.

Thank you!

Lecture notes on wy webpage https://helios2.mi.parisdescartes.fr/~rlachiez/recherche/talks/slides-hu.pdf

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