

Hyperuniform number variance bounds for Coulomb gases and Girko ensembles

Hyperuniformity and Topological Data Analysis

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Ginibre ensemble

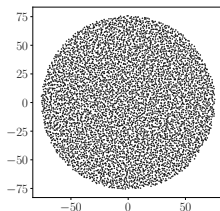


Figure: Ginibre eigenvalues

- Random **Ginibre matrix** $M = (G_{i,j})$ where the $G_{i,j}$ are i.i.d. with law $\mathcal{N}_{\mathbb{C}}(0, 1)$ (i.e. real and imaginary parts are i.i.d. $\mathcal{N}(0, 1)$)
- *Left:* Let $X_N = \text{eigenvalues}(M) \subset \mathbb{C}$. Then for $R \leq \sqrt{n}$,

$$\mathbf{E} [\#X_N \cap B(0, R)] \asymp R^2, \quad \text{Var} (\#X_N \cap B(0, R)) \asymp R.$$

[Ginibre '65]

Definition of hyperuniformity on scales $[1, N^a]$

- Let $X_N = \{x_1, \dots, x_N\} \subset \Sigma := N^{1/d}\Sigma_0$
- **Hyperuniformity of $X_N, N \rightarrow \infty$ on scales $[1, N^a]$:** for $R \in [1, N^a]$

$$\sup_{x \in \Sigma} \text{Var}(X_N(B(x, R))) \ll CR^d \quad (\approx CR^{d-\alpha} \text{ for some } \alpha > 0)$$

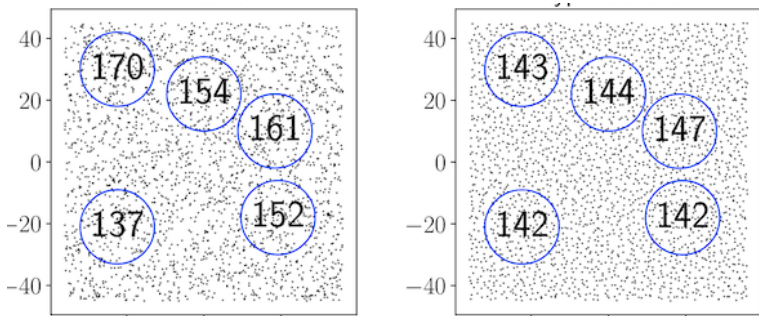


Figure: Left: Poisson / i.i.d. points. Right: “Hyperuniform disordered”

Example 1: Girko matrices (microscopic zoom)

Consider the eigenvalues $\mathbf{X}_N := \{x_1, \dots, x_N\}$ of a $N \times N$ random matrix M with i.i.d. centred entries $M_{i,j} \in \mathbb{C}, 1 \leq i, j \leq N$ with a fixed law with all moments finite and $\mathbf{E}|M_{i,j}|^2 = 1$, e.g. **complex Ginibre ensemble**:

- **Asymptotic homogeneity**: for $f \in \mathcal{C}_c^\infty(B(0,1)), R \in [1, N^{1/2}]$

$$\mathbf{X}_N(f_R) := \sum_{i=1}^N f(x_i/R) \text{ has expectation } \sim R^2 \frac{1}{\pi} \int f$$

- **[Cipolloni, Erdős, Schröder '23]** for $f \in \mathcal{C}_c^1(B(0, 1 - \varepsilon))$,
“hyperuniform / Ginibre-like” universality behaviour:

$$\sup_{N \geq 1, R \in [1, \sqrt{N}]} \text{Var}(\mathbf{X}_N(f_R)) \leq C_f < \infty, \quad f_R(x) := f(x/R).$$

- **For comparison**: If the x_i were i.i.d. / Poisson in $B(0, \sqrt{N})$

$$\text{Var}(\mathbf{X}_N(f_R)) \asymp \mathbf{E}\mathbf{X}_N(f_R) \asymp R^2$$

Girko matrices (Cont'd)

- **[Cipolloni, Erdős, Schroder '23]** hence proved hyperuniformity for smooth linear statistics. *“Extending this study to less regular f creates substantial difficulties”*.
- **[Cipolloni, Erdős, Kolupaiev '26]** For $f = 1_\Omega$ for $\Omega \subset B(0, 1)$ sufficiently smooth, with some additional (“removable”) assumption on the $X_{i,j}$, $R = N^a$ with $a \in (0, 1/2)$,

$$\text{Var}(\mathbf{X}_N(f_R)) \leq C_a R^{2-\varepsilon}$$

where $0 < \varepsilon \leq 1/40$.

$\Rightarrow$ Negligible compared to the Poisson case, *hyperuniform!*

- “Expected” rate: R^1 (like for Ginibre matrices, *class I hyperuniformity*).

Example 2. Coulomb gases /OCPs (One Component Plasmas)

- Let $\mathbf{X}_N^\beta = \{x_1, \dots, x_N\} \subset \mathbb{R}^2$ with joint density

$$Z_N^{-1} \exp(-\beta \underbrace{\sum_{i \neq j} (-\ln(\|x_i - x_j\|))}_{\text{Energy}}) \times \{\text{confining term}\}$$

- Ginibre:** case $\beta = 2$ with confinement $\prod_i \exp(-\|x_i\|^2)$
- [Leblé '23]** Proves hyperuniformity in the bulk for all $\beta > 0, R > R_0$:

$$\sup_{x: B(x, R) \subset \sqrt{N}\Sigma_0} \text{Var} \left(\mathbf{X}_N^\beta(B(x, R)) \right) \leq C_\beta \frac{R^2}{\ln(R)^{0.6}}, \quad \Sigma_0 = B(0, \rho_2 - \varepsilon)$$

- Poisson** (with the same scaling): $\text{Var}(\mathbf{X}_N(B_R)) \asymp R^2$
- What is expected:** $\asymp R^1$, *class I hyperuniformity*

Coulomb gases /OCPs (Cont'd)

- In dimension $d \geq 3$, the β -OCP has density

$$Z_N^{-1} \exp(-\beta \underbrace{\sum_{i \neq j} c_d \|x_i - x_j\|^{2-d}}_{\text{Energy}}) \times \{\text{confining term}\}$$

- [Serfaty '23] shows variance reduction for smooth linear statistics f

$$\text{Var} \left(X_N^\beta(f_R) \right) \leq C_{f,\beta} R^{2d-4}$$

for $d = 2, 3$ (3D Coulomb potential: $\|\cdot\|^{-1}$ instead of $-\ln(\|\cdot\|)$)

$d = 2$: $\text{Var} (P_n(f_R)) \leq C_{f,\beta} = C_{f,\beta} R^{d-2}$ “2-hyperuniformity”

$d = 3$: $\text{Var} (P_n(f_R)) \leq C_{f,\beta} R^2 = C_{f,\beta} R^{d-1}$ “1-hyperuniformity”

- “Treating the case of nonsmooth kernels (...) remains a (significantly more) delicate problem.”
- [Chatterjee '17] Proves 2-HU for the 3D hierarchical Coulomb gas

Variance transfer principle

Goal of this work: prove that if the rate for smooth linear statistics is in $R^{d-\alpha}$, then the rate for balls is $R^{d-\min(\alpha,1)}$ ($\times \ln(R)$ if $\alpha = 1$).

- $\alpha > 0$ is the *hyperuniformity* exponent
- Girko eigenvalues/Coulomb gases: α is conjectured to be 2
- Hence we should obtain “surface order growth”, i.e. class I hyperuniformity

$$\text{Var}(X_N(B(0, R))) \sim R^{d-1}$$

- **[Beck '87]** This is the minimal possible rate for a “homogeneous system of points”

Original definition for infinite processes

- $\mathcal{N}(\mathbb{R}^d)$: **Space of configurations**: locally finite sets, endowed with vague topology (local). A point configuration is seen as a measure

$$P(f) = \sum_{x \in P} f(x).$$

- **Point process** P : Random element of $\mathcal{N}(\mathbb{R}^d)$
- **Stationarity**: invariance in law under translations $\tau_x : y \mapsto x + y$

Definition ([Torquato, Stillinger, '03])

A stationary point process P is hyperuniform (HU) if

$$\text{Var}(P(B_R)) = o(R^d) \text{ where } B_R := B(0, R)$$

- For P a stationary Poisson process, $\text{Var}(P(B_R)) = \lambda R^d$ (not HU)

Isomorphism theorem and spectral characterisation

- Let P be a stationary point process with $\mathbf{E} [P(B(0,1))^2] < \infty$.
- There is a tempered measure S_P (the “spectral measure”) giving a Plancherel type formula

$$\text{Var}(P(f)) = \int |\hat{f}|^2 dS_P, f \in \mathcal{B}_c^b(\mathbb{R}^d) \quad (\text{bounded with compact support})$$

Hyperuniformity is a global variance cancellation phenomenon over linear statistics

Theorem (Torquato, Björklund & Hartnick)

The following are equivalent

- 1 $\text{Var}(P(B_R)) = o(R^d), R \rightarrow \infty$
- 2 $S_P(B(0, \varepsilon)) = o(\varepsilon^d), \varepsilon \rightarrow 0$
- 3 $\text{Var}(P(f_R)) = o(R^d), R > 1$ for some $f \in \mathcal{B}_c(\mathbb{R}^d)$ such that $\int f \neq 0$

Hyperuniformity exponent

- **Definition:** P is α -hyperuniform if

$$S_P(B(0, \varepsilon)) = O(\varepsilon^{d+\alpha})$$

Classical case: S_P has a density of class $\mathcal{C}^2 \Rightarrow$ 2-hyperuniform

Proposition

The three following are equivalent

- (i) P is α -hyperuniform
- (ii) $\exists f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\int f \neq 0$, $\text{Var}(P(f_R)) = O(R^{d-\alpha})$.
- (iii) For all f sufficiently regular, $\text{Var}(P(f_R)) = O(R^{d-\alpha})$

A bounded compactly supported function f is α -regular if for some $\gamma > \alpha$

$$|\hat{f}(\xi)|^2 = O(\|\xi\|^{-d-\gamma})$$

1D / 2D proven examples

- $d = 1$: β -ensembles / Sine_β / log-gases, with $\alpha = 1$ (optimal)
- $d = 2$: **Infinite Ginibre ensemble**, stationary and hyperuniform with $\alpha = 2$ (optimal)
- **[Sodin & Tsirelson '06]** Let $F(z)$ be the planar GAF (Gaussian Analytic Function, limit of *Weyl polynomials*)

$$F(z) = \sum_{k=0}^{\infty} \frac{X_k z^k}{\sqrt{k!}}, \quad X_k \sim \mathcal{N}_{\mathbb{C}}(0, 1), \text{ i.i.d.}$$

and its zeros $\mathcal{Z} = \{z : F(z) = 0\}$. Then $\mathcal{Z}$ is stationary and hyperuniform with $\alpha = 4$, this is the unique “non-toy model” with (proven) $\alpha > 2$.

- **[Joyce et al. '08, Lr '24, Lotz & Klatt '26]** Toy models p -hyperuniform for $p \rightarrow \infty$

Proof of the transfer principle and generalisation

$\varphi = 1_{B(0,1)}$ is 1-regular:

$$|\hat{\varphi}(\xi)|^2 \leq C(1 + \|\xi\|)^{-d-1}$$

- The proof is based on high / low frequency decomposition, using **translation boundedness** of the spectral measure.
- Assume $S(d\xi) \sim \|\xi\|^\alpha$ as $\xi \rightarrow 0$ for simplification

$$\begin{aligned}\mathrm{Var}(\mathbf{P}(\varphi_R)) &= \int_{\mathbb{R}^d} |\widehat{\varphi_R}|^2 dS = R^{2d} \int_{\mathbb{R}^d} |\hat{\varphi}(R\xi)|^2 S(d\xi) \\ &\approx R^{2d} \int_{B(0, \varepsilon/R)} |\hat{\varphi}(0)|^2 \|\xi\|^\alpha d\xi \\ &\quad + R^{2d} \int_{B(0, \varepsilon/R)^c} (1 + \|R\xi\|)^{-d-1} (\|\xi\|^\alpha \wedge O(1)) d\xi \\ &\leq (\dots) \leq CR^{d-\min(\alpha, 1)} \ln(R)^{\mathbf{1}_{\{\alpha=1\}}}.\end{aligned}$$

Similar result on the torus

- Let P_n be a point process on $\mathbb{T}_n := \mathbb{R}^d / (n\mathbb{Z})^d$
- Let τ_x^n the n -periodic translation on $\mathbb{T}_n$
- Say that P_n is $\mathbb{T}_n$ -stationarity if:

$$\tau_x^n P_n \stackrel{(d)}{=} P_n, x \in \mathbb{T}_n.$$

- **Example:** Coulomb gas with periodic boundary conditions:
 $P_n = \{x_1, \dots, x_n\} \subset \mathbb{T}_n$ with density

$$\propto \exp(-\psi_d(x_1, \dots, x_n))$$

where

$$\Delta_{\mathbb{T}_n} \psi_d = \delta_0 - n^{-d} \text{Leb}^d$$

Theorem (Lr '26)

Let P_n be a $\mathbb{T}_n$ -stationary point process with unit intensity.

Assume P_n is α -hyperuniform for some smooth kernel $f \in \mathcal{C}_c^\infty(\mathbb{R}^d) \setminus \{0\}$:
for $R \in [1, n^\gamma]$ ($\gamma \in [0, 1]$)

$$\text{Var}(P_n(f_R)) \leq C_f R^{d-\alpha}$$

then for $R \in [1, n^\gamma]$

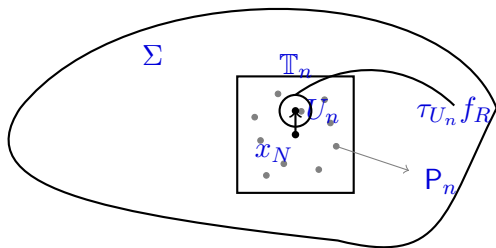
$$\sup_x \text{Var}(P_n(B(x, R))) \leq C_f R^{d-\min(\alpha, 1)} \ln(R)^{1_{\{\alpha=1\}}}$$

The proof is similar to the $\mathbb{R}^d$ case:

- The Fourier dual space of $\mathbb{T}_n$ is $\widehat{\mathbb{T}}_n := 2\pi n^{-1}\mathbb{Z}^d$
- There is a similar isomorphism theorem with a spectral measure S_n on $\widehat{\mathbb{T}}_n$. In some sense, $\widehat{\mathbb{T}}_n \xrightarrow[n \rightarrow \infty]{} \mathbb{R}^d$
- A similar high/low frequency decomposition is possible

Adaptation to “almost stationary” samples

- X_N process on $\Sigma = B(0, N^{1/d})$ (or else)
- P_n : restriction to $x_N + \mathbb{T}_n$ with some $n \ll N^{1/d}$
- Let U_n uniform on $\mathbb{T}_n$, $\tilde{P}_n := \tau_{U_n}^n P_n$ stationary by construction
- Hopefully $\text{Var}(\tilde{P}_n(\tau_x f_R)) \approx \text{Var}(P_n(\tau_x f_R))$ in law, unfortunately boundary effects kill some available scales



Consequences for infinite Coulomb gases / OCPs

Let X^β be an *infinite volume Coulomb gas*, i.e.

$$X^\beta = \lim_{N_j \rightarrow \infty} X_{N_j}^\beta$$

is a “limit point in law” on $\mathbb{R}^d$ for the vague topology. The previous method yields the following:

Theorem

- In dimension $d = 2$, any limit point X^β is stationary and X^β is 2-hyperuniform, i.e. for all $x \in \mathbb{R}^d$

$$\text{Var}\left(X^\beta(B(x, R))\right) \leq C_\beta R < \infty.$$

- In dimension $d = 3$, it gives 1-hyperuniformity if P is assumed to be stationary.

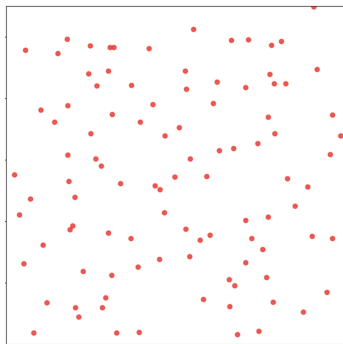
Perturbed lattices in dimension $d = 2$

This proves **finite Coulomb energy** and **finite Wasserstein distance** to Lebesgue measure by [Lr & Yorgeswharan '24, Huesmann & Leblé '25]. It means X^β is a L^2 -perturbed lattice.

- Red: Coulomb gas / OCP particles of X^β
- Blue: Lattice points $k \in \mathbb{Z}^d$
- Gray arrows: Vectors U_k such that

$$X^\beta = \{k + U_k; k \in \mathbb{Z}^d\}$$
$$\{U_k\} \stackrel{(d)}{=} \{U_{k+k_0}\}, k_0 \in \mathbb{Z}^d$$

- L^2 -perturbed lattice $\Leftrightarrow \mathbf{E}\|U_0\|^2 < \infty$.



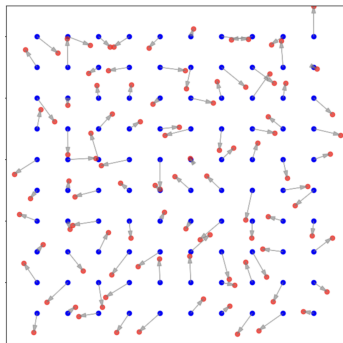
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- L^2 -perturbed lattice $\Leftrightarrow \mathbf{E}\|U_0\|^2 < \infty$.



Non-asymptotic results

Building on results of [Serfaty '23] about smooth linear statistics

Theorem (Lr '26)

Let $R \in [1, N^a]$ with $a(d+2) < 1/d$, then for some $0 < \delta < 1/d$,

- $d = 2, \beta > 0$: for $x \in B(0, (\rho_2 - \varepsilon)\sqrt{N})$,

$$\frac{1}{(N^\delta)^d} \int_{\mathbb{T}_{N^\delta}} \text{Var} \left(\mathbf{X}_N^\beta(B(x, R)) \right) \leq C_\beta R \ll R^2$$

- $d = 3, \beta > 0$: for $x \in B(0, (\rho_3 - \varepsilon)N^{1/3})$,

$$\frac{1}{(N^\delta)^d} \int_{\mathbb{T}_{N^\delta}} \text{Var} \left(\mathbf{X}_N^\beta(B(x, R)) \right) \leq C_\beta R^2 \ln(R) \ll R^3.$$

It prevents the rate from being $\gg R^{d-1} \ln(R)$ outside a vanishing set.

Results for Girko random matrices

Building on the results of [Cipolloni et al. '23] about smooth linear statistics, there is number variance hyperuniformity “almost everywhere” (but not at all scales)

Theorem (Lr '26)

Let $R \in [1, N^{\frac{1}{8}}]$, $\delta < 1/2 - 1/8$, $x \in B(0, (1 - \varepsilon)\sqrt{N})$

$$\frac{1}{N^{2\delta}} \int_{\mathbb{T}_{N^\delta}} \text{Var}(\mathbf{X}_N(B(x, R))) dx \leqslant CR^1$$

Comparison with [Cipolloni et al. '26]

- Scales $[1, N^{1/8}]$
- No uniform result in x
- But optimal rate: R^1 instead of $R^{2-\varepsilon}$.

Thank you very much!