Hyperuniformity and the variance transfer principle

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Hyperuniformity concept

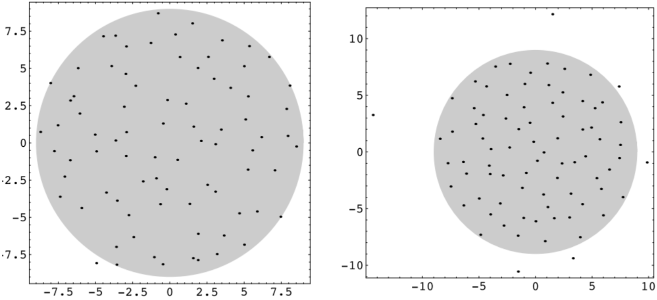


Figure 1. A typical realization of the eigenvalues of a 81×81 complex Gaussian random matrix (leftmost plot) and the zeros of a complex Gaussian random polynomial of degree 81 with variances given by (1.6). The shaded region represents the disc $|z| < 9$ which is the leading order support of the density in both cases. Outside the disc the density has a $1/r^2$ tail in the case of the zeros, whereas it falls off as a Gaussian for the eigenvalues, in keeping with the realizations in the figure.

Figure: **[Forrester, Honner '99]** Ginibre eigenvalues and Weyl zeros

Asymptotic variance reduction

- $\mathsf{X}_N = \{x_1, \dots, x_N\}$
- **Hyperuniformity of $\mathsf{X}_N, N \to \infty$ on scales $[1, N^\beta]$**: for $R \in [1, N^\beta]$

$$\mathrm{Var}\left(\mathsf{X}_N(B(x,R))\right) \ll CR^d \qquad (\approx CR^{d-\alpha} \text{ for some } \alpha > 0)$$

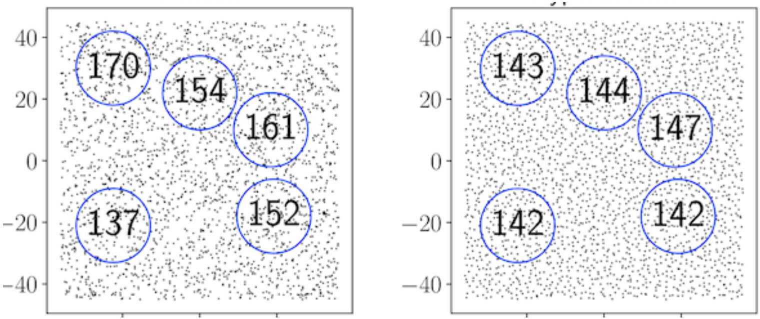


Figure: *Left:* Poisson / i.i.d. points. *Right:* "Hyperuniform disordered"

Example 1: Girko matrices (microscopic zoom)

Consider the eigenvalues $\mathsf{X}_N := \{x_1, \ldots, x_N\}$ of a $N \times N$ random matrix with i.i.d. centred entries $X_{i,j} \in \mathbb{C}, 1 \leqslant i, j \leqslant N$ with a fixed law with all moments finite and $\mathbf{E}|X_{i,j}|^2 = 1$, e.g. **complex Ginibre ensemble:**

- **Asymptotic homogeneity:** for $f \in \mathscr{C}_c^\infty(B(0,1)), R \in [1, N^{1/2}]$

$$\mathsf{X}_N(f_R) := \sum_{i=1}^N f(x_i/R) \text{ has expectation } \sim R^2 \frac{1}{\pi} \int f$$

- **[Cipolloni, Erdös, Schröder '23]** for $f \in \mathscr{C}_c^1(B(0, 1-\varepsilon))$,

$$\sup_{N \geqslant 1, R \in [1, \sqrt{N}]} \operatorname{Var}(\mathsf{X}_N(f_R)) \leqslant C_f < \infty.$$

- **For comparison:** If the x_i were i.i.d. / Poisson in $B(0, \sqrt{N})$

$$\operatorname{Var}(\mathsf{X}_N(f_R)) \asymp \mathbf{E}\mathsf{X}_N(f_R) \asymp R^2$$

Girko matrices (Cont'd)

- **[Cipolloni, Erdös, Schroder '23]** hence proved hyperuniformity for smooth linear statistics. *"Extending this study to less regular f creates substantial difficulties"*.
- **[Cipolloni, Erdös, Kolupaiev '26]** For $f = 1_\Omega$ for $\Omega \subset B(0,1)$ sufficiently smooth, with some additional ("removable") assumption on the $X_{i,j}, R = N^a$ with $a \in (0, 1/2)$,

$$\mathrm{Var}\left(\mathsf{X}_N(f_R)\right) \leqslant C_a R^{2-\varepsilon} \asymp \left(\mathbf{E}\mathsf{X}_N(f_R)\right)^{1-\varepsilon/2}$$

where $0 < \varepsilon \leqslant 1/40.$

$\Rightarrow$ Negligible compared to the Poisson case, *hyperuniform*!

- Expected rate: R^1 (like for Ginibre matrices, *class I hyperuniformity*).
- **Remark:** $\mathsf{X}_N \to \mathsf{X}$, the "infinite Ginibre process" weakly

Example 2. Coulomb gases

- Let $\mathsf{X}_N^\beta = \{x_1, \ldots, x_N\} \subset \mathbb{R}^2$ with joint density

$$Z_N^{-1} \exp(-\beta \underbrace{\sum_{i \neq j} (-\ln(\|x_i - x_j\|))}_{\text{Energy}}) \times \{\text{confining term}\}$$

- **Ginibre:** case $\beta = 2$ with confinement $\prod_i \exp(-\|x_i\|^2)$
- **[Leblé '23]** Proves hyperuniformity in the bulk for all $\beta > 0$:

$$\sup_{x : B(x,R) \subset B(0,(1-\varepsilon)\sqrt{N})} \operatorname{Var}\left(\mathsf{X}_N^\beta(B(x,R))\right) \leqslant C_\beta \frac{R^2}{\ln(R)^{0.6}}$$

- **Poisson** (with the same scaling) $\operatorname{Var}(\mathsf{X}_N(B_R)) \asymp R^2$
- **What is expected:** $\asymp R^1$ *class I hyperuniformity*

Coulomb gases (Cont'd)

- **[Serfaty '23]** shows variance reduction for smooth linear statistics f

$$\mathrm{Var}\left(\mathsf{X}_N^{\beta}(f_R)\right) \leqslant C_{f,\beta} R^{2d-4}$$

for $d = 2, 3$ (3D Coulomb potential: $\|\cdot\|^{-1}$ instead of $-\ln(\|\cdot\|)$

$$d = 2 : \mathrm{Var}\left(\mathsf{P}_n(f_R)\right) \leqslant C_{f,\beta} = C_{f,\beta} R^{d-2} \quad \text{“2-hyperuniformity”}$$
$$d = 3 : \mathrm{Var}\left(\mathsf{P}_n(f_R)\right) \leqslant C_{f,\beta} R^{2} = C_{f,\beta} R^{d-1} \quad \text{“1-hyperuniformity”}$$

- *“Treating the case of nonsmooth kernels (...) remains a (significantly more) delicate problem.”*

Variance transfer principle

Goal of this talk: prove that if the rate for smooth linear statistics is in $R^{d-\alpha}$, then the rate for balls is $R^{d-\min(\alpha,1)}$ ($\times \ln(R)$ if $\alpha = 1$).

- $\alpha > 0$ is the *hyperuniformity* exponent :
- Ginibre/Girko/Coulomb gases: α seems to be 2 (not proved in 3D)
- Hence we should obtain "surface order growth", i.e. class I hyperuniformity

$$\mathrm{Var}\left(\mathsf{X}_N(B(0,R))\right) \sim R^{d-1}$$

- **[Beck '87]** This is the minimal possible rate for a "homogeneous system of points"
- **1st part: stationary models on $\mathbb{R}^d$:** Poisson, infinite Ginibre, infinite Coulomb gases, ...
- **2nd part: large finite models:** Girko, Coulomb, ...

Original definition for infinite processes

- $\mathcal{N}(\mathbb{R}^d)$: **Space of configurations:** locally finite sets, endowed with vague topology (local). A point configuration is seen as a measure

$$\mathsf{P}(f) = \sum_{x \in \mathsf{P}} f(x).$$

- **Point process** P : Random element of $\mathcal{N}(\mathbb{R}^d)$
- **Stationarity:** invariance in law under translations $\tau_x : y \mapsto x + y$

Definition ([Torquato, Stillinger, '03])

A stationary point process P is hyperuniform (HU) if

$$\mathrm{Var}\left(\mathsf{P}(B_R)\right) = o(R^d) \text{ where } B_R := B(0, R)$$

- For P a stationary Poisson process, $\mathrm{Var}\left(\mathsf{P}(B_R)\right) = \lambda R^d$ (not HU)

Isomorphism theorem and spectral characterisation

- Let P be a stationary point process with $\mathbf{E}\left[\mathsf{P}(B(0,1))^2\right] < \infty$.
- There is a tempered measure S_P (the "spectral measure") giving a Plancherel type formula

$$\mathrm{Var}\left(\mathsf{P}(f)\right) = \int |\hat{f}|^2 d\mathsf{S}_P, f \in \mathscr{C}_c^{\mathsf{b}}(\mathbb{R}^d) \quad \text{(bounded with compact support)}$$

Hyperuniformity is a global variance cancellation phenomenon over linear statistics

Theorem (Torquato, Björklund & Hartnick, Lr)

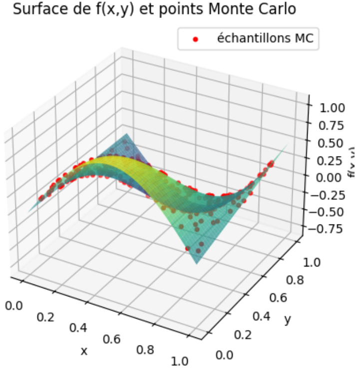
The following are equivalent

1. $Var(\mathsf{P}(B_R)) = o(R^d), R \to \infty$
2. $\mathsf{S}_{\mathsf{P}}(B(0,\varepsilon)) = o(\varepsilon^d), \varepsilon \to 0$
3. $Var(\mathsf{P}(f_R)) = o(R^d), R > 1$ *for some* $f \in \mathscr{C}_c^b(\mathbb{R}^d)$ *such that* $\int f \neq 0$

Numerical integration / Monte Carlo

Let f be a "complicated" function on a "high-dimensional" cube of dimension d. What is $\int f$?
Let $x_1, \ldots, x_n$ be random points.

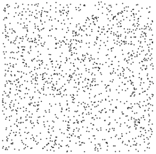
$$\int f \sim \frac{1}{n} \sum_{i=1}^{n} f(x_i)?$$



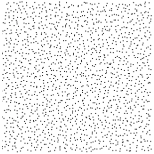
If the x_i are independent and identically distributed ("i.i.d."),

$$\left| \int_{[0,1]^d} f - \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right| \sim n^{-1/2}$$

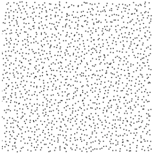
Monte Carlo integration with hyperuniform samples



i.i.d. sample: Error $\sim n^{-1/2}$



Ginibre eigenvalues: Error $\sim n^{-1}$



Weyl zeros: Error $\sim n^{-3/2}$

Hyperuniformity exponent and numerical integration

Definition

P is α-hyperuniform if

$$\mathsf{S}_{\mathsf{P}}(B(0,\varepsilon)) = O(\varepsilon^{d+\alpha})$$

$$\Leftrightarrow \exists f \in \mathscr{C}_c^\infty(\mathbb{R}^d) \text{ such that } \int f \neq 0, \mathrm{Var}\left(\mathsf{P}(f_R)\right) = O(R^{d-\alpha}).$$

- Relevant for numerical integration
- Classical case: S_{P} has a density $\mathcal{C}^2 \Rightarrow$ 2-hyperuniform

Emblematic 1D / 2D proven examples

- $d = 1$: β-ensembles / Sine_β / $\log$-gases, with $\alpha = 1$
- $d = 2$: **Infinite Ginibre ensemble**, stationary and hyperuniform with $\alpha = 2$ (optimal)
- Let $F(z)$ be the planar GAF (Gaussian Analytic Function, limit of *Weyl polynomials*)

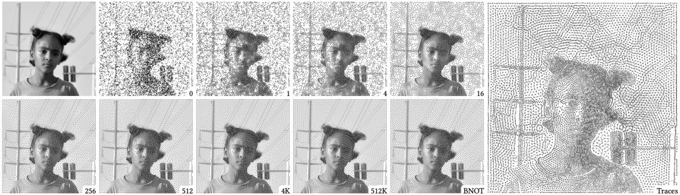
$$F(z) = \sum_{k=0}^{\infty} \frac{X_k z^k}{\sqrt{k!}}, \; X_k \sim \mathcal{N}_{\mathbb{C}}(0,1), \text{ i.i.d.}$$

  and its zeros $\mathscr{Z} = \{z : F(z) = 0\}$. Then $\mathscr{Z}$ is stationary and hyperuniform with $\alpha = 4$, this is the unique “non-toy model” with (proven) $\alpha > 2$.
- **[Joyce et al. ’08, Lr ’24]** Toy models p-hyperuniform for $p \to \infty$

Other fields

Hyperuniformity is also investigated in

- Physics (Torquato, Lebowitz, Stillinger, ...)
  - Strictly jammed packings, densest packings
  - Crystals and quasicrystals
  - Self-organising systems at criticality
- Image processing / Optimal transport under the terminology *blue noise*
- Biology (marine algae systems *Effrenium Voratum*, birds photoreceptors)
- Numerical integration, Machine learning



Proof of the transfer principle and generalisation

For $\varphi = 1_{B(0,1)}$ we have

$$|\hat{\varphi}(\xi)|^2 \leqslant C(1+\|\xi\|)^{-d-1}$$

The proof is based on high / low frequency decomposition, using **translation boundedness** of the spectral measure. Assume $\mathsf{S}(d\xi) \sim \|\xi\|^\alpha$ as $\xi \to 0$ for simplification

$$\begin{aligned}
\operatorname{Var}\left(\mathsf{P}(\varphi_R)\right) &= \int_{\mathbb{R}^d} |\widehat{\varphi_R}|^2 d\mathsf{S} = R^{2d} \int_{\mathbb{R}^d} |\hat{\varphi}(R\xi)|^2 \mathsf{S}(d\xi) \\
&\approx R^{2d} \int_{B(0,\varepsilon/R)} |\hat{\varphi}(0)|^2 \|\xi\|^\alpha d\xi \\
&\qquad\qquad + R^{2d} \int_{B(0,\varepsilon/R)^c} (1+\|R\xi\|)^{-d-1} (``\|\xi\|^\alpha \wedge O(1)'') d\xi \\
&\leqslant C R^{d-\min(\alpha,1)} \ln(R)^{\mathbf{1}\{\alpha=1\}}.
\end{aligned}$$

Similar result on the torus

- Let P_n be a point process on $\mathbb{T}_n := \mathbb{R}^d/(n\mathbb{Z})^d$
- Let τ_x^n the n-periodic translation on $\mathbb{T}_n$
- Assume $\mathbb{T}_n$-stationarity:

$$\tau_x^n \mathsf{P}_n \overset{(d)}{=} \mathsf{P}_n, x \in \mathbb{T}_n.$$

- **Example:** Coulomb gas with periodic boundary conditions: $\mathsf{P}_n = \{x_1, \dots, x_n\} \subset \mathbb{T}_n$ with density

$$\propto \exp(-\psi_d(x_1, \dots, x_n))$$

where

$$\Delta_{\mathbb{T}_n} \psi_d = \delta_0 - n^{-d} \textbf{Leb}^d$$

Theorem (Lr '26 +)

Let P_n *be a* $\mathbb{T}_n$*-stationary point process.*
Assume P_n *is* α*-hyperuniform for some smooth kernel* $f \in \mathscr{C}_c^\infty(\mathbb{R}^d) \setminus \{0\}$*:*
for $R \in [1, n^\gamma]$ *(*$\gamma \in [0,1]$*)*

$$Var(\mathsf{P}_n(f_R)) \leqslant C_f R^{d-\alpha}$$

then for $R \in [1, n^\gamma]$

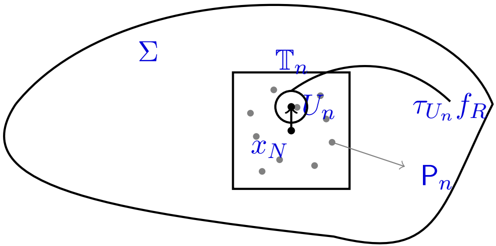
$$\sup_x Var(\mathsf{P}_n(B(x,R))) \leqslant C_f R^{d-\min(\alpha,1)} \ln(R)^{\mathbf{1}\{\alpha=1\}}$$

The proof is the same as for the stationary case:

- The Fourier dual space of $\mathbb{T}_n$ is $\widehat{\mathbb{T}_n} := 2\pi n^{-1}\mathbb{Z}^d$
- There is a similar isomorphism theorem with a spectral measure S_n on $\widehat{\mathbb{T}_n}$. In some sense, $\widehat{\mathbb{T}_n} \xrightarrow[n\to\infty]{} \mathbb{R}^d$
- A similar high/low frequency decomposition is possible

Adaptation to "almost stationary" samples

- X_N process on $\Sigma = B(0, N^{1/d})$ (or else)
- P_n : restriction to $x_N + \mathbb{T}_n$ with some $n \leqslant N^{1/d}$
- Let U_n uniform on $\mathbb{T}_n$, $\tilde{\mathsf{P}}_n := \tau^n_{U_n} \mathsf{P}_n$ stationary by construction
- Hopefully $\mathrm{Var}\left(\tilde{\mathsf{P}}_n(\tau_x f_R)\right) \approx \mathrm{Var}\left(\mathsf{P}_n(\tau_x f_R)\right)$ in law, unfortunately boundary effects kill some available scales



Results for Girko random matrices

Building on the results of **[Cipolloni et al. '23]** about smooth linear statistics, there is number variance hyperuniformity "almost everywhere" (but not at all scales)

Theorem (Lr '26 +)

Let $R \in [1, N^{\frac{1}{9}}], \delta < 1/2 - 1/9,$

$$\frac{1}{N^{2\delta}} \int_{\mathbb{T}_{N^\delta}} Var(\mathsf{X}_N(B(x,R)))\, dx \leqslant CR^1$$

Comparison with [Cipolloni et al. '26]

- Only scales $[1, N^{1/9}]$
- No uniform result in x
- But optimal rate: R^1 instead of $R^{2-\varepsilon}$.

Coulomb gases

Essentially similar results hold, building on results of **[Serfaty '23]** about smooth linear statistics for any $\beta > 0$:

Theorem (Lr '26 +)

Let $R \in [1, N^\gamma]$ *with* $\gamma(d+2) < 1/d$*, then for some* $\delta < 1/d$*,*

- $d = 2$*:*

$$\frac{1}{N^{2\delta}} \int_{\mathbb{T}_{N^\delta}} Var\left(\mathsf{X}_N^\beta(B(x,R))\right) \leqslant CR \ll R^2$$

- $d = 3$ *:*

$$\frac{1}{N^{3\delta}} \int_{\mathbb{T}_{N^\delta}} Var\left(\mathsf{X}_N^\beta(B(x,R))\right) \leqslant CR^2 \ln(R) \ll R^3.$$

Consequences for infinite Coulomb gases

Let $\mathsf{X}^\beta = \lim_{N_j\to\infty} \mathsf{X}^\beta_{N_j}$ be a "limit point in law" on $\mathbb{R}^d$ for the vague topology, i.e. a *infinite volume Coulomb gas*. When $N \to \infty$, the finite volume results on X^β_N show that X^β is hyperuniform around "almost every point x".

Theorem

- *In dimension $d = 2$,* **[Leblé '24]** *proved that any such limit point* P *is stationary. Therefore we can prove that* P *is* 2*-hyperuniform, i.e. for all* $x \in \mathbb{R}^d$

$$Var\left(\mathsf{X}^\beta(B(x,R))\right) \leqslant C_\beta R < \infty.$$

This proves **finite Coulomb energy** *and* **finite Wasserstein distance** *to Lebesgue measure by* **[Huesmann & Leblé '25]**

- *In dimension $d = 3$, it gives* 1*-hyperuniformity if* P *is assumed to be stationary.*