Hyperuniformity of random measures, transport and rigidity

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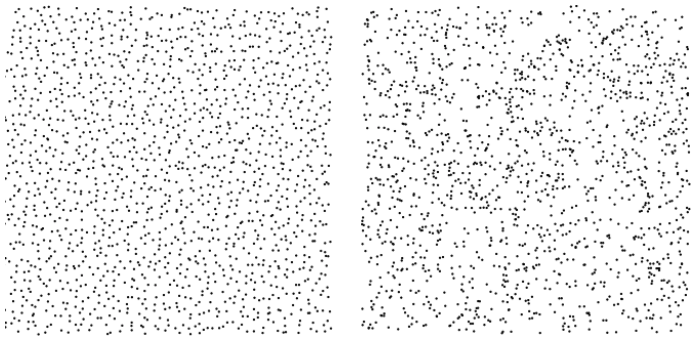
Chapter 1

Introduction and examples

When one has to generate a sample of *random points* in a window W of $\mathbb{R}^d$, the default strategy across many fields is to choose a large number n and draw n i.i.d points uniformly in W. The asymptotic mathematical object as $n \to \infty$ is the celebrated *homogeneous Poisson process*, used as a universal reference, especially in theoretical studies, due to its nice mathematical properties. Such finite (n i.i.d points) or infinite (Poisson) samples of independent points has some downsides, such as their tendency to leave large empty spaces, or on the contrary, regions cluttered with too many points (see the picture on the right, below), but this is an inevitable consequence of total randomness.

Hyperuniformity is a property exhibited by many mathematical models presenting instead a *regular* spatial arrangement, remedying some flaws of independent samples. This type of arrangement is reminiscent of the way particles subject to mutual repulsive forces would be distributed; moreover, many natural models from statistical physics, biology, or other fields, exhibit hyperuniform behavior.



A hyperuniform sample A Poisson sample

Figure 1.3 (left) shows the photoreceptor locations of a bird's eye, a class of animals renowned for their excellent long-distance vision. This sample can be categorised as hyperuniform due to its spatial statistical characteristics [40]. In image analysis and optimal transport, hyperuniformity is also present, sometimes under the term *blue noise*, because regular samples can be useful for many tasks, such as texture synthesis, dithering, or else, and variance reduction is an essential feature of blue noise samples [88, 84, 67, 19]. The rendering picture in Figure 1.2, for instance, has been obtained by replacing greyscale levels by blue noise samples with the corresponding density. For many applications, the samples should be *disordered*, i.e. non-periodic, it can otherwise cause in Monte Carlo integration or image processing undesired aliasing or structured artifacts [67].

This *regularity* is difficult to define rigorously in a non-ambiguous way, but the good fortune of mathematicians and the reason why this field of study exists is that hyperuniformity is a very natural and universal way to mathematically define a certain form of regularity, as we shall see. Roughly

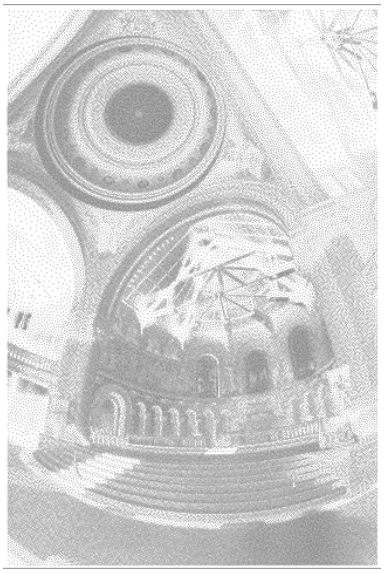


Figure 1.2: Dithering - Greyscale levels replaced by hyperuniform samples [19], ACM Trans. Graph.

speaking, a sample is hyperuniform if the variance of the number of points in each large window is smaller than if the points were independent (Poisson or i.i.d). Hyperuniformity is a simple second-order assumption, which does not involves higher order structure of the process, but it surprisingly implies many macroscopic phenomena, related to optimal transport, or rigidity. It is also quite universal since it can equivalently be defined by low variance for not-too-irregular linear statistics, and the whole theory extends to general random measures, including for instance Gaussian fields, spin systems and nodal domains. The systematic study of processes from the perspective of hyperuniformity essentially originates from theoretical physics, in particular with the team of S. Torquato at Princeton, who popularised the term *hyperuniformity,* or J. Lebowitz at Rutgers University, sometimes under the terminology of *superhomogeneity*.

Besides its usefulness and appearances in other sciences, many popular mathematical models turned out to be hyperuniform, in random matrices, statistical physics, random polynomials, quasicrystals, see the surveys [82, 17] for an in depth collection. Reading the literature gives the impression that hyperuniform point processes can be categorised in two classes: the class of lattices that eventually undergo a perturbation, having properties similar to those of crystalline structures, and the class of particle systems that look like spontaneous organisation of particles that arrange themselves due to a pairwise repulsive force, and conserve some sort of local disorder. To draw a parallel with the way animal visual receptors sample space, the latter models seem visually more disordered, somewhat like the bird photoreceptors in the figure below (left), whereas one can make a parallel between crystalline models and the regular arrangement of the eyes of insects (right). A remarkable property of disordered hyperuniform processes is that they often display the same large-scale properties as their crystalline counterparts, which is why physicists sometimes subtitle hyperuniformity as *global order and local disorder*.

The scope of this survey is to study hyperuniformity and its consequences under a mathematical perspective. We also present most stationary models for which hyperuniformity has been proven rigourously: determinantal processes, zeros of random Gaussian functions, Coulomb gases ... We give a first definition in Section 1.1 and discuss the concept of *disordered* sample, we give some emblematic examples in Sections 1.2, 1.4. In Chapter 2, the mathematical core of this book, we explore hyperuniformity from the spectral viewpoint, which allows for a practical and universal characterization in the direct space. The most natural framework for studying hyperuniformity is in fact that of (wide sense)

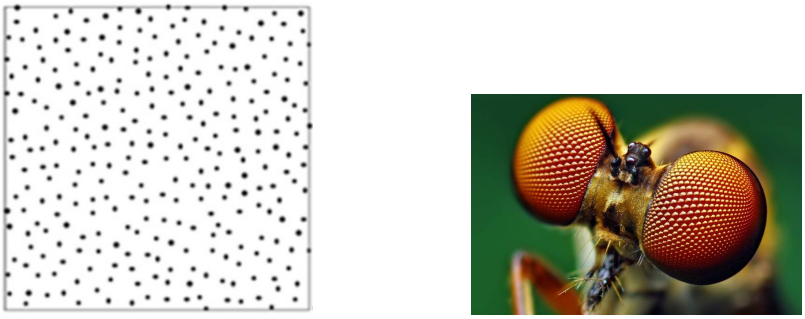


Figure 1.3: *Left.* Disordered hyperuniform receptors [40]. *Right.* Periodic "ordered" receptors

stationary random measures, generalising point processes. In this setting, we also provide universal bounds on the variance of linear statistics, useful for parameterizing measures by their hyperuniformity exponent. Section 3 is devoted to a deeper study of certain classes of hyperuniform point processes, which requires delving into the world of Gaussian analytic functions (GAFs), determinantal point processes (DPPs), and quasicrystals. Sections 4 and 5 present surprising macroscopic properties of hyperuniform processes in dimension 1 and 2: in Section 4, we show that they enjoy good optimal transport properties, which allows crystalline/periodic structures to be merged with the class of disordered hyperuniform processes into a single continuum. Section 5 deals with another property, *rigidity*, which states that for many hyperuniform processes, the number of particles in a region of space can be completely inferred by observing the process on the rest of the space. This rigidity can take more extreme forms as the hyperuniformity exponent increases, leading us to the study of *stealthy processes*, with infinite exponent, showcasing even more fascinating properties.

Context and objectives

These notes were written at the occasion of the mini-course *Hyperuniformity of random samples* given at the 2025 GeoSto conference at Grenoble-Alpes University, the slides can be found at `https://helios2.mi.parisdescartes.fr/~rlachiez/recherche/talks/slides-hu.pdf`. This version is intended to take part in the Springer series *Stochastic Geometry*, I expect to produce a longer version containing additional models and proofs, and more insights into numerical aspects.

There already exists general studies about hyperuniformity. The survey [81] lists many different physical models experiencing hyperuniformity at different orders, and presents most important properties, and some conjectures on hyperuniform and stealthy processes. The more mathematical discussion of Coste [17] contains some of the material treated here. Since its publication, there have been several theoretical advances that we report here, notably concerning spectral characterization [10], transport properties [52, 13, 37, 41, 21, 25], rigidity [21, 50, 51], linear statistics and limit theorems [58, 39, 47], Gibbs measures [20, 54, 53], and others. The excellent book [36] describes several models of hyperuniform DPPs and random GAF zeros, and provides their properties, it constitutes an essential source for this work.

1.1 Point processes formalism and first definition of HU

Even though many general results will be stated without additional cost to general random measures, the main objects of this branch of litterature are *simple point processes*. To define them properly, introduce the space of configurations $\mathscr{N} = \mathscr{N}(\mathbb{R}^d)$ which elements are the atomic measures $P = \sum_i \delta_{x_i}$, where the x_i are countably many isolated points in $\mathbb{R}^d$. A configuration P can unambiguously be

assimilated to its support and we often use set-related notation such as, for $P, P' \in \mathscr{N}$,

$$P \cup P' = \mathrm{supp}(P) \cup \mathrm{supp}(P'), P \cap P' = \mathrm{supp}(P) \cap \mathrm{supp}(P'), P \backslash A = \mathrm{supp}(P) \backslash A, ...$$

Endow $\mathscr{N}$ with the *vague* topology, generated by the mappings

$$\varphi_f : P \in \mathscr{N} \to P(f) := \int f dP$$

for f continuous with compact support. The corresponding Borel σ-algebra $\mathscr{B}(\mathscr{N})$ is alternatively generated by mappings f_{1_A} for $A \subset \mathbb{R}^d$ bounded measurable. A *simple point process*, or just *point process* in the following, is a random element P of $(\mathscr{N}, \mathscr{B}(\mathscr{N}))$. Equivalently, it is a family of measurable mappings

$$A \mapsto \mathsf{P}(A) \in \mathbb{N}$$

for $A \subset \mathbb{R}^d$ bounded measurable, and the family of laws $\{\mathsf{P}(A); A \subset \mathbb{R}^d\}$ uniquely defines the law of P as a probability measure over $\mathscr{N}$. An essential assumption for the modelisation of homogeneous structures is that of stationarity. Call τ_x the operator of shift by $x \in \mathbb{R}^d$, lifted to a set $P \subset \mathbb{R}^d$ with $\tau_x P = \{y + x; y \in P\}$, and say that a point process P is *stationary* if for all $x \in \mathbb{R}^d$ $\tau_x \mathsf{P} \overset{(d)}{=} \mathsf{P}$, using the set notation.

Let us introduce the *Poisson process*, fundamental brick in the realm of point processes. Given a non-zero non-negative locally finite measure μ on $\mathbb{R}^d$ without atoms, P^μ is defined as the unique process satisfying

$$\mathsf{P}^\mu(A) \overset{(d)}{=} \mathrm{Poi}(\mu(A)), A \subset \mathbb{R}^d,$$

where $\mathrm{Poi}(\lambda)$ denotes the law of a Poisson variable with parameter $\lambda \in \mathbb{R} \cup \{\infty\}$ ($\mathrm{Poi}(\infty) = \infty$ a.s. by convention). One way to explicitly build P^μ is to start from i.i.d. variables uniform in the ball B_n centred in 0 with volume n, i.e. $\mathsf{P}^\mu_n := \{X_1^{(n)}, \dots, X_n^{(n)}\}$ with law $\frac{1}{\mu(B_n)} \mu 1_{B_n}$ (for n sufficiently large). We have for $A \subset \mathbb{R}^d$ bounded,

$$\#\{k : X_k^{(n)} \in A\} \xrightarrow[n\to\infty]{\mathrm{Law}} \mathrm{Poi}(\mu(A)),$$

hence P^μ exists in $\mathscr{N}$ as the weak limit of the P^μ_n in the vague topology. Let $\mathscr{L}^d$ be Lebesgue measure. To obtain a stationary model, one must necessarily choose $\mu = \lambda \mathscr{L}^d$ for some $\lambda > 0$, and λ is called the intensity of P. More generally, for any stationary point process P, the intensity λ is defined by

$$\lambda = \frac{\mathbf{E}\mathsf{P}(A)}{\mathscr{L}^d(A)}, A \subset \mathbb{R}^d \text{ bounded non-negligible},$$

and this definition does not depend on A; the finiteness of λ is by no means automatic, but we will implicitly assume it by default. Since we mainly conduct here second order analyses of such processes, we will in fact always assume *local square integrability* (denoted by L^2_{loc}), i.e. $\mathbf{E}\mathsf{P}(B)^2 < \infty$ for B bounded.

As a Poisson variable, the variance of the number of points in the ball B_R centered in 0 with radius $R > 0$ for a Poisson process is the volume of the ball

$$\mathrm{Var}\left(\mathsf{P}^{\lambda\mathscr{L}^d}(B_R)\right) = \mathrm{Var}\left(\mathrm{Poi}(\mathscr{L}^d(B_R))\right) = \mathscr{L}^d(B_R) = \kappa_d R^d \text{ with } \kappa_d = \mathscr{L}^d(B_1).$$

In general, a random measure with variance proportionnal to the volume on large domains is said to be *extensive*, and is in fact expected for most natural stationary point processes where particles only interact locally.

The study of perturbed lattices, random matrices, particle systems, random polynomials, and many other natural objects, that will be the main topic of the current work, made emerge a huge class of stationary processes where there is no extensivity, and some cancellation seems to equilibrate fluctuations of points, in what we call a *hyperuniform*, or *superhomogeneous* behaviour. The reasons for this compensation are not always clear, and generally different for each system.

Definition 1.1. A stationary point process P of $\mathbb{R}^d$ is hyperuniform if

$$\lim_{R\to\infty} \frac{\mathrm{Var}\left(\mathsf{P}(B_R)\right)}{\mathscr{L}^d(B_R)} = 0. \tag{1.1}$$

1.2 Perturbed lattices

The most basic example of a hyperuniform infinite sample is the *shifted lattice*, i.e. in $\mathbb{R}^d$

$$\mathsf{Z}^d := \{\mathbf{k} + U; \mathbf{k} \in \mathbb{Z}^d\},$$

where U has the uniform distribution on $[0,1]^d$, denoted by $\mathscr{U}_{[0,1]^d}$. The shift by U ensures stationary, i.e. invariance under $\mathbb{R}^d$ translations. This model is not very rich from the mathematical point of view, it still serves as a reference or as a counter-example for many phenomena. Any other Bravais lattice, i.e. obtained through a linear mapping applied to $\mathbb{Z}^d$, would do as well, for simplicity we mostly consider Z^d. We shall introduce the more general concept of *independently perturbed lattice* (IPL).

Example 1.1 (IPL)**.** For μ a probability measure on $\mathbb{R}^d$, let $\mathsf{Z}^{d,\mu} := \{\mathbf{k} + U + U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$ where the $U_{\mathbf{k}}$ are i.i.d. with law μ, called IPL with law μ.

The hyperuniformity of $\mathsf{Z}^{d,\mu}$ is not trivial, especially when $\mu = \delta_0$, i.e. $\mathsf{Z}^{d,\mu} = \mathsf{Z}^d$, where it is related to *Gauss's circle problem*. A general proof in the spectral domain is a corollary of Theorem 2.1. We can still give some geometric intuition when the $U_{\mathbf{k}}$ are bounded and not deterministic: there are approximately $O(R^{d-1})$ points close to ∂B_R, hence likely to cross the boundary under application of the shift U, and they would cross approximately independently of one another. The variance of the number of particles inside is the sum of variances of indicators for such points, which therefore gives a sum with $O(R^{d-1})$ uniformly bounded terms, which is indeed negligible with respect to R^d.

This model can be refined by introducing dependency among the $U_{\mathbf{k}}$, but to ensure stationarity we will always require the perturbations to form a stationary field of $\mathbb{Z}^d$, i.e. for $\mathbf{m} \in \mathbb{Z}^d$, $\{\tau_{\mathbf{m}} U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\} \overset{(d)}{=} \{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$. At chapter 4, we will see that hyperuniformity persists if the assumption of independence of the $U_{\mathbf{k}}$ is dropped, as long as the U_k form a sufficiently mixing field. More surpsisingly, we will see that, conversely, most hyperuniform processes can be written as a (non-mixing) stationary perturbating field applied to a lattice. In this framework, the U_k can be interpreted as a transport between Z^d and the obtained point process P.

1.3 Disordered samples

Like chicken photoreceptors (Figure 1.3), many hyperuniform processes observed in physics or biology seem to be *disordered*. Physicist sometimes present disorder as the absence of peaks in the spectrum, which can be reminiscent of an underlying periodic structure. Another often used requirement is *isotropy*, where no direction is priviledged: $\mathsf{P} = \sum_i \delta_{x_i}$ is isotropic if for any orthogonal matrix O of size d, $O\mathsf{P} := \sum_i \delta_{Ox_i} \overset{(d)}{=} \mathsf{P}$. It is sometimes additionally assumed that $\mathsf{C} - \delta_0$ has a density, and/or has finite total mass. The latter assumption, and most of the results in this survey, pertain to second order analysis, i.e. variance and covariance behaviour. Still, one might have order at this level and disorder from a more global perspective, see for instance the example of cloaked lattices [56], see Example 2.4. It is not hard to build counter-examples which satisfy the above properties but cannot be categorised as *disordered*, but they are probably physically unnatural. A more satisfying mathematical concept is that of *mixing*. This property is another interpretation of disorder where the behaviour of the model at distant locations should be asymptotically independent.

Definition 1.2. Say that a stationary point process P is mixing if for $A, B \subset \mathbb{R}^d$ Bounded Borel sets,

$$\mathbf{P}(\mathsf{P}(A) = 0, \mathsf{P}(\tau_x B) = 0) \xrightarrow[x\to\infty]{} \mathbf{P}(\mathsf{P}(A) = 0)\mathbf{P}(\mathsf{P}(B) = 0)$$

The fact that empty intersection events characterise the law comes from the fact that the σ-algebra is generated by the corresponding indicators. Mixing extends to general events Ω, Ω' of $\mathscr{B}(\mathscr{N})$ (see [18, Lm. 12.3.II]). If P is mixing, we have

$$\mathbf{P}(\mathsf{P} \in \Omega, \tau_x \mathsf{P} \in \Omega') \to \mathbf{P}(\mathsf{P} \in \Omega)\mathbf{P}(\mathsf{P} \in \Omega').$$

This definition is not completely satisfactory either as some models, such as the stationary Poisson line intersection process (see [42] and references therein), satisfies it and still exhibits very long range dependency. We introduce at Section 2.5 the concept of Brillinger mixing, which seems ideal from many points of view, but hard to verify in practice.

1.4 Three emblematic examples

We present here three examples that emerge from different branches of mathematics and can be considered disordered. The first examples come from random matrices, more precisely they are the scaling limit of points in the bulk of the eigenvalues of two prominent models. Two of them, the GUE and Ginibre ensemble, are also *determinantal processes*, which will lead us to introduce this very important class at Chapter 3. The third example comes from the unrelated field of random polynomials and functions. It still bears a flavour similar to the Ginibre ensemble in that they are naturally defined on the Complex plane, through Gaussian Standard Complex Variables, and are connected to the theory of analytic functions through the complex covariance $C(z,w) = e^{z\bar{w}}$. Together with the Ginibre ensemble, they really are the two seminal examples for which have been uncovered in first the universal properties of hyperuniform processes such as rigidity or good transport properties, partly because they are tractable, up to a certain point, among the jungle of all physically relevant point processes. Beyond hyperuniformity, showing that they are stationary is actually non trivial in both cases, in some sense they are just at the right place in the world of particle models, between relevancy and tractability.

We say a random complex variable G is a *Standard Complex Gaussian* (SCG), denoted $G \sim \mathscr{N}_{\mathbb{C}}(0,1)$, if it has density

$$\frac{1}{\pi} e^{-|z|^2}, z \in \mathbb{C}.$$

Equivalently, $G = X + iY$, where X, Y are i.i.d. with law $\mathscr{N}(0, 1/2)$. The simplicity of this definition, without square root or factor 2, and the easy computation of the normalisation constant with Gauss's integral, sometimes identifies it as more natural than real Gaussian variables.

1.4.1 The Sine_β processes.

Let $\beta > 0$. Consider the random vector $(\Lambda_1, \dots, \Lambda_n)$ on $\mathbb{R}^n$ with joint density

$$\propto \prod_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n \exp(-\beta \lambda_i^2/4), \tag{1.2}$$

where the symbol $\propto$ means *proportionnal to*, which essentially allows to avoid mentionning the renormalising constant. This density can be rewritten $\propto \exp(-\beta H(\lambda_1, \dots, \lambda_n))$ with the Hamiltonian

$$H(\lambda_1, \dots, \lambda_n) = -\frac{1}{2} \sum_{i \neq j} \ln(|\lambda_i - \lambda_j|) + \frac{1}{4} \sum_i \lambda_i^2.$$

This can be interpreted in terms of a system, called β-ensemble, where particles are individually attracted to 0 due to the *confinment term* $\exp(-\beta\lambda_i^2/4)$ term, and this tendancy is compensated by the pairwise *repulsion terms* $|\lambda_i - \lambda_j|^\beta$, that favor configurations where particles are not too close from one another. A fundamental point is that $\mathsf{P}_n^\beta := \{\Lambda_1, \dots, \Lambda_n\}$ can also be seen as the set of eigenvalues of a random matrix:

- If $\beta = 1$, P_n^1 has the same law as the set of eigenvalues of the Gaussian Orthogonal Ensemble (GOE). The GOE is the random matrix $\mathsf{M}^{1,(n)} = (\mathsf{M}_{i,j})_{1\leqslant i,j\leqslant n}$ in the space $\mathscr{S}_n(\mathbb{R})$ of $n \times n$ symmetric matrices, where the $\mathsf{M}_{i,i}$ are i.i.d. with law $\mathscr{N}(0,2)$, and $\mathsf{M}_{j,i} = \mathsf{M}_{i,j}, 1 \leqslant i < j \leqslant n$ are i.i.d. with law $\mathscr{N}(0,1)$ (proved at Section 3.3.1 through a change of variables). The reason for a different variance on the diagonal is that the density of $\mathsf{M}^{1,(n)}$ in $\mathscr{S}_n(\mathbb{R})$ at some $M \in \mathscr{S}_n(\mathbb{R})$ has a neat expression in terms of M's eigenvalues $\lambda_1, \dots, \lambda_n$: the density is by definition

$$\begin{aligned} \propto \prod_{i<j} \exp(-M_{i,j}^2/2) \prod_i \exp(-M_i^2/4) &= \prod_{i\neq j} \exp(-M_{i,j}^2/2)^{1/2} \prod_i \exp(-M_i^2/4) \\ &= \exp(-\mathrm{Tr}(MM^T)/4) \\ &= \exp(-\sum_{i=1}^n \lambda_i^2/4). \end{aligned} \tag{1.3}$$

  This expression differs from (1.2) as it is the density of the matrix itself, not its eigenvalues (see Section 3.3.1). It is clear under this form that the law of $\mathsf{M}^{1,(n)}$ is invariant under conjugation by the orthogonal group, which is the reason for the name *orthogonal ensemble*: for O an orthogonal matrix,

$$O\mathsf{M}^{1,(n)}O^T \overset{(d)}{=} \mathsf{M}^{1,(n)}.$$

- If $\beta = 2$, the Λ_i's are the eigenvalues of the Gaussian Unitary Ensemble (GUE), the random matrix $\mathsf{M}^{2,(n)} = (\mathsf{M}_{i,j})_{i,j}$ which entries are independent complex Gaussian variables with variance 1 on the diagonal and 2 on the upper diagonal. We then define a Hermitian model through $\mathsf{M}_{j,i} := \overline{\mathsf{M}_{i,j}}$ for $i < j$. Similarly as for the GOE, the matrix $\mathsf{M}^{2,(n)}$ has a density in each Hermitian matrix H

$$\propto \exp(-\sum_i \lambda_i^2/2) = \exp(-\mathrm{Tr}(H\bar{H}^T)/2),$$

  invariant under the conjugation by a unitary matrix.

- For any $\beta > 0$, the β-ensemble has been showed by [23] to constitute the eigenvalues of an explicit matrix model $\mathsf{M}^{\beta,(n)}$. The case $\beta = 4$ involves matrices of quaternions and is called the Gaussian Symplectic Ensemble (GSE), but we will not explicit further cases $\beta \notin \{1, 2\}$.

Recently, Valko and Viràg [85] derived the construction for each $\beta > 0$ of the *Brownian carrousel*, a set of SDEs which limit points form a point process of $\mathbb{R}$, and which is the weak limit of the β-ensembles as $n \to \infty$. Under the formulation (1.2), the mean number of particles per unit volume goes to infinity, which is why a rescaling by $\sqrt{n}$ is necessary:

Theorem 1.1 ([85])**.** For $\beta > 0$, there is a stationary point process $\mathsf{P}^\beta \subset \mathbb{R}$, called Sine_β process, such that

$$\sqrt{n}\mathsf{P}_n^\beta \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}^\beta.$$

Furthermore, P^β is hyperuniform.

The scaling $\sqrt{n}$ is not immediate to justify from (1.2). Let us compare with i.i.d. points $X_1, \dots, X_n$ uniform on $[-n, n]$, where indeed the mean number of points per unit volume remains constant:

$$\mathbf{E}\sum_i X_i^2 \asymp n^3,$$

which matches the rescaled eigenvalues

$$\mathbf{E}\sum_i (\sqrt{n}\lambda_i)^2 = n\mathbf{E}\sum_{i,j} (M_{i,j}^{\beta,(n)})^2 \asymp n^3.$$

Most proofs of those results are pretty involved, and often parts of classical textbooks about random matrices, so we will mostly omit them. We still provide at Section 3.3.1 a proof that for $\beta = 1$, (1.2) indeed is the density of the eigenvalues of $\mathsf{M}^{1,(n)}$, to illustrate the fundamental link between statistical physics and random matrices. The process Sine_1 turns out to be a member of the class of *Pfaffian point processes* [36]. The case $\beta = 2$ is also special as Sine_2 process is a member of the class of Determinantal Point Processes (DPPs), important in the theory of hyperuniformity, which we will prove at Section 3.3.4. Determinantal point processes, central in random matrix theory, are probably the main source of mathematically tractable hyperuniform point processes in any dimension.

1.4.2 The Ginibre ensemble

The next example is again a system of particles, and at the same time the eigenvalues of a random matrix model, but in dimension 2, more naturally in $\mathbb{C}$. Let $\mathsf{G}_{i,j} \sim \mathscr{N}_{\mathbb{C}}(0,1)$ i.i.d, $1 \leqslant i,j \leqslant n$, and the (non-Hermitian) random matrix $\mathrm{Gin}_n = (\mathsf{G}_{i,j})_{1\leqslant i,j\leqslant n}$. Let $\mathsf{P}_n^{\mathrm{Gin}} \subset \mathbb{C}$ the random subset of $\mathbb{C}$ formed by the a.s. distinct n eigenvalues of Gin_n. A change of variable yields that $\mathsf{P}_n^{\mathrm{Gin}}$ yields an interpretation in terms of statistical physics, namely it corresponds to the equilibrium state of n particles with the so-called *Coulomb interaction potential*:

Proposition 1.1. $\mathsf{P}_n^{\mathrm{Gin}}$ has density

$$\propto \prod_{1\leqslant i<j\leqslant n} |z_i - z_j|^2 \exp(-\sum_i |z_i|^2). \tag{1.4}$$

Here again, the density (1.4) translates an antagonism between an individual confinment term and a repulsive pairwise interaction.

Theorem 1.2. The point processes P_n converge weakly in the vague topology to a point process $\mathsf{P}^{\mathrm{Gin}} \subset \mathbb{C}$ that is stationary, isotropic, hyperuniform.

Note the absence of rescaling, which can again be justified by comparing with i.i.d. variables $X_1, \dots, X_n$ uniform on $B_{\sqrt{n}}$:

$$\mathbf{E}\sum_i X_i^2 \asymp n^2 \asymp \mathbf{E}\sum_{i,j} G_{i,j}^2 = \mathbf{E}\mathrm{Tr}(\mathrm{Gin}_n \overline{\mathrm{Gin}}^T).$$

The proof of Theorem 1.2 is at Section 3.3.3, it relies as for Sine_2 in dimension 1 on the fact that P_n is a determinantal point process, those two proofs are actually very similar.

1.4.3 Zeros of the planar GAF

Another important class of point processes, or more generally random measures, is that of nodal sets of random functions, i.e. $\mathsf{P} = \{x : \mathsf{F}(x) = 0\} \subset \mathbb{R}^d$ for some random $\mathsf{F} : \mathbb{R}^d \to \mathbb{R}^q$. In general these systems are extensive, i.e. they present no hyperuniformity [49, 28]. A notable exception is the zero set of the planar Gaussian Analytic Function (GAF). Let $\mathsf{G}_k, k \geqslant 1$, i.i.d. $\mathscr{N}_{\mathbb{C}}(0,1)$ distributed variables, and the random function

$$\mathsf{F}^{\mathrm{Pl}}(z) = \sum_{k\geqslant 1} \frac{\mathsf{G}_k}{\sqrt{k!}} z^k,$$

where a.s. the series converges absolutely. Let its zero set be

$$\mathsf{P}^{\mathsf{GAF}} = \{z : \mathsf{F}^{\mathrm{Pl}}(z) = 0\}. \tag{1.5}$$

Theorem 1.3. $\mathsf{P}^{\mathsf{GAF}}$ is a stationary hyperuniform isotropic point process.

A surprising point is that the law of GAF is not invariant under $\mathbb{C}$-translations, but its zero set is. This can be explained by the fact that for $v \in \mathbb{C}$, $\tau_v \mathsf{F}^{\mathrm{Pl}} := \mathsf{F}^{\mathrm{Pl}}(v + \cdot) \overset{(d)}{=} \exp(\varphi(v, \cdot))\mathsf{F}^{\mathrm{Pl}}$ for some deterministic function $\varphi : \mathbb{C}^2 \to \mathbb{C}$. From this identity, it is clear that the zero sets of F^{Pl} and $\tau_v \mathsf{F}^{\mathrm{Pl}}$ have the same law. One can also define $\mathsf{P}^{\mathsf{GAF}}$ as the weak limit $\mathsf{P}_n^{\mathsf{GAF}}$ of the zeros of the n-degree polynomial $\mathsf{F}_{\mathsf{GAF},n}(z) = \sum_{k=1}^{n} \frac{\mathsf{G}_k}{\sqrt{k!}} z^k$. More background and results about GAFs and a proof are provided at Section 3.2, based on [36].

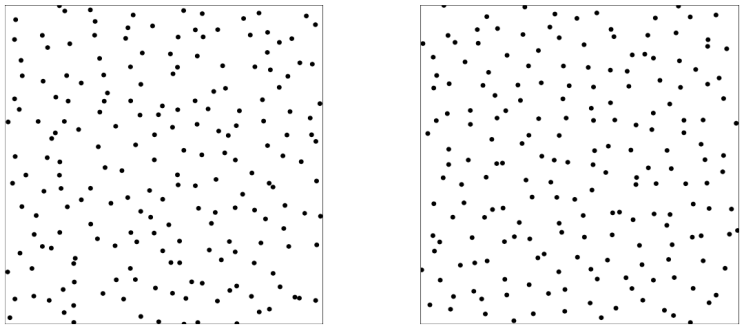


Figure 1.4: *Left.* Ginibre ensemble. *Right.* GAF zeros.

Chapter 2

Mathematical hyperuniformity of random measures

Physicists often define the hyperuniformity of a point process as the property that its Fourier transform vanishes at 0. We provide here the mathematical material to justify this assertion in full generality and deduce other characterisations of hyperuniformity easier to handle technically, and prepare some of the proofs of Chapter 3.

Even though point processes provide the main motivation and featured examples, the theory can be applied most generally in the framework of *wide sense stationary measures*, which will allow us to illustrate technical considerations with examples drawn from random Gaussian fields, spin systems, or nodal lines. We also give universal lower bounds on the variance and Central Limit theorems, and other insights about general hyperuniformity.

2.1 Wide sense stationary random measures

We consider the space $\mathscr{M}(\mathbb{R}^d)$ of complex-valued measures on $\mathbb{R}^d$, endowed with the vague topology, generated by the mappings $\varphi_f : M \in \mathscr{M}(\mathbb{R}^d) \mapsto M(f) = \int f dM$, for f a continuous complex-valued function with compact support, and the corresponding Borel σ-algebra $\mathscr{B}(\mathscr{M}(\mathbb{R}^d))$. A Locally Square Integrable random measure (denoted L^2_{loc} measure) is a random element M of $\mathscr{M}(\mathbb{R}^d)$ such that $\mathbf{E}(\mathsf{M}(A)^2) < \infty$ for A bounded. We extend the notation $\mathsf{M}(f) = \int f d\mathsf{M}$ to the class of bounded functions with compact support $\mathscr{C}^b_c(\mathbb{R}^d)$, or other classes when available, and say M is stationary, or *strongly* stationary, if $\mathsf{M}(\tau_x f) \overset{(d)}{=} \mathsf{M}(f)$ for $x \in \mathbb{R}^d$. See the fundamental books [8, 69] or the more recent work [10] for a justification of the facts presented below. A disintegration theorem gives the existence of the *covariance measure* C, sometimes called *auto-correlation,* characterised by

$$\operatorname{Cov}(\mathsf{M}(f), \mathsf{M}(g)) = \int f(x)\bar{g}(x+y)\mathsf{C}(dy)dx, f, g \in \mathscr{C}^b_c(\mathbb{R}^d). \tag{2.1}$$

When f, g, M have complex values, recall that we consider the complex covariance $\operatorname{Cov}(U, V) = \mathbf{E}U\bar{V} - \mathbf{E}U\mathbf{E}\bar{V}$. Taking for f, g approximations of Dirac masses in resp. 0 and some $y \in \mathbb{R}^d \backslash \{0\}$, we see that $\mathsf{C}(dy)$ measures the covariance between infinitesimal masses around 0 and y. Despite its name, C is well-defined as a signed measure only on bounded subsets of $\mathbb{R}^d$ (see Example 2.3), it is formally considered as a tempered distribution.

The covariance measure is semi-definite positive in the sense that

$$\int f(x)\bar{f}(x+y)\mathsf{C}(dy)dx = \operatorname{Var}(\mathsf{M}(f)) \geqslant 0$$

for $f \in \mathscr{C}_c^b(\mathbb{R}^d)$, hence Bochner's theorem yields the existence of a non-negative measure S such that

$$\mathrm{Var}\,(\mathsf{M}(f)) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{f}(u)|^2 \mathsf{S}(du) \tag{2.2}$$

where the Fourier transform is defined by

$$\hat{f}(u) := \int_{\mathbb{R}^d} f(x) e^{-iu\cdot x} dx.$$

Note that this identity extends to Schwarz functions f, i.e. such that $f, \hat{f}$ and their derivatives are integrable against any function $\|x\|^\alpha, \alpha > 0$. S is called the *spectral measure*, or *structure factor* when it is continuous with respect to $\mathscr{L}^d$. One can also view C as a tempered distribution on $\mathbb{R}^d$, and then $\mathsf{S} = \mathscr{F}\mathsf{C}$, where $\mathscr{F}$ denotes the Fourier transform on tempered distributions.

More axiomatically and without mention to $\mathscr{M}(\mathbb{R}^d)$, we can define a *wide sense, or weakly, stationary random measure* (WSRM) as a collection of complex-valued square integrable random variables $\mathsf{M}(f)$ that satisfy $\mathsf{M}(f+g) = \mathsf{M}(f) + \mathsf{M}(g),\ f, g \in \mathscr{C}_c^b$ and (2.2) for some non-negative measure S. It yields $\mathrm{Var}\,(\mathsf{M}(\tau_x f)) = \mathrm{Var}\,(\mathsf{M}(f))$. The property (2.2) might seem secondary, but it is really what we need for most purposes, we give several examples below.

The more investigated examples of linear statistics are ball indicators $f = 1_{B_R}, R > 0$. Understanding the behaviour of their Fourier transform is essential to study fluctuations of the number variance of point processes, i.e. the variance of the number of points in B_R, as $R \to \infty$.

Lemma 2.1. We have

$$\widehat{1_{B_1}}(u) = \|u\|^{-d/2} B_{d/2}(u)$$

where $B_{d/2}$ is the Bessel function of order $d/2$, it implies in particular for some $C_d > 0, c_d \in \mathbb{R}$ [1, Sec. 9.2]

$$\widehat{1_{B_1}}(u) = C_d \|u\|^{-\frac{d+1}{2}} \sin(\|u\| - c_d)(1 + o_{u\to\infty}(1)). \tag{2.3}$$

A first consequence of this lemma is that not any non-negative measure S can be the spectral measure of a L^2_{loc} (wide-sense) stationary random measure M; S is locally finite and its growth at ∞ is controlled by $\|u\|^{d+1}$:

Lemma 2.2. For any L^2_{loc} wide sense stationary random measure M, the spectral measure S satisfies

$$\int_{\mathbb{R}^d} (1 + \|u\|)^{-d-1} \mathsf{S}(du) < \infty. \tag{2.4}$$

This lemma also provides a proof that S is a tempered measure.

Example 2.1 (Gaussian processes)**.** Any finite measure S is the Fourier transform of some continuous covariance function C, and there exists a random Gaussian process $G(x), x \in \mathbb{R}^d$, and the corresponding random measure $\mathsf{M}(dx) = G(x)dx$, characterised by

$$\mathrm{Cov}\,(G(x), G(y)) = \mathsf{C}(x - y),$$

see for instance [3, Th. 5.4.2]. Minimal regularity assumptions on C (or tail decay on S) imply that G can be chosen to have regular sample paths, see for instance [3, Th.1.4.2].

Example 2.2 (Point processes)**.** A L^2_{loc} random measure P taking only integer values is called a point process as it can a.s. be represented as $\mathsf{P} = \sum_i n_i \delta_{x_i}$ for some isolated points x_i and $n_i \in \mathbb{N}^*$. As in the previous chapter, we shall generally require here that the process is *simple*, i.e. $n_i = \mathsf{P}(\{x_i\}) = 1$, so that P can be unambiguously associated with its support, we sometimes abusively write $\mathsf{P}(A) = \#\mathsf{P} \cap A$. Local square integrability implies that $\mathrm{supp}(\mathsf{P})$ is a.s. locally finite.

The most important example is certainly the unit intensity homogeneous Poisson point process (defined at Section 1.1), which satisfies $\mathrm{Var}\,(\mathsf{P}(A)) = \mathscr{L}^d(A)$, hence (2.2),(2.1) readily imply $\mathsf{C} = \delta_0, \mathsf{S} = \mathscr{L}^d$. This is the traduction that there is no interaction between different locations (the Dirac mass in 0 is an artifact of the atomic nature of point processes). A general disordered point process, supposed to have asymptotic independence for distant points, is expected to have a covariance measure of the form $\mathsf{C} = \delta_0 + g\mathscr{L}^d$ for some integrable g, and a structure factor $\mathsf{S} = \mathsf{s}\mathscr{L}^d$ where $\mathsf{s}-1$ is expected to be integrable.

Example 2.3 (Shifted lattices)**.** Following up on Section 1.2, we have for test functions f, g

$$\begin{aligned}\mathbf{E}\mathsf{Z}^d(f)\mathsf{Z}^d(g) &= \sum_{\mathbf{k},\mathbf{m}\in\mathbb{Z}^d} \mathbf{E}f(\mathbf{k}+U)g(\mathbf{m}+U)\\ &= \sum_{\mathbf{k},\mathbf{m}\in\mathbb{Z}^d} \int_{[0,1]^d} f(\mathbf{k}+u)g(\mathbf{m}+u)du\\ &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_{[0,1]^d} f(\mathbf{k}+u) \sum_{\mathbf{l}\in\mathbb{Z}^d} g(\mathbf{k}+\mathbf{l}+u)\end{aligned}$$

hence we have

$$\mathrm{Cov}\left(\mathsf{Z}^d(f), \mathsf{Z}^d(g)\right) = \int f(x)g(x+y) \sum_{\mathbf{l}\in\mathbb{Z}^d} \delta_{\mathbf{l}}(y)dx - \int f(x)g(x+y)dxdy$$

and the covariance is $\mathsf{C} = \sum_{\mathbf{l}\in\mathbb{Z}^d} \delta_{\mathbf{l}} - \mathscr{L}^d$ (remark that $\mathsf{C}(\mathbb{R}^d)$ is not well defined). We then use the Poisson summation formula

$$\sum_{\mathbf{l}\in\mathbb{Z}^d} f(\mathbf{l}) = \sum_{\mathbf{k}\in 2\pi\mathbb{Z}^d} \hat{f}(\mathbf{k})$$

to have by the Plancherel formula, with $\mathsf{S} = \mathscr{F}\mathsf{C}$,

$$(2\pi)^d\langle \mathsf{S}, f\rangle = \langle \mathsf{C}, \hat{f}\rangle = \sum_{\mathbf{l}\in\mathbb{Z}^d} \hat{f}(\mathbf{l}) - \int \hat{f} = \sum_{\mathbf{k}\in 2\pi\mathbb{Z}^d} (2\pi)^d f(\mathbf{k}) - (2\pi)^d f(0),$$

i.e. $\mathsf{S} = \sum_{\mathbf{k}\in 2\pi\mathbb{Z}^d\setminus\{0\}} \delta_{\mathbf{k}}$. We will see at the next section that this form of the spectral measure, in particular the gap around 0, neatly proves the hyperuniformity of Z^d, a fact that is not obvious through direct geometric computations.

Example 2.4 (Independently perturbed lattices)**.** Following up on Example 1.1, let us now give the spectral measure for the perturbed lattice $\mathsf{Z}^{d,\mu}$, where μ is a probability measure on $\mathbb{R}^d$. Let $\psi(u) = \int e^{-iu\cdot t}d\mu(t)$. We have

$$\mathsf{S}(du) = (1 - |\psi(u)|^2)du + \sum_{\mathbf{m}\in 2\pi\mathbb{Z}^d\setminus\{0\}} |\psi(\mathbf{m})|^2 \tag{2.5}$$

This is a particular case of the more general Proposition 4.1 where a point process is perturbed by clusters. We can observe that the periodic structure of the lattice is present through the atomic component in the second term, while the continuous component expresses the slight disorder introduced in the system.

As for shifted lattices with no perturbations, we observe that the spectral measure vanishes around 0. Still using the next section, this shall imply the hyperuniformity of such models. A nice observation by Klatt and Torquato [56] is that if μ is $\mathscr{U}_{[0,1]^d}$, then the singular component vanishes. It means that the periodic structure is not detectable by a second order analysis. It is still present at higher orders, in the sense of factorial moment measures defined at Section 3.1; more generally it is likely not mixing, and the cloaking of the second order periodic structure does not kill the anisotropy of $\mathbb{Z}^d$.

2.2 Spectral characterisation of hyperuniformity

Coste [17] derived the spectral characterisation of hyperuniformity of a weakly disordered L^2_{loc} point process, i.e. when C is integrable or with constant sign [17, Prop. 2.2]: hyperuniformity is equivalent to $\mathsf{S}(B_\varepsilon) = o(\varepsilon^d)$ as $\varepsilon \to 0$; Bjorklünd and Hartnick [10] removed this assumption. This is in particular useful to show that hyperuniformity can be equivalently characterised using smooth linear statistics instead of discontinuous ball indicators. For $f : \mathbb{R}^d \to \mathbb{C}$, let $f_R(x) = f(x/R), R > 0$. To give an optimal statement, recall that (2.2) holds for Schwarz functions and bounded measurable functions with compact support, but often it also holds for a wider class of functions. Without discussing this further, we call **S-admissible** an integrable function $f$such that (2.2) holds for all $f_R, R > 0$, but the two afore-mentionned classes are sufficient for most purposes.

Theorem 2.1. Let a wide sense stationary random measure M with spectral measure S. The three following are equivalent.

- (i) M is hyperuniform, i.e. $\mathrm{Var}\,(\mathsf{M}(B_R)) = o(R^d)$.
- (ii) there exists f S-admissible such that $\int f \neq 0$ and $\mathrm{Var}\,(\mathsf{M}(f_R)) = o(R^d)$.
- (iii) We have *spectral hyperuniformity*:

$$\lim_{\varepsilon \to 0} \frac{\mathsf{S}(B_\varepsilon)}{\varepsilon^d} = 0.$$

Point (iii) immediately implies that (independently perturbed) shifted lattices $\mathsf{Z}^{d,\mu}$ from Examples 2.3,2.4 are hyperuniform, since the structure factor vanishes at the origin. Importantly, the class of regular functions admissible for hyperuniformity, i.e. satisfying (ii), includes the indicator of a sphere by Lemma 2.1, but not the indicator of all shapes, as for instance a direct geometric reasoning yields the hyperfluctuating variance

$$\mathrm{Var}\left(\mathsf{Z}^d([-n, n+1/2]^d)\right) \asymp n^{2(d-1)}$$

for $n \in \mathbb{N}$ going to infinity. Geometrically, the large variance comes from the possibility of large groups of n^{d-1} points to cross the border of a large cube in the same direction without being compensated. This irregularity hence does not come from the sharp corners of the rectangles, rather from its flat edges; a similar reasoning yields that the indicator of the rectangle with "rounded corners" $W = \{x + y : x \in [-10, 10]^d, y \in B(0, 1)\}$ does not satisfy (ii) either. [10, Theorem 3.6] shows that the number variance cancellation holds for so-called *Fourier smooth* shapes. Possible number variance fluctuations are further discussed at Section 2.3.

Proof. We intensively use the scaling identity $\widehat{f_R} = R^d \hat{f}(R\,\cdot)$ for $f \in L^1(\mathbb{R}^d)$. We immediately have (i) implies (ii) since $1_{B_1} \in \mathsf{C}^b_c(\mathbb{R}^d)$.

- (ii)$\Rightarrow$ (iii). Since f is integrable, $\hat{f}$ is continuous with $\hat{f}(0) = \int f \neq 0$. Hence there is $a, \kappa > 0$ such that $\kappa 1_{B_a} \leqslant |\hat{f}|$. Then by (2.2)

$$\kappa^2 \mathsf{S}(B_{a/R}) \leqslant \int_{B_{a/R}} R^{-2d} |\hat{f}_R|^2 \mathsf{S}(du) \leqslant R^{-2d} \int_{\mathbb{R}^d} |\hat{f}_R|^2 \mathsf{S}(du) = (2\pi)^d R^{-2d} \mathrm{Var}\,(\mathsf{M}(f_R)) = o(R^{-d})$$

  by assumption, which gives (iii).

- (iii) $\Rightarrow$ (i). We have with $f = 1_{B_1}$ and Lemma 2.1, using (2.2), for some $c < \infty$,

$$\begin{aligned}(2\pi)^d \mathrm{Var}\,(\mathsf{M}(f_R)) &= \int |\hat{f}_R|^2 \mathsf{S}(du) \\ &\leqslant R^{2d} \sup |\hat{f}|^2 \mathsf{S}(B_{10/R}) + R^{2d} c \int_{B_1 \setminus B_{10/R}} (1 + \|u\| R)^{-d-1} \mathsf{S}(du) + c R^{2d} \int_{B_1^c} (\|u\| R)^{-d-1} \mathsf{S}(du).\end{aligned}$$

Lemma 2.2 shows that the last term is in $O(R^{d-1})$, and (iii) implies that the first term is in $o(R^d)$, hence we must show that the second term is in $o(R^d)$. It would be easy under the simplifying assumption that S has a density $\mathsf{s}(u) \to 0$ as $u \to 0$. In the general case, one needs the identity, for a bounded test function φ and a probability measure μ,

$$\int \varphi(u)\mu(du) = \int_0^{\sup|\varphi|} \mu(\{u : \varphi(u) > t\})dt,$$

applied to $\mu = c1_{B_1\setminus B_{10/R}}\mathsf{S}$ with the right renormalising constant c, and $\varphi(u) = (1 + \|u\|R)^{-d-1}$. Since $1 + \|u\|R < 2\|u\|R$ for $\|u\| > 10/R$, it yields for some finite c

$$\begin{aligned}(2\pi)^d \mathrm{Var}\left(\mathsf{M}(f_R)\right) \leqslant & cR^{2d}\int_0^1 \mathsf{S}(\{u \in B_1\setminus B_{10/R} : 2\|u\|R < t^{-\frac{1}{d+1}})\})dt + o(R^d)\\ \leqslant & cR^d \int_{R^{-d-1}}^1 R^d \mathsf{S}\left(B_{t^{-\frac{1}{d+1}}/2R}\right) dt + cR^{2d}R^{-(d+1)}\mathsf{S}(B_1) + o(R^d).\end{aligned}$$

By (iii), $R^d\mathsf{S}(B_{t^{-\frac{1}{d+1}}/2R}) = t^{-\frac{d}{d+1}}\xi(t^{-\frac{1}{d+1}}/2R)$ where $\lim_{v\to 0}\xi(v) = 0$. Then Lebesgue's theorem yield that $R^{-d}\mathrm{Var}\left(\mathsf{M}(f_R)\right) \to 0$, as desired.

□

2.3 Universal variance lower bounds and non-spherical windows

A celebrated result of [7] shows that for a deterministic point configuration P, the fluctuations of the number of points in a large window B_R are at least of the order $\sqrt{R^{d-1}}$, which leads in general to a variance lower bound of the order R^{d-1} for arbitrarily large R. This principle is not restricted to atomic measures, as we shall see here, stating the generalisation of [14, Theorem 1.1].

Theorem 2.2 (Beck)**.** Let M a wide sense stationary random measure that is not identically 0 a.s.. Then for some $c > 0$, for $\mathbf{R}$ sufficiently large,

$$\int_1^{\mathbf{R}} \frac{\mathrm{Var}\left(\mathsf{M}(B_R)\right)}{R^{d-1}} dR \geqslant c\mathbf{R}$$

which in particular yields

$$\limsup_{R\to\infty} \frac{\mathrm{Var}\left(\mathsf{M}(B_R)\right)}{R^{d-1}} > 0.$$

Proof. Let $f = 1_{B_1}$. By Lemma 2.1, $|\hat{f}|^2$ is larger than $(1 + \|u\|)^{-d-1}$ "on average", i.e. there is $\kappa > 0, \rho_0 > 0, \mathbf{R}_0 > 0$ such that $\mathsf{S}(B^c_{\rho_0}) > 0$ and for $u \notin B_{\rho_0}, \mathbf{R} > \mathbf{R_0}$,

$$\int_1^{\mathbf{R}} R^{d+1}|\hat{f}(Ru)|^2 dR \geqslant \kappa \int_1^{\mathbf{R}} R^{d+1}(1 + \|u\|R)^{-d-1}dR. \tag{2.6}$$

We then have by (2.2), for $\mathbf{R} > \mathbf{R_0}$,

$$\begin{aligned}(2\pi)^d \int_1^{\mathbf{R}} R^{1-d}\mathrm{Var}\left(\mathsf{M}(f_R)\right) dR \geqslant & \int_{\mathbb{R}^d\setminus B_{\rho_0}} \int_1^{\mathbf{R}} R^{d+1}|\hat{f}(Ru)|^2 dR\mathsf{S}(du)\\ \geqslant & \kappa \int_{\mathbb{R}^d\setminus B_{\rho_0}} \int_1^{\mathbf{R}} R^{d+1}(1 + \|u\|R)^{-d-1}dR\mathsf{S}(du)\\ = & \kappa \int_{\mathbb{R}^d\setminus B_{u_0}} \|u\|^{-d-2}\int_{\|u\|}^{\|u\|\mathbf{R}} \rho^{d+1}(1+\rho)^{-d-1}d\rho\mathsf{S}(du)\\ \geqslant & \kappa \int_{\mathbb{R}^d\setminus B_{\rho_0}} \|u\|^{-d-2}(1 + 1/\rho_0)^{-d-1}\|u\|(\mathbf{R}-1)\mathsf{S}(du).\end{aligned}$$

□

The proof above actually works for any kernel f satisfying (2.6). Since the ball is the most regular shape in many aspects, one could imagine that it has the lowest Fourier transform in some sense, hence this lower bound could hold for any shape. This is in fact not the case for rectangular shapes, which Fourier transform is indeed larger in some directions, but also smaller in others. The class of admissible shapes, called *Fourier smooth,* is discussed around [10, Theorem 3.6]. In the complex plane, Sodin, Wennman and Yakir [77] give variance asymptotics for Jordan domains with rectifiable boundary for a class of weakly disordered point processes. Let us make some further remarks:

- (vanishing lim inf) Björklund and Bylehn [14, Prop. 2.2.3] show that, iff $d \equiv 1 \bmod 4$,
$$\liminf_{R\to\infty} \frac{\mathrm{Var}\left(\mathsf{Z}^d(B_R)\right)}{R^{d-1}} = 0.$$
- (General shape dependence) The lim sup bound is invalid on rectangular windows when one studies non-atomic random measures, even with short range dependencies. Consider for instance the spectral measure $\mathsf{S} = 1_S\mathscr{L}^d$ where $S = \{(u_i) \in \mathbb{R}^d : \forall i, 2 > |u_i| > 1\}$, and the unique centred Gaussian field which spectral measure is S (Example 2.1). One can directly show that the variance is bounded: from (2.2)
$$\begin{aligned}(2\pi)^d \mathrm{Var}\left(\mathsf{M}([-R,R]^d)\right) &= \int_S R^{2d} |\prod_{i=1}^d \sin(Ru_i)/(Ru_i)|^2 du \\ &\leqslant \int_S c\prod_i u_i^{-2} du < \infty.\end{aligned}$$
One can refine this example by taking S with support on all $\mathbb{R}^d$, as long as it vanishes sufficiently fast close to the axes and at infinity.
- (shape-dependance for point processes) In a private communication, M. Bylehn mentions that there are some *admissible orthogonal transformations* $\mathbf{O}$ such that the previous lim sup bounds does not hold for the rotated lattice $\mathbf{O}\mathsf{Z}^d$, using a bound of Skryganov [73]:
$$\mathrm{Var}\left(\mathbf{O}\mathsf{Z}^d([-R,R]^d)\right) = O(\ln(R)^{2(d-1)}).$$
- (no shape-dependence for disordered point processes) Nazarov & Sodin [62, Lemma 1.6] show that under the *weak disorder* assumption that $\mathsf{C} - \delta_0$ is integrable for some point process P, for any bounded window W with non-empty interior, $\mathrm{Var}\left(\mathsf{P}(RW)\right)$ satisfies Beck's lower bound.

Remark 2.1. When the variance of a point process is subextensive, i.e. in $o(R^d)$, then under mild integrability conditions, it behaves necessarily in R^{d-1}, see [57, Proposition 2].

2.4 Hyperuniformity exponent and classification

The speed of decay of the structure factor in 0 actually matters, for the decay of smooth linear statistics, but also for other phenomena, such as rigidity (Chapter 5). It is traditionally said that some number $\alpha > 0$ is a hyperuniformity exponent of S (or M) if $\mathsf{S}(du) \sim c\|u\|^\alpha du$ as $u \to 0$ for some $c > 0$. We shall more generally say without requiring a density that S admits hyperuniformity index $\alpha > 0$ if $\mathsf{S}(B_\varepsilon) = O(\varepsilon^{d+\alpha})$ as $\varepsilon \to 0$.

Proposition 2.1. Let S the spectral measure of a L^2_{loc} wide sense stationary random measure M. Assume S admits exponent $\alpha > 0$. Let f an $\mathsf{S}-$admissible function such that for some $\beta \geqslant (d+1)/2$, as $u \to \infty$,
$$|\hat{f}(u)| = O(\|u\|^{-\beta}).$$

(For the unit ball indicator, $\beta = (d+1)/2$.) Then

$$\mathrm{Var}\,(\mathsf{M}(f_R)) = O(R^{2d}) \times \begin{cases} R^{-(d+\alpha)\vee 2\beta)} \text{ id } d+\alpha \neq 2\beta. \\ R^{-d-\alpha}\ln(R)) \text{ if } d+\alpha = 2\beta. \end{cases}$$

If conversely $\mathrm{Var}\,(\mathsf{P}(f_R)) = O(R^{d-\alpha})$ holds for some S-admissible function f with $\int f \neq 0$, then S admits exponent α.

A version of this result for Schwartz functions appears in [58] under an integrability assumption on S. Variance estimation for linear statistics is also a central topic in [47], under the running assumption that the covariance measure C is integrable, or has finite higher order moments. They notice in particular that the decay exponent is even when the covariance is integrable. Due to the reciprocal relation with the hyperuniformity exponent exhibited by the previous proposition, one can also see it as a consequence of the symmetry of S. They also give sufficient conditions for S to admit exponent α, and discuss the class of admissible kernels f.

Proof. The proof is strongly similar to the step (iii) implies (i) in the proof of Theorem 2.1, with $\mathsf{S}(B_\varepsilon) = O(\varepsilon^{d+\alpha})$: Using a usual (2.2) and Lemma 2.2,

$$\begin{aligned}(2\pi)^d R^{-2d}\mathrm{Var}\,(\mathsf{M}(f_R)) \leqslant& \sup|\hat f|^2 \mathsf{S}(B_{10/R}) + c\int_{B_1\setminus B_{10/R}} (1+\|u\|R)^{-2\beta}\mathsf{S}(du) + c\int_{B_1^c}(\|u\|R)^{-2\beta}\mathsf{S}(du)\\ \leqslant& O(R^{-d-\alpha}) + (\mathrm{II}) + O(R^{-2\beta}).\end{aligned}$$

As before, (II) is easily dealt with when S has a density s satisfying $\mathsf{s}(u) \leqslant c\|u\|^\alpha$ close to 0, with the change of variables $v = Ru$. In the general case, proceed also as in (iii) $\Rightarrow$(i) but with 2β instead of $d+1$. It yields for some $c < \infty$ changing at each line

$$\begin{aligned}(\mathrm{II}) \leqslant& c\int_{R^{-2\beta}}^1 \mathsf{S}(B_{t^{-\frac{1}{2\beta}}/R})dt + c\int_0^{R^{-2\beta}} \mathsf{S}(B_1)dt\\ \leqslant& c\int_{R^{-2\beta}}^1 (t^{-1/2\beta}/R)^{d+\alpha}dt + cR^{-2\beta}\\ \leqslant& c\begin{cases} R^{-\alpha-d}(R^{-2\beta})^{1-(d+\alpha)/2\beta} + cR^{-2\beta} \leqslant \quad R^{-2\beta} + cR^{-2\beta} \text{ if } d+\alpha \neq 2\beta \\ R^{-d-\alpha}O(\ln(R)) + R^{-2\beta} \text{ otherwise.}\end{cases}\end{aligned}$$

For the converse, as in (ii) implies (iii), $|\hat f| \geqslant \kappa 1_{B_a}$ for some $\kappa, a > 0$, and

$$(2\pi)^d \mathrm{Var}\,(\mathsf{M}(f_R)) \geqslant \int_{B_{a/R}} R^{2d}|\hat f(Ru)|^2 d\mathsf{S}(u) \geqslant \kappa^2 R^{2d}\mathsf{S}(B_{a/R})$$

hence indeed S admits index α if $\mathrm{Var}\,(\mathsf{M}(f_R)) = O(R^{d-\alpha})$. □

Example 3.1 and Theorem 4.3 provide examples of point processes with arbitrary decay for S around 0. The optimal exponent α is related to a classification of hyperuniform point processes relevant in the physics litterature [82], depending on the number variance behaviour (i.e. for $f = 1_{B_1}$ and $2\beta = d+1$):

- Class I if $\mathrm{Var}\,(\mathsf{P}(B_R)) = O(R^{d-1})$, corresponding to $\alpha > 1$. Recall that by Beck's theorem (Theorem 2.2), R^{d-1} is also the smallest possible magnitude for the number variance.
- Class II, when $\alpha = 1$, which yields $\mathrm{Var}\,(\mathsf{P}(B_R)) = O(R^{d-1}\ln(R))$
- Class III, when $\alpha \in (0,1)$, giving $\mathrm{Var}\,(\mathsf{P}(B_R)) = O(R^{d-\alpha})$.

Let us make two remarks:

- This classification encompasses most "useful models", i.e. arising in a genuine physical or biological phenomenon, but some mathematical hyperuniform systems are not represented here, when the reduced variance decreases to 0 at a logarithmic scale, see Example 3.1 or examples in [21].
- By Beck's theorem, using the variance on a large ball indeed restricts the variance range at R^{d-1} and above, but when one uses smoother linear statistics, by Proposition 2.1, one can discriminate more efficiently between hyperuniform systems. For instance, The GAF zeros $\mathsf{P}^{\mathsf{GAF}}$ and Ginibre process $\mathsf{P}^{\mathrm{Gin}}$ of Section 1.4 are both in class I, but they have respective optimal indexes 4 and 2, and in agreement exhibit different macroscopic properties, for instance regarding rigidity ([31], see Theorem 5.1). See also Section 4.1.1, which yields examples for arbitrarily large α. The extreme case $\alpha = \infty$ corresponds to stealthy processes, discussed at Section 5.2.

Let us conclude with a lemma that yields that most point processes have exponent at most 2.

Lemma 2.3. Let μ a symmetric non-negative finite measure which is not supported by a hyperplane. Then there is $\sigma > 0, \rho_0 > 0$ such that for $\|u\| < \rho_0$,

$$\left|\int (1 - e^{-iu\cdot x})\mu(dx)\right| \geqslant \sigma\|u\|^2.$$

This conclusion also holds if the assumption of symmetry is dropped, if instead μ is assumed to have a finite second moment and be centred, i.e. $\int x d\mu = 0$.

Proof. Let $R > 0$ such that

$$\int_{B(0,R)} \|x\|^2 \mu(dx) > 0$$

and $\rho > 0$ such that $1 - \cos(t) \geqslant t^2/4$ for $|t| \leqslant R\rho$, and let $u \in B(0,\rho)$.

We have

$$\begin{aligned}\mathsf{s}(u) := \int (1 - e^{iu\cdot x})\mu(dx) &= \int (1 - \cos(x\cdot u))\mu(dx) \\ &\geqslant \int_{B(0,R)} \frac{|x\cdot u|^2}{4}\mu(dx). \end{aligned} \tag{2.7}$$

Call $C_\varepsilon(u)$ the cone of x such that $|x \cdot u| \geqslant \varepsilon\|x\|\|u\|$ (and $C_0(u) = \mathbb{R}^d$). Define the function

$$\Sigma(v,\varepsilon) := \int_{C_\varepsilon(v)\cap B(0,R)} \|x\|^2 \mu(dx), v \in \partial B(0,1).$$

Since by assumption μ is not supported by the hyperplane orthogonal to $v \in \partial B(0,1)$, $\Sigma(v,\varepsilon_v) > 0$ for some $\varepsilon_v > 0$. We wish to show by contradiction that

$$\exists \sigma_0 > 0, \varepsilon > 0 : \forall v \in \partial B(0,1), \Sigma(v,\varepsilon) \geqslant \sigma_0.$$

If it is not the case, there is $v_n \in \partial B(0,1)$ such that $\Sigma(v_n, 1/n) \leqslant 1/n$. By compacity we can choose v_n that converges to some v, and it yields a contradiction when $C_{\varepsilon_v}(v) \subset C_{1/n}(v_n)$. In consequence, for $u \in B(0,\rho)$, we can conclude the proof with

$$\mathsf{s}(u) \geqslant \varepsilon^2\|u\|^2\sigma_0/4.$$

□

This lemma implies that stationary determinantal processes have exponent at most 2 (see Theorem 5.2), and with (2.5) yields the following corollary for some IPLs:

Corollary 2.1. Let μ a non-null probability measure on $\mathbb{R}^d$ which is symmetric.Then the perturbed lattice $\mathsf{Z}^{d,\mu}$ has exponent at most 2.

Even if μ is supported by a hyperplane, doing the same analysis on a subspace of minimal dimension supporting μ, we have $\mathsf{s}(u) \geqslant \sigma|u_i|^2$ at least for one coordinate i, which forbids exponent more than 2.

2.5 CLTs and Brillinger mixing

We investigate the Central Limit Theorem for the mass of a large ball $\mathsf{M}(B_R)$ for a wide sense stationary random measure$\mathsf{M}$. The cumulants method, classical for extensive systems, fortunately extends to hyperuniform systems, due to the universality of Beck's lower bound (Theorem 2.2). Recall that the m-th cumulant of a real random variable X is the m-th order derivative in 0, when it exists, of the log of the moment generative function $K_X(t) = \ln \mathbf{E}e^{tX}$, i.e.

$$\kappa_m(X) = \frac{d^m}{dt^m} K_X(t)|_{t=0}.$$

We have a familiar interpretation for low order cumulants: for $\bar{X} = X - \mathbf{E}X$,

$$\begin{aligned}
\kappa_1(X) =& \mathbf{E}X \\
\kappa_2(X) =& \mathrm{Var}\,(X) \\
\kappa_3(X) =& \mathbf{E}\bar{X}^3 \\
\kappa_4(X) =& \mathbf{E}\bar{X}^4 - \mathrm{Var}\,(X)^2 .
\end{aligned}$$

For instance for $X \sim \mathscr{P}(\lambda), \lambda > 0$, we have easily $K_X(t) = \lambda(e^t - 1) = \lambda \sum_{k\geqslant 1} \frac{t^k}{k!}$, hence for all m

$$\kappa_m(X) = \lambda.$$

The Gaussian variables are characterised as those variables such that $\kappa_k(X) = 0$ for $k \geqslant 3$, indeed $K_X(t) = t^2$ for $X \sim \mathscr{N}(0,1)$. Marcinkiewicz refined this result by showing that a variable is Gaussian as soon as only finitely many cumulants do not vanish.

Recall that a sequence of variables $X_n, n \geqslant 1$ converge to a standard Gaussian variable X if all moments converge, i.e. for each $m \geqslant 1$, $\mathbf{E}X_n^m \to \mathbf{E}X^m$ as $n \to \infty$. Since moments are linear combination of cumulants, the convergence still holds if for each $m \geqslant 1$, $\kappa_m(X_n) \to \kappa_m(X)$. Using Marcinkiewicz theorem, one can show that this convergence holds if one only assumes $\mathrm{Var}\,(X_n) \to 1$ and $\kappa_m(X_n) \to 0$ for all $m \geqslant m_0$, for some $m_0 \geqslant 3$, see for instance [78, Lemma 3].

Define in general

$$\widetilde{\mathsf{M}(B_R)} = \frac{\mathsf{M}(B_R) - \mathbf{E}\mathsf{M}(B_R)}{\sqrt{\mathrm{Var}\,(\mathsf{M}(B_R))}}.$$

The previous method applies to the number of points of a Poisson point process in a large ball because $\mathsf{P}(B_R) \overset{(d)}{=} \mathrm{Poiss}(\mathscr{L}^d(R^d))$. Hence $\mathrm{Var}\,(\mathsf{P}(B_R)) \asymp R^d$ and

$$\kappa_m\left(\widetilde{\mathsf{P}(B_R)}\right) = \mathrm{Var}\,(\mathsf{P}(B_R))^{-m/2}\, \kappa_m(\mathsf{P}(B_R)) \asymp R^{-md/2} R^d,$$

it indeed goes to 0 for $m \geqslant 3$. More generally, it applies to many standard and hyperuniform random measures:

Theorem 2.3. Let M a wide sense stationary random measure in dimension $d \geqslant 2$ having finite moments of all orders on a compact set. Assume that for some $m_0 \geqslant 3$, for $m \geqslant m_0$, the cumulants have Poison / sub-Poisson decay

$$\kappa_m(\mathsf{M}(B_R)) = O(R^d). \tag{2.8}$$

Then we have the CLT for some sequence $R_n \to \infty$

$$\widetilde{\mathsf{M}(B_{R_n})} \to \mathscr{N}(0,1).$$

In dimension 1, if (2.8) holds and $\mathrm{Var}\,(\mathsf{M}(B_R)) \geqslant cR^\alpha$ with $\alpha > 0$, as $R \to \infty$,

$$\widetilde{\mathsf{M}(B_R)} \to \mathscr{N}(0,1).$$

Proof. Let $X_n = \mathsf{M}(B_{R_n}), \widetilde{X_n} = \widetilde{\mathsf{M}(B_R)}, n \geqslant 1$. Recalling Beck's Theorem 2.2, $\mathrm{Var}\,(X_n) \geqslant cR_n^{d-1}$ for some $R_n \to \infty$ and

$$\kappa_m(\widetilde{\mathsf{M}(B_{R_n})}) = \frac{\kappa_m(\mathsf{M}(B_{R_n}))}{\mathrm{Var}\,(\mathsf{M}(B_{R_n}))^{m/2}} \leqslant c\frac{R_n^d}{R_n^{(d-1)m/2}}.$$

We see that indeed for $d > 1$ and m sufficiently large, the right hand side goes to 0, and Marcinkiewicz's theorem allows to conclude to the CLT. In dimension $d = 1$, the bound is $R^{d-\alpha k/2}$, and we can conclude similarly. □

Remark that the R^{d-1} lower bound in Theorem 2.2 actually holds for a set of radii R with positive Lebesgue density, it is not just a marginal sequence of R_n's. For other linear statistics $\mathsf{M}(f_R)$, Beck's bound might not hold, see in particular Proposition 2.1. It can still happen that there is a CLT even if the variance is bounded or, surprisingly, goes to 0, see for instance [75], but this is rather exceptionnal. See [47, Section 4.1] for a discussion, and for more general results than Theorem 2.3.

Assumption (2.8) is the manifestation of a property which is expected for strongly disordered point processes, named *Brillinger mixing*, but hard to prove apart from some well understood classes such as Poisson, determinantal, and permanental processes, or zeros of random Gaussian fields. One can strictly weaken this assumption to $\kappa_m(\mathsf{M}(B_R)) = o(R^{m(d-1)/2})$ (for m above some $m_0 \in \mathbb{N}$) but there is no immediate interpretation for the relevancy of such an hypothesis.

This theorem has been applied successfully to many linear statistics over point processes, but also to random measures. In [11], the authors consider more generally a geometric functional over a point process P under the form

$$X_n = \sum_{x \in \mathsf{P} \cap B_n} \xi(x, \mathsf{P})$$

for some *score function* ξ that does not only depend on the location x. It can be interpreted as a linear statistic over the stationary random measure

$$\mathsf{M} = \sum_{x \in \mathsf{P}} \delta_x \xi(x, \mathsf{P}).$$

Under some assumptions of stabilisation and dependency decay related to Brillinger mixing on P and ξ, they are able to show a CLT for $\mathsf{M}(B_R)$.

2.6 Non-Euclidean hyperuniformity

A rescaling yields that the hyperuniformity of a point process P on $\mathbb{R}^d$ can be equivalently stated by

$$\lim_{r \to \infty} \frac{\mathrm{Var}\,(\mathsf{P}_r(B_1))}{\mathbf{E}\mathsf{P}_r(B_1)} = 0;$$

where $\mathsf{P}_r(\cdot) := \mathsf{P}(r\cdot)$. We can exploit this to define asymptotic hyperuniformity for a sequence of point processes which are finite and, obviously, do not satisfy stationarity.

Definition 2.1. Let P_n random measures supported by some $K \subset \mathbb{R}^d$. Say the family $\{\mathsf{P}_n; n \geqslant 1\}$ is *asymptotically hyperuniform* over $K \subset \mathbb{R}^d$ if for each "smooth" compact $B \subset K$

$$\lim_n \frac{\mathrm{Var}\,(\mathsf{P}_n(B))}{\mathbf{E}\mathsf{P}_n(B)} = 0.$$

This can be relevant for finite models, such as Coulomb / Riesz gases, or eigenvalues of random matrices. For such models, particles near the *edge* behave in general differently. The good framework is to choose K containing a.s. the *bulk*, i.e. a number of particles in $O((1-\varepsilon)n)$ for some $\varepsilon > 0$ that

seem to behave in a homogeneous way, see Theorem 3.2 for Coulomb gases.

This definition is not adapted to random measures of a different nature. For instance, if M is a hyperuniform random measure of one-dimensional object of $\mathbb{R}^2$ (lines, cuves, etc...), then $\mathsf{M}_n(B) = \sqrt{n}\mathscr{L}^2(B)$. Hence the renormalisation in the definition of hyperuniformity must be adapted to the intrinsic dimension of the model.

Chapter 3

Hyperuniform point processes

In this chapter, we develop the examples from the introduction and give some proofs. We start by defining factorial moment measures, central in the study of determinantal processes. We then discuss the planar GAF zeros, as it is a striking and isolated example, easy to define formally. Random matrices and DPPs form the core of hyperuniform point processes, but require more context and preparation. Non-integrable Gibbs measures are discussed after, but their high mathematical complexity leaves less room for rigourous results, let alone proofs. We conclude with models presenting an aperiodic order, such as quasicrystals.

3.1 Factorial moment measures

We saw that the law of a point process $\mathsf{P} = \sum_i \delta_{x_i}$ is characterised by the laws $\mathsf{P}(f)$ for $f \in \mathscr{C}_c^b(\mathbb{R}^d)$. Factorial moment measures give a more analytic way to decompose P's law in projections of orders $1, 2, ...$ and characterise it in the same way that the law of a reasonable random variable is characterised by its moments of every order. Let μ_{P}^m the *m-th factorial moment measure* of P, characterised through non-negative test functions $f \in \mathsf{C}_c^b((\mathbb{R}^d)^m)$ applied to m-tuples of distinct x_i's with

$$\mathbf{E} \sum_{i_1,\dots,i_m \text{ distinct}} f(x_{i_1}, \dots, x_{i_m}) = \int f \mu_m^{\mathsf{M}}.$$

The number of terms in the sum is determined by the number of Dirac masses in $\mathsf{P} = \sum_i \delta_{x_i}$. When μ has a density with respect to $\mathscr{L}^m$, it is denoted by ρ_{P}^m.

- For $k = 1$, one retrieves the intensity $\mu_{\mathsf{P}}^1(A) = \mathbf{E}\mathsf{P}(A)$ for $A \subset \mathbb{R}^d$. If P is stationary, μ_{P}^1 is invariant under translations, hence $\mu_{\mathsf{P}}^1 = \lambda \mathscr{L}^d$, with the **intensity** $\lambda \geqslant 0$.
- For $1 \leqslant k \leqslant m$, μ_{P}^k is proportionnal to the projection of μ_{P}^m on k arguments (recall that those measures are symmetric).

  projection
- The first and second order properties of P can be equivalently described by the couple $(\lambda, \rho_{\mathsf{P}}^2)$ or by the couple (λ, C) (or obviously the couple (λ, S)): combine (2.1) with

$$\begin{aligned}
\operatorname{Cov}(\mathsf{P}(f), \mathsf{P}(g)) =& \mathbf{E} \sum_{i,j} f(x_i)\bar{g}(x_j) - \mathbf{E}(\mathsf{P}(f))\,\mathbf{E}(\mathsf{P}(\bar{g})) \\
=& \mathbf{E} \sum_{i \neq j} f(x_i)\bar{g}(x_j) - \int f(x)\lambda dx \int \bar{g}(x)\lambda dx \\
=& \int f \otimes \bar{g} \rho_{\mathsf{P}}^2 + \lambda \int f\bar{g} - \lambda^2 \int f \int \bar{g}. \qquad (3.1)
\end{aligned}$$

- The *factorial* in the name refers to the formula obtained for a simple point process P when $f = 1_{B^m}$ for a bounded set B:

$$\mu_{\mathsf{P}}^m(B^m) = \mathbf{E} \sum_{x_1 \in B \cap \mathsf{P}} \sum_{x_2 \in B \cap \mathsf{P} \setminus x_1} \dots \sum_{x_m \in B \cap \mathsf{P} \setminus \{x_1, \dots, x_{m-1}\}} 1 = \mathbf{E}\mathsf{P}(B)^{(m)} \tag{3.2}$$

where for a number $x \geqslant 0, x^{(m)} := x(x-1)\dots(x-m)$. Notice that the Newton formula

$$\mathbf{1}\{k = 0\} = (1-1)^k = \sum_{m=0}^{\infty} (-1)^m \frac{k^{(m)}}{m!}$$

gives the inclusion-exclusion formula whenever the sum converges absolutely

$$\mathbf{P}(\mathsf{P}(B) = 0) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \mu_{\mathsf{P}}^m(B^m). \tag{3.3}$$

This can be useful as, by standard results on *random closed sets,* the law of a simple point process P can equivalently be characterised by the values $\mathbf{P}(\mathsf{P}(B) = 0), B \subset \mathbb{R}^d$, called *capacity functional* [60].

3.1.1 Law characterisation and convergence

It is classical that the law of a real random variable X is characterised by its moments $\mathbf{E}X^m, m \geqslant 1$ if $\mathbf{E}\exp(t|X|) < \infty$ for some $t > 0$. Similarly, if some point process P satisfies $\mathbf{E}\exp(t\mathsf{P}(A)) < \infty$ for all A bounded for some $t > 0$ (depending on A), in which case we say that P *has some exponential moments*, then the law of the $\mathsf{P}(A), A$ bounded, and hence the law of P, is characterised by the moments $\mathbf{E}\mathsf{P}(A)^m, m \geqslant 1, A$ bounded. In turn, the $\mathbf{E}\mathsf{P}(A)^m, m \geqslant 1$ can be recovered from the $\mu_{\mathsf{P}}^m, m \geqslant 1$ with (3.2). We hence proved the following:

Proposition 3.1. The law of a point process P having some exponential moments is characterised by the $\mu_{\mathsf{P}}^m, m \geqslant 1$.

If for instance the factorial moment measures are known to satisfy $\mu_{\mathsf{P}}^m(B^m) \leqslant c_B^m$ for some $c_B < \infty$, it implies finite exponential moments on B:

$$\mathbf{E}\exp(t\mathsf{P}(B)) \leqslant c \sum_m \frac{|tc_B|^m}{m!} < \infty, \tag{3.4}$$

hence under such an assumption for all B, the μ_{P}^m uniquely define a distribution (if they define a distribution at all). This will in particular allow to define properly the class of DPPs in Section 3.3.2 through their factorial moment measures.

Similarly, the convergence between random variables $X_n \to X$ for X with some exponential moment is implied by the convergence of the m-th moment $\mathbf{E}X_n^m \to \mathbf{E}X^m$ for each $m \geqslant 1$. Recall that the convergence between point processes $\mathsf{P}_n \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}$ is implied by $\mathsf{P}_n(A) \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}(A)$ for each bounded A. Hence if for all A bounded, for all $m \geqslant 1, \mathbf{E}\mathsf{P}_n(A)^m \to \mathbf{E}\mathsf{P}(A)^m$, we have indeed the weak convergence $\mathsf{P}_n \to \mathsf{P}$. Finally, since $\mathbf{E}\mathsf{P}(A)^m$ is a linear combination of the $\mu_{\mathsf{P}}^k(A), 1 \leqslant k \leqslant m$, we have:

Proposition 3.2. If for some random measures P_n, P with P having some exponential moments, we have $\mu_{\mathsf{P}_n}^m(A^m) \to \mu_{\mathsf{P}}^m(A^m)$, for each compact A, then $\mathsf{P}_n \xrightarrow[n\to\infty]{\text{Law}} \mathsf{P}$.

3.1.2 Repulsivity and negative dependence

Hyperuniform processes are sometimes believed to exhibit local repulsion, probably due to the fact that the most famous examples, DPPs and zeros of random functions, indeed experience a natural local

repulsion. Mecke's formulas exactly mean that for P a homogeneous Poisson process with intensity $\lambda = 1$, $\rho_{\mathsf{P}}^m = (\mathscr{L}^d)^m$, hence what is considered *repulsion* for a point process P is when the difference $\rho_{\mathsf{P}}^m - (\mathscr{L}^d)^m$, measuring somehow the deviation to neutrality, takes negative values, which indicates negative dependancy. We say in particular that P is completely repulsive if for all $m \geqslant 1$, $\rho_{\mathsf{P}}^m \leqslant \mathscr{L}^d$ as measures. One often talks about local repulsion when the inequality holds locally for $m = 2$. It holds much more rarely at large scales, and for all m; the class of DPPs is probably the sole tractable class of useful models having such a property, the GAF zeros do not [36]. Repulsivity can be seen as negative dependence, in the sense that a positive mass at some location discourages mass in other locations.

The reduced variance at large scale still induces some negative dependance, there should necessarily be compensation of large batch of particles. For instance in a large rectangular window W that can be decomposed in two disjoint congruent rectangles W_1, W_2, if there is a large concentration of particles in some half, the concentration in the other half should be below average to ensure the low discrepancy guaranteed by hyperuniformity. This phenomenon is quantified by negative asymptotic correlations

$$\lim_{R\to\infty} \mathrm{Corr}(\mathsf{P}(RW_1), \mathsf{P}(RW_2)) < 0.$$

This phenomenon has been first formally studied in [2] in the discrete setting, and recently in the continuum setting in [47, 39, 77].

3.2 Zeros of the planar Gaussian analytic function

We give here some more context about Gaussian fields and Gaussian Analytic Functions (GAFs), based on the excellent reference [36]. This section is a summary of some results of their Section 2 under the angle of Euclidean Gaussian fields and point processes.

In general, a Gaussian field $\mathsf{F} : \mathbb{R}^d \to \mathbb{R}^q$ is a collection of random vectors $\mathsf{F}(x) = (\mathsf{F}(x), \dots, \mathsf{F}_q(x)) \in \mathbb{R}^q, x \in \mathbb{R}^d$ such that for $(x_1, \dots, x_m) \in (\mathbb{R}^d)^m$, $(\mathsf{F}(x_1), \dots, \mathsf{F}(x_m))$ is a $(\mathbb{R}^q)^m$-valued Gaussian vector. In particular each coordinate field $\{\mathsf{F}_i(x), x \in \mathbb{R}^d\}, 1 \leqslant i \leqslant q$ defines a Gaussian signed measure as in Example 2.1. We saw that each finite measure S on $\mathbb{R}^d$ uniquely defines the law of a stationary Gaussian field $\mathsf{F} : \mathbb{R}^d \to \mathbb{R}$ with covariance function $\mathsf{C} = \mathscr{F}^{-1}\mathsf{S}$. For F a vector-valued field, each of the $q(q+1)/2$ pairwise covariance functions $\mathsf{C}_{i,j}(x, y) = \mathrm{Cov}\left(\mathsf{F}_i(x), \mathsf{F}_j(y)\right), i \leqslant j$, must be specified to determine the law of F uniquely in the class of Gaussian fields.

The class of GAFs is a subclass of the class of Gaussian fields from $\mathbb{C}$ to $\mathbb{C}$ (with the identification $\mathbb{C} \equiv \mathbb{R}^2$) such that a.s. each sample path is a.s. an analytic function. Hence one should in principle specify the 3 covariance functions $\mathsf{C}_{1,1}, \mathsf{C}_{2,2}, \mathsf{C}_{1,2}$ to characterise the law of the field. The class of GAFs crucially holds an additional requirement: each finite linear combination $\sum_j a_j \mathsf{F}(z_j)$ must be a complex Gaussian variable, which means it should have i.i.d. (Gaussian) real and imaginary parts (this in fact means that the FIDI are so-called *complex Gaussian vectors*). Hence GAFs are not just Gaussian fields which are a.s. analytic. A very convenient and fundamental gain from this requirement is that the law of a centred GAF F is uniquely determined by its *complex covariance*

$$\mathsf{C}(z, w) = \mathbf{E}\mathsf{F}(z)\overline{\mathsf{F}(w)},$$

replacing the $\mathsf{C}_{i,j}$ of the real Euclidean representation. By [36, Lemma 2.2.3], a general recipe to construct GAFs is to use models of the form

$$\mathsf{F}(z) = \sum_{k=1}^{\infty} Z_k \psi_k(z)$$

for i.i.d. $\mathscr{N}_{\mathbb{C}}(0, 1)$-distributed Z_k and analytic functions ψ_k such that a.e.

$$\sum_k |\psi_k(z)|^2 < \infty.$$

Such a representation can generally be obtained using the theory of compact operators. We are interested here in the planar GAF, defined by

$$\mathsf{F}^{\mathrm{Pl}}(z) = \sum_{k=1}^{\infty} Z_k \frac{z^k}{\overline{k!}},$$

and the analycity of F^{Pl} yields that its zero set $\mathsf{P}^{\mathsf{GAF}}$ contains a.s. isolated points, it is indeed a point process. An easy computation gives the complex covariance

$$\mathsf{C}(z,w) = \mathbf{E}\mathsf{F}^{\mathrm{Pl}}(z)\overline{\mathsf{F}^{\mathrm{Pl}}(w)} = e^{z\bar{w}}. \tag{3.5}$$

Proof that the $\mathsf{P}^{\mathsf{GAF}}$ is stationary and isotropic. The statistical invariances of $\mathsf{P}^{\mathsf{GAF}}$ come from the following conjugation property under shifts and rotations:

Lemma 3.1. For $\theta \in \mathbb{R}, z, w, v \in \mathbb{C}$

$$\begin{aligned}\mathsf{C}(e^{i\theta}z, e^{i\theta}w) =& \mathsf{C}(z,w),\\ \mathsf{C}(z+v, w+v) =& e^{i\varphi(z,v)}\mathsf{C}(z,w)e^{-i\varphi(w,v)}\end{aligned}$$

where $\varphi(z,v) = -iz\bar{v}/2 + i\bar{z}v/2 \in \mathbb{R}$ because $\bar{\varphi} = \varphi$.

The proof of this lemma is a straightforward computation. The invariance under rotation of $\mathsf{P}^{\mathsf{GAF}}$ comes from the fact that the rotated field $z \mapsto \mathsf{F}^{\mathrm{Pl}}(e^{i\theta}z)$ has the same complex covariance $\mathsf{C}(z,w)$ as F^{Pl}, hence they have the same law, and their zero sets have the same law as well.

As for invariance under complex translations, $\tau_v\mathsf{F}^{\mathrm{Pl}}$ has the same covariance as the GAF $\mathsf{F}_1^v : z \mapsto e^{\varphi(z,v)}\mathsf{F}^{\mathrm{Pl}}(z)$, hence these two GAFs have the same law. Since F and F_1^v have the same zero set, it indeed yields that $\mathsf{P}^{\mathsf{GAF}}$ and $\tau_v\mathsf{P}^{\mathsf{GAF}}$ have the same law, and $\mathsf{P}^{\mathsf{GAF}}$ is indeed stationary. □

Remark 3.1 (Non-Euclidean GAFs)**.** The theory of GAFs is very general and takes its full power on domains endowed with a non-euclidean metric, such as the sphere or the hyperbolic disk. It yields the *spherical* and *hyperbolic* families $\mathsf{F}_L^{\mathrm{Sph}}, \mathsf{F}_L^{\mathrm{Hyp}}$ of GAFs, each invariant under natural isometries, parametrized by a density parameter $L > 0$, which is an analogue of the scaling $z \mapsto \mathsf{F}^{\mathrm{Pl}}(Lz)$ on the complex plane. A further link with DPPs was uncovered by Peres and Viràg [66]: curiously, the zero set of $\mathsf{F}_1^{\mathrm{Hyp}}$ is a DPP on the unit disk, and it is the unique DPP among the zeros of all aforementionned GAFs.

Remark 3.2. $\mathsf{P}^{\mathsf{GAF}}$ and $\mathsf{P}^{\mathrm{Gin}}$ troubly share quite many features, especially locally. Krishnapur and Viràg [46] give an interesting explanation: $\mathsf{P}^{\mathrm{Gin}}$ can be written as the zero set of a GAF with a random complex covariance function (this is stronger than being the zero set of a random analytic function, which is true for any point process by Weierstrass's theorem).

3.2.1 Hyperuniformity of $\mathsf{P}^{\mathsf{GAF}}$

Let us give a proof of hyperuniformity that shows how the harmonic nature of analytic functions is at the core of the hyperuniformity behaviour of $\mathsf{P}^{\mathsf{GAF}}$, borrowed from [75], and originating in the pioneed work of Forrester and Horner [26].

Lemma 3.2. There is a constant C such that for f of class $\mathcal{C}^2$ with compact support,

$$\mathrm{Var}\left(\mathsf{P}^{\mathsf{GAF}}(f)\right) \leqslant C\|\Delta f\|^2_{L^2(\mathbb{C})}.$$

This entails that as $R \to \infty$,

$$\mathrm{Var}\left(\mathsf{P}^{\mathsf{GAF}}(f_R)\right) = O(R^{-2}).$$

The hyperuniformity therefore follows directly from Theorem 2.1-(ii). This also shows by Proposition 2.1 that $\mathsf{P}^{\mathsf{GAF}}$ has an hyperuniformity index $\alpha \geqslant 4$ (one can prove that the exponent is exactly 4

with the converse statement). This is exceptionnal in the sense that, apart from shifted lattices, other known hyperuniform processes have index at most $\alpha = 2$, even perturbed lattices, due in particular to Lemma 2.3.

Proof. Morally, the proof follows from the a.s. analycity of GAF and the stationarity of $\mathsf{P}^{\mathsf{GAF}}$. It relies on two claims: a (non-random) analytic function F on $D \subset \mathbb{C}$ with zero set P satisfies

$$P(f) = \frac{1}{2\pi} \int_D \Delta f(z) \ln |F(z)| dz \tag{3.6}$$

and the zero set P of a GAF F satisfies for $f \in \mathsf{C}_c^b$

$$\mathrm{Var}\,(\mathsf{P}(f)) = \frac{1}{4\pi^2} \int \Delta f(z) \Delta f(w) \mathrm{Cov}\left(\ln |\tilde{\mathsf{F}}(z)|, \ln |\tilde{\mathsf{F}}(w)|\right) dzdw. \tag{3.7}$$

For the first claim, the starting point is the harmonicity of the log on the complex plane: $\Delta \ln(|\cdot|) = \frac{1}{2\pi}\delta_0$ in the distributional sense, i.e. for $f \in \mathsf{C}_c^2(\mathbb{C})$,

$$f(0) = \frac{1}{2\pi} \int \Delta f(z) \ln(|z|) dz. \tag{3.8}$$

Hereafter, fix f and denote by Λ its support. A non null holomorphic function F has finitely many zeros z_i in Λ, and the logarithm has an analytic determination on Λ, hence F can be written

$$F(z) = e^{g(z)} \prod_i |z - z_i| dz, z \in \Lambda,$$

for some analytic function g. Therefore with $P = \sum_i \delta_{z_i} \in \mathscr{N}(\mathbb{C})$, the smooth linear statistic can be expressed

$$P(f) = \sum_{i=1}^n f(z_i) = \frac{1}{2\pi} \sum_{i=1}^n \int \Delta f(z) \ln(|z_i - z|) dz = \frac{1}{2\pi} \int \Delta f(z) \ln |F(z) e^{-g(z)}| dz = \frac{1}{2\pi} \int \Delta f(z) \ln |F(z)| dz$$

which yields (3.6), exploiting the fact that the real part of an analytic function g is harmonic:

$$\int \Delta f(z) \mathscr{R} g(z) dz = 0.$$

Let us apply this to a GAF F. Denoting by $\kappa(z)^2 = \mathbf{E}|\mathsf{F}(z)|^2 =, \tilde{\mathsf{F}}(z) = \mathsf{F}(z)/\kappa(z)$ is a complex Gaussian variable with constant variance, hence $\ell := \mathbf{E} \ln |\tilde{\mathsf{F}}(z)|$ is constant as well, and

$$\begin{aligned}
2\pi \mathbf{E}\mathsf{P}(f) &= \int_\Lambda \Delta f(z) \mathbf{E} \ln |\mathsf{F}(z)| dz = \int_\Lambda \Delta f(z) \mathbf{E} \ln |\tilde{\mathsf{F}}(z)| dz + \int \Delta f(z) \ln(\kappa(z)) dz = 0 + \int_\Lambda \Delta f(z) \ln(\kappa(z)) dz \\
4\pi^2 \mathbf{E}|\mathsf{P}(f)|^2 &= \int_{\Lambda^2} \Delta f(z) \Delta f(w) \mathbf{E}[\ln |\mathsf{F}(z)| \ln |\mathsf{F}(w)|] dzdw \\
&= \int_{\Lambda^2} \Delta f(z) \Delta f(w) \mathbf{E}[\ln |\tilde{\mathsf{F}}(z)| \ln |\tilde{\mathsf{F}}(w)|] dzdw + \int \Delta f(z) \Delta f(w) \ln(\kappa(z)) \ln(\kappa(w)) dzdw + 0 + 0 \\
&= \int \Delta f(z) \Delta f(w) \mathrm{Cov}\left(\ln |\tilde{\mathsf{F}}(z)|, \ln |\tilde{\mathsf{F}}(w)|\right) dzdw + 0 + 0 + \int \Delta f(z) \Delta f(w) \ln(\kappa(z)) \ln(\kappa(w)) dzdw
\end{aligned}$$

which yields (3.7). Let us apply this to F^{Pl}. The final idea is that $|\tilde{\mathsf{F}}^{\mathrm{Pl}}(z)|, \ln |\tilde{\mathsf{F}}^{\mathrm{Pl}}(w)|$ have a small correlation when z, w are far away. We have the general inequality ([36, Lemma 3.5.2]) for $\mathscr{N}_{\mathbb{C}}(0,1)$ variables Z, Z', $\mathrm{Cov}\,(\ln |Z|, \ln |Z'|) \leqslant \frac{1}{2} |\mathbf{E} Z \overline{Z'}|^2$, hence

$$\mathrm{Cov}\left(\ln |\tilde{\mathsf{F}}^{\mathrm{Pl}}(z)|, \ln |\tilde{\mathsf{F}}^{\mathrm{Pl}}(w)|\right) \leqslant \frac{1}{2} |\mathbf{E} \tilde{\mathsf{F}}^{\mathrm{Pl}}(z) \overline{\tilde{\mathsf{F}}^{\mathrm{Pl}}(w)}|^2.$$

By (3.5), the right hand side is $\frac{1}{2}e^{-|z-w|^2} =: \sigma(z-w)$. Hence by Cauchy-Schwarz inequality

$$4\pi^2 \mathrm{Var}\left(\mathsf{P}^{\mathsf{GAF}}(f)\right) = \int \Delta f(z)\Delta f(w)\sigma(z-w)dzdw \leqslant \|\Delta f\|_{L^2(\mathbb{C})}\|\Delta f \star \sigma\|_{L^2(\mathbb{C})} \leqslant \|\Delta f\|^2_{L^2(\mathbb{C})}\|\sigma\|^2_{L^1(\mathbb{C})}$$

which concludes the proof.

□

3.2.2 Other hyperuniform Gaussian nodal measures

We conclude the study of planar GAF zeros by showing that they form the unique such stationary set up to a rescaling, by a rigidity principle discovered by Sodin:

Theorem 3.1 ([74])**.** Let F, G two GAFs on some domain $D \subset \mathbb{C}$, such that their respective zero sets $\mathsf{P}^{\mathsf{F}}, \mathsf{P}^{\mathsf{G}}$ have the same intensity: $\rho^1_{\mathsf{P}^{\mathsf{F}}} = \rho^1_{\mathsf{P}^{\mathsf{G}}}$. Then there exists a deterministic function $\varphi : D \to \mathbb{C}$ not vanishing on D with $\mathsf{F} \equiv \varphi\mathsf{G}$ a.s..

Hence for $\lambda > 0$, there is a unique set of GAF zeros P with $\rho^1_{\mathsf{P}} = \lambda\mathscr{L}^d$, obtained by properly rescaling $\mathsf{P}^{\mathsf{GAF}}$. To the author's knowledge, no other Gaussian field with stationary hyperuniform zeros has been found, such a phenomenon is known to be impossible for one-dimensionnal stationary fields [49]. The complex version of Berry's Gaussian random wave model, experiencing many cancellation phenomena, has hyperfluctuating zeros [87]. The general cancellation phenomena for Gaussian random measures and their chaotic projections has been analysed in [28].

In [32], inspired by the Euclidean GAF and motivated by signal theory, the authors exhibit charged particles located on the zero set of a *Gaussian Weyl-Heisenberg field* F, i.e. a random measure of the form

$$\mathsf{M} = \sum_{z:\mathsf{F}(z)=0} \delta_z \kappa_z,$$

where κ_z is the sign of the Jacobian determinant $\det D\mathsf{F}(z)$. Such measures (with or without charges) are stationary, and the authors give their intensity and prove that under some assumptions, the global charge experiences *screening*, which implies a hyperuniform behaviour of class I (Section 2.4), see their Theorem 1.14.

3.3 Random matrices and determinantal point processes

We provide here partial proofs of Theorems 1.1 and 1.2. It is the occasion to make more explicit the fundamental link between random matrix models and particle systems such as log gases and OCPs, through astute changes of variables. Determinantal processes in the continuous space arise mainly as eigenvalues of random matrix models. We are interested here in stationary examples (GUE, Ginibre) where the key point is to prove that they are projector DPPs.

3.3.1 Change of variables

It is claimed in the introduction that the β-ensembles for $\beta = 1, 2$ in dimension $d = 1$ and the Ginibre ensemble can be written out as the set of eigenvalues of resp. the GOE, GUE, and Ginibre random matrix models. As an elementary and prototypical example, we provide here an argument for the GOE ensemble, similar reasonings are possible for other matrix models, but more involved. We refer to the monographs [5, 59, 36] for other models.

Partial proof that the eigenvalues of $\mathsf{M}^{1,(n)}$ *have density* (1.2) *for* $\beta = 1$. Since $\mathsf{M}^{1,(n)}$ is a symmetric matrix, there exist a random matrix Q_n in the orthogonal space $\mathscr{O}_n$ and a random matrix $\mathsf{D}_n =$

$\mathrm{diag}(\Lambda_1,\dots,\Lambda_n)$ in the space $\mathscr{D}_n$ of diagonal matrices such that $\mathsf{M}^{1,(n)} = \mathsf{Q}_n\mathsf{D}_n\mathsf{Q}_n^T$. D_n is assimilated to $\mathbb{R}^n$ with the Euclidean metric and endowed with $\mathscr{L}^n$. We in fact give a full proof except for the fact that $\mathsf{M}^{1,(n)}$ is a.s. in $\mathcal{S}_n^*$ the set of symmetric matrices with distinct eigenvalues, hence D_n is a.s. in $\mathcal{D}_n^*$ the class of diagonal matrices with distinct eigenvalues, but it is a plausible fact given that the entries of $\mathsf{M}^{1,(n)}$ are independent Gaussian, see for instance [59, 5].

Our interest is the exact density of D_n. Let us first prove that the law of Q_n can be chosen to be σ, the unique probability measure on $\mathscr{O}_n$ that is invariant under the action of the multiplication by any $Q \in \mathscr{O}_n$, or *Haar measure*. Let Q with law σ and independent of $(\mathsf{M}^{1,(n)}, \mathsf{Q}_n, \mathsf{D}_n)$. The expression (1.3) yields that the law of $\mathsf{M}^{1,(n)}$ is invariant under the conjugation action $M \mapsto QMQ^T$ for $Q \in \mathscr{O}_n$, hence $\mathsf{Q}\mathsf{Q}_n$ is another possible random orthogonal matrix in the decomposition of $\mathsf{M}^{1,(n)}$. Since $\mathsf{Q}\mathsf{Q}_n$ also has law σ, it proves the claim, and we indeed choose Q_n with law σ independent of D_n. Hence the law of $(\mathsf{Q}_n, \mathsf{D}_n)$ is $\sigma \times \mu_1$ for some measure μ_1 on $\mathcal{D}_n^*$ that we seek to explicit.

The idea of the proof is to compute the Jacobian of the mapping $(Q, D) \mapsto M = QDQ^T$ for $(Q, D) \in \mathscr{O}_n \times \mathscr{D}_n^*$. It is tricky to directly perform a change of variables on the non-Euclidean manifold $\mathscr{O}_n$, we therefore locally linearise it first with the space of antisymmetric matrices $\mathscr{A}_n$, assimilated to $\mathbb{R}^{n(n-1)/2}$ and endowed with $\mathscr{L}^{n(n-1)/2}$, thanks to the exponential mapping. Recall that for Q in a neighbourhood U' of I_n, $Q = \exp(A)$ for some $A \in \mathscr{A}_n$ (close to 0). Call ν the law on U' induced on $\mathscr{A}_n$ by σ, i.e. for a test function ψ supported by $\exp(U')$

$$\mathbf{E}\psi(\mathsf{Q}) = \int \psi(Q)\sigma(dQ) = \int \psi(\exp(A))\nu(dA). \tag{3.9}$$

We introduce the mapping $\Gamma : \mathscr{A}_n \times \mathscr{D}_n^* \to \mathscr{S}_n^*$

$$\Gamma(A, D) := \exp(A) D \exp(A)^T.$$

We saw that Γ is surjective onto ..., it is clearly $\mathcal{C}^\infty$. Let us compute the absolute Jacobian determinant $J_\Gamma(\cdot)$ in $(0, D_0)$.

Lemma 3.3. For $D_0 = \mathrm{diag}(\lambda_1,\dots,\lambda_n) \in \mathscr{D}_n^*$, the Jacobian matrix of Γ has absolute determinant

$$J_\Gamma(0, D_0) = c_n \prod_{i<j} |\lambda_i - \lambda_j| \text{ for some } c_n > 0.$$

In particular, it does not vanish. By the inverse function theorem, Γ is a $\mathcal{C}^\infty$-diffomorphism on a neighbourhood V of $(0, D_0)$ contained in $U \times U_{D_0}$. We can therefore perform a change of variables, for φ supported by $\Gamma(V)$,

$$\int_{\mathscr{S}_n^*} \varphi(M)dM = \int_V \varphi(\Gamma(A, D)) J_\Gamma(A, D) d\mathscr{L}^{n(n-1)/2}(A) d\mathscr{L}^n(D). \tag{3.10}$$

On the other hand, by (1.3) and (3.9),

$$\begin{aligned}
\mathbf{E}\varphi(\mathsf{M}^{1,(n)}) &\propto \int_{\mathscr{S}_n^*} \varphi(M) e^{-\mathrm{Tr}(MM^T)/4} dM \\
&= \int e^{-\mathrm{Tr}(DD^T)/4} \int \varphi(QDQ^T)\sigma(dQ)d\mu_1(D) \\
&= \int \varphi(\Gamma(A, D)) e^{-\mathrm{Tr}(DD^T)/4} d\nu(A) d\mu_1(D).
\end{aligned}$$

We can therefore identify with (3.10)

$$e^{-\mathrm{Tr}(DD^T)/4} J_\Gamma(A, D) d\mathscr{L}^{n(n-1)/2}(A) d\mathscr{L}^n(D) \propto d\nu(A) e^{-\mathrm{Tr}(DD^T)/4} d\mu_1(D).$$

It yields that J_Γ is separable on V and μ_1 has a density f around D_0. With Lemma 3.3, we have $f(D_0) \propto \prod_{i<j} |\lambda_i - \lambda_j|$, and this is valid for any $D_0 \in \mathscr{D}_n^*$. For a test function $\tilde{\varphi}$ on U_{D_0}, and $\varphi(A, D) := \tilde{\varphi}(D)$,

$$\begin{aligned}\mathbf{E}\tilde{\varphi}(\mathsf{D}_n) =& \mathbf{E}\varphi(\mathsf{M}^{1,(n)}) \\ \propto& \int \tilde{\varphi}(D) e^{-\mathrm{Tr}(DD^T)} d\nu(A) d\mu_1(D) \\ \propto& \int \tilde{\varphi}(\lambda_1, \dots, \lambda_n) e^{-\sum_i \lambda_i^2/4} \prod_{i<j} |\lambda_i - \lambda_j| d\lambda_1 \dots d\lambda_n,\end{aligned}$$

which concludes the proof.

Let us finally prove Lemma 3.3. We see Γ as a function from $\mathbb{R}^{n(n-1)/2} \times \mathbb{R}^n$ to $\mathbb{R}^{n(n+1)/2}$, and denote its components $\Gamma_{i,j}, i \leqslant j$. Let $\delta = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathscr{D}_n$, $H = (H_{i,j})_{1 \leqslant i,j \leqslant n} \in U$.

$$\begin{aligned}\Gamma(H, D_0 + \delta) =& (I + H + o(H))(D_0 + \delta)(I - H + o(H)) \\ =& \Gamma(0, D_0) + \delta + HD_0 - D_0 H + o((H, \delta)).\end{aligned}$$

Note that $HD_0 - D_0 H = (H_{i,j}(\lambda_j - \lambda_i))_{1 \leqslant i,j \leqslant n}$ vanishes on the diagonal, and at the opposite δ is supported by the diagonal. For $i < j$, we can read the partial derivatives on the lower diagonal

$$\begin{aligned}\frac{\partial \Gamma_{i,j}}{\partial H_{k,l}}(0, D_0) =& \lambda_i - \lambda_j \text{ iff } (i,j) = (k,l), \text{ for } k < l \\ \frac{\partial \Gamma_{i,j}}{\partial \lambda_k}(0, D_0) =& 0, 1 \leqslant k \leqslant n,\end{aligned}$$

and for $1 \leqslant i \leqslant n$

$$\begin{aligned}\frac{\partial \Gamma_{i,i}}{\partial \lambda_k}(0, D_0) =& \delta_{k,i} \\ \frac{\partial \Gamma_{i,i}}{\partial H_{k,l}}(0, D_0) =& 0, k < l.\end{aligned}$$

Seeing the Jacobian matrix $\nabla\Gamma$ as blocks of dimension $n(n-1)/2$ or n, the $n \times n$ block gives determinant 1, the $n(n-1)/2$ block is diagonal and gives $\prod_{i<j}(\lambda_i - \lambda_j)$, and the other blocks vanish; this gives the desired expression.

□

3.3.2 Determinantal processes

In order to prove results for the GUE and Ginibre process, we must derive some basic definitions and concepts related to the theory of determinantal processes. We give a simplified treatment in the context of hyperuniformity on a continuous space.

Definition 3.1. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ measurable. A point process P on $\mathbb{R}^d$ is a DPP with kernel K if it admits factorial moments densities of the form, for $m \geqslant 1$,

$$\rho_{\mathsf{P}}^m(x_1, \dots, x_m) = \det((K(x_i, x_j))_{1 \leqslant i,j \leqslant m}), x_1, \dots, x_m \in \mathbb{R}^d. \tag{3.11}$$

The first example is the homogeneous Poisson process with intensity $\lambda > 0$, for which $K(x, y) = \lambda \mathbf{1}\{x = y\}$. More generally, $\rho_{\mathsf{P}}^1(x) = K(x, x) \in \mathbb{R}_+$ on $\mathbb{R}^d$. For existence and unicity questions, it is enough to perform a local analysis, in the sense that P is a DPP with kernel K if and only if for each bounded B, $\mathsf{P}1_B$ is a DPP with kernel $K1_{B \times B}$, and unicity and existence of P need only be solved on

such B. Implicitly, P should have local moments of every order, and we shall henceforth assume that K is locally square integrable as it will be fruitful to consider the associated operator in $L^2(B)$

$$\mathsf{L}_K f(x) := \int f(y)K(x,y)dy.$$

Remark that by (3.11), each minor of K has a positive determinant, hence each submatrix $(K(x_i,x_j))$ is necessarily semi-definite positive. Therefore, Hadamard's inequality yields for compact measurable $B \subset \mathbb{R}^d$

$$|\rho^m_{\mathsf{P}}(B^n)| \leqslant \int_{B^n} \prod_i K(x_i,x_i)dx_1 \dots dx_m = \left(\int_B K(x,x)dx\right)^m \leqslant \mathscr{L}^d(B)^{m/2}\left(\int_B K(x,x)^2\right)^{m/2} < \infty.$$

By (3.4) and Proposition 3.1, such a DPP has exponential moments and is uniquely defined by equations (3.11). This is not an obligation, but in general, to have tractable existence and unicity results, one requires K to be Hermitian, i.e. for $x,y \in \mathbb{R}^d$, $K(y,x) = \bar{K}(x,y)$.

A particularly important class is that of *canonical kernels*, defined to be of the form

$$K(x,y) = \sum_{k\geqslant 1} a_k \varphi_k(x)\bar{\varphi}_k(y) \tag{3.12}$$

where the $\varphi_k, k \geqslant 1$ form an orthonormal family. The semi-definite positiveness of K implies that $a_k \in \mathbb{R}_+$, and it is also necessary that $a_k \leqslant 1$, see [36, Th. 4.5.5].

Those are typically the form of kernels for finite point processes P_n coming from finite matrix models. When all (non-zero) a_k's equal 1, the kernel is easily seen to enjoy the *reproducing* property:

Definition 3.2. Say that K is reproducing if for $x,y \in \mathbb{R}^d$,

$$K(x,y) = \int_{\mathbb{R}^d} K(x,z)K(z,y)dz.$$

This property is more conceptually seen as a projection property in the L^2 space. If (3.12) is satisfied, for $f \in L^2(\mathbb{R}^d)$ with compact support, $\mathsf{L}_K f$ is the projection of f onto the space spanned by the φ_k. For a general reproducing kernel

$$\mathsf{L}_K(\mathsf{L}_K f)(x) = \int K(x,z)(\int f(y)K(z,y)dy)dz = \int f(y)\int K(x,z)K(z,y)dz = \int f(y)K(x,y)dy = \mathsf{L}_K f(x).$$

As a counterexample, the Poisson kernel $K(x,y) = \mathbf{1}\{x=y\}$ neither satisfies (3.12) nor is reproducing.

This reproducing property is useful here for at least two reasons. Remember first that the $\rho^k_{\mathsf{P}}, k \leqslant m$ can be seen as projections of ρ^m_{P}, but the fact that the defining DPP property (3.11) holds for ρ^m_{P} does not implies automatically that it holds for $\rho^k_{\mathsf{P}}, k \leqslant m$, except for reproducing kernels:

Proposition 3.3 ([24])**.** Let K a reproducing kernel and P a point process such that (3.11) holds for some $m \geqslant 1$. Then (3.11) holds for all $1 \leqslant k \leqslant m$.

Hence the reproducing property saves us a lot of effort in the proof that a point process is determinantal, as we will see with the GUE and Ginibre ensembles.

Proof. It suffices to show it for $k=1$ by induction. We decompose the set of permutations Σ_n in Σ^1_n for which $\sigma(n) = n$ and Σ^*_n the complement. For $\sigma \in \Sigma^1_n$, let $\tilde{\sigma}$ the restriction of σ to $[\![1, n-1]\!]$. For $\sigma \in \Sigma^*_n$, call $c_n(\sigma)$ the cycle containing n, and $\tilde{\sigma} \in \Sigma_{n-1}$ bypassing n (i.e. $\tilde{\sigma}(\sigma^{-1}(n)) = \sigma(n)$, others

values do not change). Then

$$\begin{aligned}
&\int \det((K(z_i,z_j))_{i,j\leqslant n})d\mu(z_n)\\
&= \sum_{\sigma\in\Sigma_n^1}\varepsilon(\sigma)\prod_{i<n}K(i,\sigma(i))\int K(z_n,z_n)\mu(dz_n) + \sum_{\sigma\in\Sigma_n^*}\varepsilon(\sigma)\prod_{i\notin c_n(\sigma)}K(z_i,z_{\sigma(i)})\int\prod_{i\in c_n(\sigma)}K(z_i,z_{\sigma(i)})\mu(dz_n)\\
&= c\sum_{\sigma\in\Sigma_n^1}\varepsilon(\tilde\sigma)\prod_{i\leqslant n-1}K(z_i,z_{\tilde\sigma(i)})\\
&\quad + \sum_{\sigma\in\Sigma_n^*}\varepsilon(\sigma)\prod_{i\notin c_n(\sigma)}K(z_i,z_{\sigma(i)})\prod_{i\in c_n(\sigma)\setminus\{n,\sigma^{-1}(n)\}}K(z_i,z_{\sigma(i)})\times\int K(z_{\sigma^{-1}(n)},z_n)K(z_n,z_{\sigma(n)})\mu(dz_n)\\
&= c\sum_{\tilde\sigma\in\Sigma_{n-1}}\varepsilon(\tilde\sigma)\prod_{i\leqslant n-1}K(z_i,z_{\tilde\sigma(i)}) + \sum_{\sigma\in\Sigma_n^*}\underbrace{\varepsilon(\sigma)}_{-\varepsilon(\tilde\sigma)}\prod_{i\notin c_n(\sigma)}K(z_i,z_{\tilde\sigma(i)})\underbrace{\prod_{i\in c_n(\sigma)\setminus\{n,\sigma^{-1}(n)\}}K(z_i,z_{\sigma(i)})\times K(z_{\sigma^{-1}(n)},z_{\sigma(n)})}_{=\prod_{i\in c_n(\sigma)\setminus\{n\}}K(z_i,z_{\tilde\sigma(i)})}\\
&= c\det((K(z_i,z_j))_{i,j\leqslant n-1}) - \sum_{\sigma'\in\Sigma_{n-1}}\#\{\sigma:\tilde\sigma=\sigma'\}K_z^{\sigma'}.
\end{aligned}$$

To conclude, notice that for each $\tilde\sigma$, there are $N-1$ ways to choose where to insert index n in permutation $\tilde\sigma$, which yields the result with the constant $C_{n,n-1} = (c-(N-1))$. Then one can iterate. □

Let us now focus on stationary DPPs. Assume up to applying a rescaling that the intensity is 1, i.e. $\rho^1_{\mathsf{P}}(x) = K(x,x) = 1$. Since by (3.11), we have the density $\rho^2_{\mathsf{P}}(x,y) = (1-|K(x,y)|^2)$, stationarity yields that it only depends on $x-y$, we therefore define $\kappa(x) = |K(0,x)|$. A direct computation combining (3.1) and (2.2) gives the spectral measure $\mathsf{S} = \mathsf{s}\mathscr{L}^d$ with the structure factor

$$\mathsf{s} = 1 - \mathscr{F}\kappa^2. \tag{3.13}$$

As a consequence, $\widehat{\kappa^2}(u) \leqslant 1$. Also, $\mathsf{s}(0) = 1-\widehat{\kappa^2}(0)$ and reproducing kernels yield hyperuniform DPPs:

Proposition 3.4. Let P a stationary DPP with locally square integrable reproducing kernel K. Then $\mathsf{s}(0) = 0$ and P is hyperuniform. Furthermore, $\mathsf{s}(u) \geqslant \sigma\|u\|^2$ for some $\sigma > 0$, hence the hyperuniformity exponent is at most 2.

Proof. s is uniformly continuous as κ^2 is integrable by assumption. The reproducing and Hermitian properties yield

$$\widehat{\kappa^2}(0) = \int\kappa^2 = \int K(0,x)K(x,0)dx = K(0,0) = 1$$

which yields hyperuniformity by Theorem 2.1. The lower bound is a consequence of Lemma 2.3. □

3.3.3 Ginibre as a DPP

Recall from Chapter 1 that Gin_n is the random element of $\mathscr{M}_n(\mathbb{C})$ with density proportionnal to $\exp(-Tr(HH^*)), H\in\mathscr{M}_n(\mathbb{C})$.

Lemma 3.4. The point process Gin_n is a DPP with kernel

$$K_n(z,w) = \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(z\bar w)^k}{k!}e^{-\frac{|z|^2+|w|^2}{2}}$$

Remark 3.3. We will see that K_n is of the canonical form (3.12) with n terms and all a_k equal to 1 because the $\varphi_k(z) := \frac{1}{\sqrt{k!}} z^{k-1} e^{-|z|^2/2}$ are orthonormal. This characterises processes with a.s. n points because for $m > n$, the matrix $(K_n(z_i, z_j))_{1 \leqslant i,j \leqslant m}$ has rank n, hence $\rho^m_{\mathrm{Gin}_n} \equiv 0$, confirming that Gin_n has never more than n points, and

$$\mathbf{E}\#\mathrm{Gin}_n = \int K(z,z)dz = n$$

confirms $\#\mathrm{Gin}_n = n$ a.s.. If for some other point process P one of the a_k was in $(0,1)$, we would have a random number of points because $\mathbf{E}\#\mathsf{P} < n$ but $\rho^n_{\mathsf{P}} \not\equiv 0$. In fact, for a canonical kernel, one can represent the number of points as a sum of independent Bernoulli variables with parameters a_k (see [36, Theorem 4.5.3]).

Remark 3.4. We only consider DPPs with some kernel K on $\mathbb{R}^d$ endowed with Lebesgue measure but it is easily seen that one can equivalently consider a DPP with kernel $\tilde{K}(x,y) = K(x,y)\varphi(x)\bar{\varphi}(y)$ for some $\varphi : \mathbb{R}^d \to \mathbb{C}$ on $\mathbb{R}^d$ endowed with $|\varphi(x)|^2 dx$, in the sense that for $x_1, \dots, x_m \in \mathbb{R}^d$

$$\det((K(x_i,x_j))_{1\leqslant i,j\leqslant m})|\varphi(x_1)|^2 \dots |\varphi(x_n)|^2 = \det((\tilde{K}(x_i,x_j))_{1\leqslant i,j\leqslant m}). \tag{3.14}$$

Proof. The starting point is the Van Der Monde determinant. For $\mathsf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$,

$$\prod_{i<j}(z_i - z_j) = \det((z_i^{k-1})_{1\leqslant i,k\leqslant n}).$$

Let us multiply each column by a scalar α_k to be defined later and take the square modulus: let $M(\mathsf{z}) = (\alpha_k z_i^{k-1})_{1\leqslant i,k\leqslant n}$, then

$$\prod_{i<j}|z_i - z_j|^2 \propto |\det(M(\mathsf{z}))|^2 = \det(M(\mathsf{z})M(\mathsf{z})^*) = \det(\tilde{K}_n(z_i,z_j))$$

with

$$\tilde{K}_n(z,w) := \sum_{k=1}^n |\alpha_k|^2 z^{k-1}\bar{w}^{k-1}.$$

We admit here the density representation (1.4), it can be proved in a similar (but more intricate) manner than for the GOE, see Section 3.3.1. Using Remark 3.4, it proves that $\rho^n_{\mathsf{P}^{\mathrm{Gin}}}$ has the DPP form (3.11) with the kernel $K_n(z,w) = \tilde{K}_n(z,w)e^{-\frac{|z|^2+|w|^2}{2}}$. We already saw that $\rho^m_{\mathsf{P}} \equiv 0$ for $m > n$, and for $m \leqslant n$ we wish to apply the reproducing property, which concludes the proof thanks to Proposition 3.3.

More precisely we prove that K_n has the canonical form (3.12) with $\varphi_k(z) = \alpha_k z^{k-1} e^{-|z|^2/2}$. Let us check that the φ_k form an orthonormal family for the choice $\alpha_k := (\pi(k-1)!)^{-1/2}$:

$$\int_{\mathbb{C}} \varphi_k(z)\varphi_j(z)dz = \alpha_k\bar{\alpha}_j \int z^{k-1}\bar{z}^{j-1}e^{-|z|^2}dz = \alpha_k\bar{\alpha}_j \int_0^\infty \int_0^{2\pi} \rho^{k+j-2}e^{i\theta(k-j)}e^{-\rho^2}\rho d\rho d\theta$$

in polar coordinates. We see in particular that it vanishes for $k = j$ due to the angular integral, and for $k = j$, it gives with the change of variables $u = \rho^2$

$$2\pi|\alpha_k|^2 \int_0^\infty \rho^{2k-1}d\rho e^{-\rho^2} = \pi|\alpha_k|^2\Gamma(k) = 1.$$

This concludes the proof of Lemma 3.4.

□

Turning back to the proof of Theorem 1.2, we have the convergence $K_n(z,w) \to K(z,w) := \pi^{-1} e^{z\bar{w}}$ uniformly on each compact. Hence by Proposition 3.2, $\mathsf{P}_n^{\mathrm{Gin}}$ converges to $\mathsf{P}^{\mathrm{Gin}}$, the DPP with kernel K, weaky in the weak topology. The invariance of $\mathsf{P}^{\mathrm{Gin}}$ under rotations is inherited from the invariance of $\mathsf{P}_n^{\mathrm{Gin}}$ under rotations, because the random matrix Gin_n is invariant under the action of the orthogonal group.

Finally, for the stationarity of P, we recall Lemma 3.1 (which also proves isotropy):

$$K(z+v, w+v) = e^{i\varphi(z,v)} K(z,w) e^{-i\varphi(w,v)}$$

for some real function φ. As already observed for the planar GAF, the kernel K is not invariant under the action of shifts, but it does not prevent $\mathsf{P}^{\mathrm{Gin}}$ to be stationary because for each $m \in \mathbb{N}$,

$$\rho^m_{\mathsf{P}^{\mathrm{Gin}}}(z_1+w, \dots, z_m+w) = \det(K(z_i+w, z_j+w)_{i,j\leqslant m}) = \sum_\sigma \varepsilon(\sigma) \prod_i K(z_i+w, z_{\sigma(i)}+w)$$

where the sum is over permutations of $\{1,\dots,m\}$, and with $i' = \sigma(i)$

$$\prod_i K(z_i+w, z_{i'}+w) = \prod_i e^{i\varphi(z_i,w)} K(z_i, z_{i'}) e^{-i\varphi(z_{i'},w)} = \quad \exp\left(i\sum_i \varphi(z_i,w) - i\sum_i \varphi(z_{i'},w) \right) \prod_i K(z_i, z_{i'})$$

and the first exponential equals 1. Therefore the kernels ρ^{P}_m are invariant under the action of shifts, which proves by Proposition 3.1 that the point processes $\tau_w \mathsf{P}^{\mathrm{Gin}}$ and $\mathsf{P}^{\mathrm{Gin}}$ have the same law, i.e. $\mathsf{P}^{\mathrm{Gin}}$ is stationary.

3.3.4 GUE as a DPP

We see from (1.2) that the GUE has the same density form as Ginibre, but on $\mathbb{R}$ instead of $\mathbb{C}$. Everything works the same in the previous proof, except that the φ_k are not orthogonal on $\mathbb{R}$. Going back to the Van Der Monde determinant, we have another unexploited freedom: we can substract from each column a linear combination of columns with a smaller index:

$$\det\left(\left(\frac{1}{\sqrt{k!}} \lambda_i^{k-1} \right)_{1\leqslant i,k\leqslant n} \right) = \det\left(\underbrace{\left(\frac{1}{\sqrt{k!}} \lambda_i^{k-1} + \sum_{j=1}^{k} a_{j,k} \lambda_i^{j-1} \right)}_{=:P_k(\lambda_i)}{}_{1\leqslant i,k\leqslant n} \right).$$

We apply a Gramm-Schmidt orthonormalisation procedure to ensure that the P_k are orthogonal on $\mathbb{R}$. In fact, they are a renormalised version of the Hermite polynomials. It then proves with the same computations that $\mathsf{P}_n^{\mathrm{GUE}}$ is a DPP with kernel

$$K_n(\lambda, \lambda') = \sum_{k=1}^{n} P_k(\lambda) P_k(\lambda').$$

We leave as a black box the following result: Uniformly on each compact of $\mathbb{R}$,

$$K_n(\lambda, \lambda') \to K(\lambda, \lambda') := c \frac{\sin(\lambda - \lambda')}{\lambda - \lambda'}.$$

One can consult for instance [5]. Hence using Proposition 3.2 and the exact same reasoning than for the Ginibre process, $\mathsf{P}_n^{\mathrm{GUE}}$ converges to an infinite stationary hyperuniform process called the Sine process (this is in fact easier than for Ginibre because here the kernel itself is invariant under shifts, not only the factorial moment measures).

3.3.5 Pfaffian processes

We mention for completeness the existence of Pfaffian processes, of which the GOE process is an instance, but we shall not develop them here, see for instance [17, Section 3.3] and references therein.

3.4 Systems of particles

The β-ensembles, GOE, GUE, and Ginibre models, descend from a more general class of models in statistical physics defined through a Hamiltonian. Consider a pairwise potential, i.e. a function $\varphi : (\mathbb{R}^d)^2 \to \mathbb{R}$, and the energy function

$$E^\circ(x_1, \dots, x_n) := \sum_{i \neq j} \varphi(x_i - x_j).$$

A *Gibbs measure* with energy H° is roughly speaking a system of particles that tend to arrange themselves randomly while keeping a low-energy configuration. Since the more the particle shift apart at infinity the lowest is, in general, the energy, one must add a confinment term to ensure the particles have an antagonising force to localise them. This could be a hard confinment in a ball (or another shape), with radius $\sim n^{1/d}$ so that each particle has approximately a constant volume for itself. There are also smoother ways to confine the particles, we will typically consider an energy term of the form

$$E(x_1, \dots, x_n) = E^\circ(x_1, \dots, x_n) + \sum_i V_n(x_i)$$

and V_n is called a confinment potential, it is supposed to have compact sublevel sets. A hard confinment penalisation consists formally in choosing $V_n(x_i) = \infty \times \mathbf{1}\{\|x_i\| > n^{1/d}\}$ (with $\infty \times 0 = 0$), but this is less usual in the mathematics litterature. A frequent and convenience choice is $V(x_i) \propto \|x_i\|^2$, and it emerges naturally in the theory of random matrices as we saw with previous models. We indicate [55, 71] as a gentle introduction to the mathematical aspects of such topics.

The deterministic configurations minimising this energy are called *ground states*, or *optimal configurations*, and are highly ordered. To add some randomness among low energy configurations and reflect disordered states of matter such as gases and liquids, one balances the energy by an entropy term, favoring more disordered random states, parametrized by some temperature $T > 0$, or the inverse temperature $\beta = 1/T$. The minimisers of this combined quantity called *free energy* are random point configurations called Gibbs measure at inverse temperature $\beta > 0$. The Gibbs measure P_n^H with n particles and energy H is defined by the density

$$\rho_H^n(x_1, \dots, x_n) \propto \exp(-\beta H(x_1, \dots, x_n)). \tag{3.15}$$

The particles of P_n^H tend to approach global minimisers of the energy in the low temperature regime $\beta \to \infty$, and at the opposite converge towards independent "totally disorded" processes with density $\propto \prod_i e^{-\sum_i V(z_i)}$ when $\beta \to 0$ in the "high temperature regime".

A popular example is s-Riesz gases, $s \in \mathbb{R}$, where $\varphi(x-y) = \|x-y\|^{-s}$. When $s > d$, φ is integrable at ∞ and the model is said to be *short range*. It seems that hyperuniformity cannot happen as the energy is extensive, i.e. proportionnal to the volume, see Ginibre inequality [55, (28)] for finite systems. In this case, there is an unambigously defined infinite limiting point process P^H, i.e.

$$\mathsf{P}_n^H \xrightarrow[n\to\infty]{} \mathsf{P}^H$$

where P^H can also be described locally through DLR equations, and Dereudre and Flimmel [20] confirms that stationary locally interacting Gibbs measure are never hyperuniform.

A particular case is obtained with *Coulomb gases*, also known under the terminology *jellium*, or *One Component Plasmas* (OCPs). Call φ_d the Coulomb potential in dimension d, i.e.

$$\varphi_d(x) = \begin{cases} -\ln(\|x\|) \text{ if } d = 2 \\ \frac{1}{d-2}\|x\|^{2-d} \text{ if } d = 1 \text{ or } d \geqslant 3, \end{cases}$$

easily seen to satisfy in the distributional sense for some $\kappa_d > 0$

$$\Delta\varphi_d(x) = \kappa_d \delta_0, \tag{3.16}$$

in dimension 2 this fact has already been useful for GAF zeros at (3.8). Call $\mathsf{P}_n^{d,\beta} \in \mathscr{N}(\mathbb{R}^d)$ the simple point process with exactly n points and inverse temperature $\beta \geqslant 0$ which density is

$$\rho_{d,\beta}^n(x_1,\dots,x_n) \propto \exp(-\beta \sum_{i\neq j} \varphi_d(x_i - x_j)) \exp(-\beta \sum_i \|x_i\|^2).$$

It indeed corresponds to Boltzmann equation (3.15) with potential φ_d. We recover some previously encountered examples: In Dimension 1, we obtain the β-ensembles, that converge resp. to the stationary hyperuniform Sine_β processes as $n \to \infty$ (Theorem 1.1). In Dimension 2, for $\beta = 2$, we exactly have the Ginibre distribution.

Hence the "interesting" problem is wether long range Riesz gases are hyperuniform, which is, in general, expected [55], and proved in dimension 1 [85]. In dimension $d \geqslant 2$, the only rigourous result concerns Coulomb gases with n particles in dimension 2. The long range interactions make it complicated to even define unambiguously an infinite model, the following result is rather asymptotic hyperuniformity for large finite *bulk subsystems*, i.e. at distance $\sim n^{1/d}$ from the disk boundary.

Theorem 3.2 (Leblé)**.** For $n \geqslant 1$, let $x_n \in \mathbb{R}^2, R_n > 0$ such that for some $\varepsilon > 0$, $B(x_n, R_n) \subset B(0, \sqrt{n\pi}(1-\varepsilon))$. Then

$$\mathrm{Var}\left(\mathsf{P}_n^H(B_{x_n,R_n})\right) = o(R_n^2).$$

The question might actually be easier to solve for infinite stationary models using smooth linear statistics with Theorem 2.1-(ii). The (ill-posed) question remains:

Question 3.1. Are Coulomb systems in dimension $d \geqslant 3$ hyperuniform?

3.5 Quasi-periodic models

Hyperuniformity is important because it emerges in many different phenomena, especially in condensed matter physics and statistical physics. Mathematically, disordered particle systems have attracted most of the activity, but many other models, e.g. hard spheres, quasicrystals, or others, would deserve more attention. To introduce the topic of aperiodic order, let us start with the following toy example, which bears some similarity with subsequent more physical examples.

Example 3.1 (irrationally shifted lattices)**.** Let some positive numbers $\mathbf{a} = \{a_m > 0, m \geqslant 1\}$, mutually irrational, and i.i.d. variables $U_m \sim \mathscr{U}_{[0,1]^d}, m \geqslant 1$. Let the model obtained by summing independent rescaled versions of the lattice

$$\mathsf{P}^{\mathbf{a}} = \sum_{m\geqslant 1} \sum_{\mathbf{m}\in\mathbb{Z}^d} \delta_{a_m(U_m+\mathbf{m})}.$$

Let us assume that $\sum_m a_m^{-d} < \infty$ to have finite intensity and local square integrability. Since the spectral measure is linear in the random measure, we obtain the spectral measure of $\mathsf{P}^{\mathbf{a}}$ by using Example 2.3 at the proper scale:

$$\mathsf{S}^{\mathbf{a}} = \sum_m a_m^{-2d} \sum_{\mathbf{m}\in 2\pi\mathbb{Z}^d\setminus\{0\}} \delta_{a_m^{-1}\mathbf{m}}.$$

Remark that by carefully choosing the a_m, one can have an arbitrary decay of $\mathsf{S}^{\mathbf{a}}(B_\varepsilon)$ as $\varepsilon \to 0$, and hence an arbitrary exponent of hyperuniformity, see Section 2.4.

Quasicrystals are broadly speaking non-periodic atomic measures whose spectrum is purely atomic, their study emerged after experimental discoveries in physics in the 80s and is related to many fields, including crystallography, aperiodic tilings, almost periodicity, see the mathematical monograph [6] and references therein. Such objects are traditionally assumed to be homogeneous in space, and it is thus natural to consider random constructions that are invariant under translations ([10, 64]).

Let us introduce the invariant random quasicrystal model of Bjorklund and Hartnick [10], called *cut-and-project* process, obtained from a higher dimensional tilted lattice that is projected onto a band with a finite width. Let $d, d' \in \mathbb{N}^*, \Gamma = \Gamma_0 + U$ a shifted lattice of $\mathbb{R}^d \times \mathbb{R}^{d'}$, with $U \sim \mathscr{U}_{[0,1]^{d+d'}}$ and Γ_0 a lattice containing 0. Consider for $W \subset \mathbb{R}^{d'}$ compact the obtained projection

$$\mathsf{P}^{\Gamma_0,d,W} = \{x : (x,y) \in \Gamma, y \in W\}.$$

Often, W is implicitly assumed to be the unit ball of $\mathbb{R}^{d'}$ and omitted in the notation.

Those models possess the following properties, defining a *mathematical quasicrystal*:

- *Uniformly discrete*, i.e. $\inf_{x \neq y \in \mathsf{P}^{\Gamma_0,d}} \|x - y\| > 0$ a.s.,
- *relatively dense*, i.e. for some $r > 0$, $\cup_{x \in \mathsf{P}^{\Gamma_0,d}} B(x,r) = \mathbb{R}^d$ a.s.,
- The spectral measure S is purely atomic with a dense support.

Bkorklund and Hartnick [10] show that $\mathsf{P}^{\Gamma_0,d}$ is sometimes hyperuniform, depending on the diophantine properties of Γ. We refer to [10] for a precise definition, but an interesting class of lattices is that of *arithmetic lattices*, which elements are badly approximable by rational tuples. A typical example is $\Gamma_0 = \{(a + b\sqrt{2}, a - b\sqrt{2}); a, b \in \mathbb{Z}^2\} \subset \mathbb{R}^2$, in the same way that algebraic numbers (i.e. solutions of polynomial equations with integer coefficients, such as $\sqrt{2}$, $\frac{1+\sqrt{5}}{2}$, etc...) are the worst approximable numbers of $\mathbb{R}$. The idea is that the diophantine properties of the dual of a lattice will determine its hyperuniformity behaviour, and bad approximation properties give a more hyperuniform behaviour. Interestingly, we observe in [48] the same phenomenon for nodal domains of Gaussian fields which spectral measure has irrational atoms. Below we assume that $W = B_1$ for simplicity, and call S the structure factor of $\mathsf{P}^{\Gamma_0,d}$.

Theorem 3.3 ([10]).

- If the dual lattice of Γ_0 is arithmetic,

$$\lim_{\varepsilon \to 0} \frac{\mathsf{S}(B_\varepsilon)}{\varepsilon^d} = o(\varepsilon^{\frac{d}{d'}}).$$

- For $\mathscr{L}^{(d+d')^2}$-a.e. $(d+d') \times (d+d')$ real matrix A, the lattice $\Gamma_0 = A\mathbb{Z}^{d+d'}$ yields a hyperuniform process such that for each given $\delta > 0$

$$\frac{\mathsf{S}(B_\varepsilon)}{\varepsilon^d} = o(\varepsilon^{\frac{d}{d'} - \delta}).$$

- For $d = d' = 1$, there exists a lattice Γ_0 such that for every $\alpha > 0$,

$$\limsup_{\varepsilon \to 0} \frac{\mathrm{Var}\,(\mathsf{S}(B_\varepsilon))}{\varepsilon^\alpha} = \infty.$$

  It implies by Theorem 2.1-(iii) that $\mathsf{P}^{\Gamma_0,1}$ is not hyperuniform.

This spectrum of different second order behaviours in fact reflects the spectrum of diophantine properties of real numbers:

- the worst approximable numbers, such as the algebraic numbers, are characterised as those $x \in \mathbb{R}$ such that for some $c > 0$

$$\forall q \in \mathbb{N}^*, \inf_{p \in \mathbb{Z}} |x - \frac{p}{q^2}| > \frac{c}{q^2}$$

  they form a negligible set of $\mathbb{R}^d$.

- For $\delta > 0$, $\mathscr{L}^1$-a.e. $x \in \mathbb{R}$ is $(2+\delta)$-approximable, in the sense that for some $c > 0$, for infinitely many $q \in \mathbb{N}^*$,

$$\inf_{p \in \mathbb{Z}} |x - \frac{p}{q}| < cq^{-2-\delta}$$

- There is a negligible set of numbers x with good approximation properties, such as the *Liouville numbers*, i.e. such that for every $\delta > 0$, there are infinitely many $q \in \mathbb{N}^*$ with

$$|x - \frac{p}{q}| < cq^{-2-\delta}.$$

Chapter 4

Perturbed lattices, matching and optimal transport

We saw that an easy way to produce hyperuniform samples is to start from a lattice and perturb its points by i.i.d. variables $U_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d$, but the underlying periodic structure remains detectable in the resulting point process (see Example 2.4). A softening of this procedure is to allow for dependency between the perturbations $U_{\mathbf{k}}$, and in this chapter we first explore how much dependent can these variables be and still maintain hyperuniformity. To maintain stationarity, we assume that the field $\{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$ is invariant under $\mathbb{Z}^d$-translations; in particular the $U_{\mathbf{k}}$ are identically distributed

In the previous chapter, we saw hyperuniform models who have in appearance nothing to do with a lattice structure, they seem to form a class fundamentally disconnected from the class of perturbed lattices. We will see that, somehow surprisingly (at least in dimension 2), these disordered processes can in most cases be written as dependently perturbed lattices, forming in some sort a converse to the assertion that perturbed lattices are hyperuniform. To formulate and prove this result, we invoke the theory of optimal transport. It ultimately proves that perturbed lattices and disordered hyperuniform processes are not a dichotomy, they form two sides of a continuous class, except in dimension 1 where a stronger requirement than hyperuniformity is required.

This is another illustration of the formula *global order and local disorder*, in that a hyperuniform point process, even when it seems locally disordered, such as for a DPP or the zeros of a random function, exhibits at large scale the same order than a lattice.

4.1 Perturbed lattices are hyperuniform point processes

We already saw that when a lattice undergoes i.i.d. perturbations, it is hyperuniform (Example 2.4 and Theorem 2.1-(ii)). When the perturbations are dependent, Dereudre et al. [21] proved that it still holds as long as they have a finite second order moment in dimension 1 or 2. In the following, $\mathsf{U} = \{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$ is a stationary field over $\mathbb{Z}^d$, independent of the shifted lattice $\tau_U \mathbb{Z}^d$ where $U \sim \mathscr{U}_{[0,1]^d}$. Denote the resulting point process by

$$\mathsf{Z}^{d,\mathsf{U}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} \delta_{U + U_{\mathbf{k}}}.$$

Theorem 4.1 ([21])**.** Assume $\mathbf{E}\|U_k\|^2 < \infty$. Then

- In dimension $d = 1$, $\sup_R \mathrm{Var}\left(\mathsf{Z}^{d,\mathsf{U}}(B_R)\right) < \infty$, i.e. $\mathsf{Z}^{d,\mathsf{U}}$ is *Class 1* hyperuniform.
- In dimension $d = 2$, $\mathsf{Z}^{d,\mathsf{U}}$ is hyperuniform, but the variance decay can be arbitrarily slow.
- In dimension $d \geqslant 3$, for every $\varepsilon > 0$, there exists a stationary field U with $\|U_{\mathbf{k}}\| < \varepsilon$ a.s. and $R^{-d}\mathrm{Var}\left(\mathsf{Z}^{d,\mathsf{U}}(B_R)\right) \to \infty$.

[21] also give counter-examples showing that these results are sharp in general. The general idea is still that the more the perturbation is statistically disordered (extreme disorder being the independence), the more it has the chance to maintain the underlying lattice structure, but also the less likely is the obtained point process to be mixing. In some sense the mixing properties of U and P go in reverse directions. This is confirmed by the results of [25, 41], which ensures that hyperuniformity is maintained in any dimension as long as the perturbating field U is sufficiently mixing. Flimmel [25, Theorem 2] uses the α-mixing coefficients, for U a stationary field over $\mathbb{Z}^d$,

$$\alpha(n) := \sup\{|\mathbf{P}(\Omega) - \mathbf{P}(\Omega')| \ : \ \Omega \in \sigma(U_{\mathbf{k}}; \mathbf{k} \in A), \Omega' \in \sigma(U_{\mathbf{k}'}; \mathbf{k}' \in B); \ A, B \subset \mathbb{R}^d : d(A,B) \geqslant n\}, n \in \mathbb{N}.$$

Klatt et al. [41, Th. 5.5] introduce in the flavor of β-mixing, with $\mathbf{P}_X$ the law of X,

$$\beta(\mathbf{m}) = \sup_{A \subset \mathbb{R}^d \times \mathbb{R}^d \text{ Borel}} |\mathbf{P}_{(U_0, U_{\mathbf{m}})}(A) - \mathbf{P}_{U_0}^{\otimes 2}(A)|, \mathbf{m} \in \mathbb{Z}^d.$$

Theorem 4.2. If $\sum_{\mathbf{m}} \alpha(\|\mathbf{m}\|) < \infty$ or $\sum_{\mathbf{m}} \beta(\mathbf{m}) < \infty$, then $\mathsf{Z}^{d,\mathsf{U}}$ is hyperuniform.

[25] contains a stronger version under some moment assumptions, and [41] actually considers any stationary random measure (not only point processes), and any kind of perturbation, as long as stationarity is in order.

4.1.1 Lattices perturbed by clusters and arbitrary exponent

A similar class of dependent perturbations is obtained when each point is replaced by a whole finite point process, as considered in [50]. This is not formally a perturbed lattice as studied above, but the resulting points can still be seen as the result of a perturbation applied to a lattice. We again refer the reader to [41] for a general theory of perturbed random measures. Let us give here a result where the point processes attached to different points are independent.

Let μ a probability distribution on $\mathscr{N}(\mathbb{R}^d)$, and $\mathsf{P} = \sum_i \delta_{x_i}$ a simple stationary point process. Let $\mathsf{P}_i, i \geqslant 1$, i.i.d. point processes with law μ, and

$$\mathsf{P}^\mu := \sum_i \sum_j \tau_{x_j} \mathsf{P}_i.$$

Proposition 4.1. Let P a wide sense stationary point process with spectral measure S and assume $\mathbf{E}\#\mathsf{P}_1^2 < \infty$. Let the random Fourier-Stieljes transform $\varphi(u) = \int e^{iu\cdot t} d\mathsf{P}_1(t)$. Then P^μ has the spectral measure

$$\mathsf{S}_\mu = \left(\mathbf{E}|\varphi(u)|^2 - |\mathbf{E}\varphi(u)|^2\right) \mathscr{L}^d(du) + \mathbf{E}|\varphi(u)|^2 \mathsf{S}.$$

We recover (2.5) when $\#\mathsf{P}_1 = 1$ a.s. We see that in general, if P is hyperuniform, S vanishes in 0, hence also S_μ vanishes in 0, and by Theorem 2.1, P^μ is still hyperuniform. Remark that Lemma 2.3 does not apply to this form of structure factor, and we can indeed exploit this to have a point process with arbitrarily high hyperuniformity exponent.

Among stationary models mathematically treated, the only exception to the behaviour $\mathsf{s}(u) \geqslant \sigma\|u\|^2$ are GAF zeros (Lemma 3.2) and shifted lattices. For the latter, at the opposite, the spectral measure vanishes in a neighbourhood of the origin; it is a member of the class of *stealthy processes*, which are the topic of Section 5.2. To obtain an intermediary model, that is not formally periodic, but still presents a high level of rigidity, one can use a union of irrational shifted lattices, see above. We give here another class of models, with slightly more disorder, where the points of a lattice are perutbed by randomly rotated clusters.

Let $\rho > 0, p \in \mathbb{N}^*$. Let $\mathsf{P}_i = \sum_{j=1}^p \delta_{2i\pi j\rho + \theta_i}$ where the θ_i are i.i.d. uniform on $[0, 2\pi]$, and let μ the common law of the P_i, and let $\mathsf{P} = \mathsf{Z}^d$, recall the construction of P^μ from above.

Theorem 4.3 ([50], Theorem 3)**.** The point process P^μ is hyperuniform and if p is a prime number, its spectral measure satisfies as $u \to 0$

$$\mathsf{S}_p(u) \sim \frac{\|\rho u\|^{2p}}{(p-1)!^2} du,$$

hence P^μ has hyperuniform exponent $2p$.

See also [27], giving guidelines to design other examples in any dimension.

4.1.2 The partial matching process: a dependently perturbed lattice

Klatt, Last and Yogeshwaran [43] consider a special construction that can be seen as a dependently perturbed lattice. Let P a homogeneous Poisson process with intensity $a > 1$, let $\mathsf{Z}^d = \tau_U \mathbb{Z}^d$ the usual shifted lattice with $U \mathscr{U}_{[0,1]^d}$ independent from P (Section 1.2). Then build an injective mapping $T : \mathbb{Z}^d \to \mathsf{P}$ that is invariant under translations, i.e. $\tau_x T \overset{(d)}{=} T, x \in \mathbb{R}^d$, by performing a *stable matching:*

- First match $x \in \mathsf{Z}^d, y \in \mathsf{P}$ if they are mutual nearest neighbours, i.e. y is the closest point from x in P and vice-versa, and put $y = T(x)$. Call $\mathsf{Z}^d_{\text{match}}$ such matched points from Z^d and $\mathsf{P}_{\text{match}}$ the matched points from P.
- Remark that $\mathsf{Z}^d \backslash \mathsf{Z}^d_{\text{match}}, \mathsf{Z}^d_{\text{match}}, \mathsf{P} \backslash \mathsf{P}_{\text{match}}, \mathsf{P}_{\text{match}}$ are still stationary point processes. Repeat the procedure by matching mutual nearest neighbours of $\mathsf{Z}^d \backslash \mathsf{Z}^d_{\text{match}}$ and $\mathsf{P} \backslash \mathsf{P}_{\text{match}}$, putting each time $y = T(x)$ if y and x are matched.
- Then repeat the procedure by matching iteratively mutual nearest neighbours that have not been matched previously.

This procedure never terminates globally, but each point of Z^d is eventually matched after finitely many iterations, hence T is well defined. We then have the following result. For $x = \mathbf{k} + U \in \mathsf{Z}^d$, define $U_{\mathbf{k}} = T(x) - x$ and $\mathsf{U} = \{U_{\mathbf{k}}; k \in \mathbb{Z}^d\}$, so that indeed $T(\mathsf{Z}^d) = \mathsf{Z}^{\mathsf{U}}$ is a dependently perturbed lattice.

Theorem 4.4 (KLY)**.** $T(\mathsf{Z}^d)$ is a stationary hyperuniform perturbed lattice and the $U_{\mathbf{k}}$ have an exponential tail: for some finite $c > 0$,

$$\mathbf{P}(\|U_0\| > r) \leqslant c e^{-cr^d}.$$

The proof can be deduced from Theorem 4.2 by the same authors. In [43], they furthermore prove that $T(\mathsf{Z}^d)$ is number rigid, a concept that is introduced at Chapter 5.

Remark 4.1. A similar procedure can be conducted with $a = 1$, and in this case the points of P are also eventually exhausted, so that T is a bijection between $\mathbb{Z}^d$ and P, but the matching properties are much less agreable; in particular, $\|U_0\|$ has an infnite d-th moment , whereas other Poisson matchings perform much better [34, 35].

4.2 Hyperuniform point processes are perturbed lattices

One of the motivations for studying hyperuniform point processes is the low variance for linear statistics (Proposition 2.1), and more generally that they are "evenly" distributed across space, at least they seem so visually. We try to formalise here this impression with the concept of allocation, and more generally of optimal transport. As we will see, this provides some kind of converse statement to Theorem 4.1, in that many hyperuniform processes can be seen as a perturbed lattices.

From the point of view of transport, a sample of points is well distributed if one can divide up the space in cells of equal volume such that each cell can be associated to a nearby point of the

process nearby, the "center of the cell". One can think for instance of points as school locations in a city C_n, and Lebesgue measure represents the distribution of the population, the goal being that each school is associated with an equal volume of population and that each citizen should not have to travel too far to go to the school it has been assigned to (which is not necessarily the closest school). The ideal repartition occurs when points (or schools) are distributed periodically as in a lattice, we rather investigate here disordered samples. We show in the figure below three samples of points and a corresponding partition of a sphere in cells of equal volume using the so-called gravitationnal allocation [63].

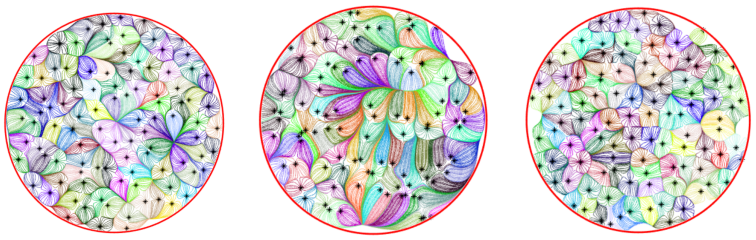


Figure 4.1: Sending kids to school. *Left.* Ginibre eigenvalues. *Middle.* Poisson points. *Right.* GAF zeros. *Ilustration by D. Hawat*

4.2.1 Optimal transport, matching and allocation

We propose in this chapter a quantification of this concept in terms of matching and allocation. Let $p \geqslant 1$.

- Let $\mathbf{C}_n = [0,n)^d$, and let P_n a sample of n^d points in $\mathbf{C}_n$. Call *allocation* a measurable mapping $T : \mathbf{C}_n \to \mathsf{P}_n$ such that a.s. the volume of the set of points sent to each $x \in \mathsf{P}_n$ is exactly 1, i.e. $\mathscr{L}^d(T^{-1}(\{x\})) = 1$, $x \in \mathsf{P}_n$. Call p-allocation cost of T

$$\mathsf{C}_p(T) = \int_{\mathbf{C}_n} \|x - T(x)\|^p dx,$$

  and call optimal p-allocation cost $\mathsf{W}^p_{p,\mathrm{alloc}}(\mathbf{C}_n, \mathsf{P}_n)$ the infimum of $\mathsf{C}_p(T)$ over such allocations T, the notation W^p_p is explained below.

- Let $\mathsf{Z}_n = \mathbf{C}_n \cap \mathbb{Z}^d$ the lattice points of $\mathbf{C}_n$, note that $\#\mathsf{Z}_n = n^d$. Call p-matching cost of a bijection $\sigma : \mathsf{Z}_n \to \mathsf{P}_n$ the quantity

$$\mathsf{C}_p(\sigma) = \sum_{\mathbf{k} \in \mathsf{Z}_n} \|\sigma(\mathbf{k}) - \mathbf{k}\|^p.$$

  Call optimal p-matching cost $\mathsf{W}^p_{p,\mathrm{match}}(\mathsf{Z}_n, \mathsf{P}_n)$ of Z_n to P_n the minimum of $\mathsf{C}_p(\sigma)$ over all such bijections σ.

The concept of matching is in fact very close to that of perturbed lattices, as it means that there are variable $U_{\mathbf{k}} := \sigma(\mathbf{k}) - \mathbf{k}$ such that $\mathsf{P}_n = \{\mathbf{k} + U_{\mathbf{k}}; \mathbf{k} \in \mathsf{Z}_n\}$. In general, of course, such $U_{\mathbf{k}}$ would not be independent.

The matching and allocation costs are obviously different in general, but in the large sample asymptotics, we will see that they have the same magnitude for a given stationary point process P restricted to $\mathbf{C}_n$. From this perspective, the process P will be stationary, i.e. invariant under $\mathbb{R}^d$-translations, and accordingly the field $\mathsf{U} := \{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$ will be required to be invariant under $\mathbb{Z}^d$-translations. To avoid some technicalities for now, we study more generally a family of point processes $\mathsf{P}_n, n \geqslant 1$, and study whether there is explosion of the mean p-transport cost per particle.

4.2.2 Linear cost for large finite samples

The concepts of matching and allocation costs above are both instances of the concept of p-Wasserstein distance in optimal transport. Given two non-negative measures μ, ν with same mass on $\mathbb{R}^d$, call *coupling* of μ, ν a measure M on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$M(\cdot \times \mathbb{R}^d) = \mu, \;\; M(\mathbb{R}^d \times \cdot) = \nu.$$

Then define the p-th order Wasserstein distance by

$$\mathsf{W}_p^p(\mu, \nu) = \inf_{M \text{ coupling}} \int \|x - y\|^p M(dx, dy). \tag{4.1}$$

Remark that this quantity is symmetric in μ and ν.

Call $\mathscr{L}_n^d$ the Lebesgue measure restricted to $\mathbf{C}_n$. Note that $\#\mathsf{Z}_n = \#\mathsf{P}_n = \mathscr{L}_n^d(\mathbf{C}_n) = n^d$, which is the reason why we employ a straight cubic window in this chapter. An allocation $T : \mathbf{C}_n \to \mathsf{P}_n$ induces the coupling

$$M(A \times B) := \sum_{y \in \mathsf{P}_n \cap B} \mathscr{L}_n^d(A \cap T^{-1}(\{y\})),$$

the mass of points from A sent to B. A matching $\sigma : \mathsf{Z}_n \to \mathsf{P}_n$ induces similarly a coupling

$$M(A \times B) = \sum_{x \in \mathsf{Z}_n \cap A, y \in \mathsf{P}_n \cap B} \mathbf{1}\{y = T(x)\}.$$

Hence the optimal matching and allocation costs correspond to infima taken over specific classes of couplings, hence

$$\mathsf{W}_p^p(\mathscr{L}_n^d, \mathsf{P}_n) \leqslant \mathsf{W}_{p,\text{alloc}}^p(\mathscr{L}_n^d, \mathsf{P}_n), \;\; \mathsf{W}_p^p(\mathsf{Z}_n, \mathsf{P}_n) \leqslant \mathsf{W}_{p,\text{match}}^p(\mathsf{Z}_n, \mathsf{P}_n).$$

We in fact have the reverse inequality

$$\mathsf{W}_p^p(\mathsf{Z}_n, \mathsf{P}_n) = \mathsf{W}_{p,\text{match}}^p(\mathsf{Z}_n, \mathsf{P}_n), \; W_p^p(\mathscr{L}_n^d, \mathsf{P}_n) = W_{p,\text{alloc}}^p(\mathscr{L}_n^d, \mathsf{P}_n),$$

see for instance [70, Thm. 1.7]. We expect all those costs to be similar, which we formalise in the next result.

Proposition 4.2. Let $\mathsf{P}_n, n \geqslant 1$ point processes with resp. n^d points in $\mathbf{C}_n$, and $p \geqslant 1$. There is a universal constant $c > 0$ such that

$$\mathsf{W}_p^p(\mathscr{L}_n^d, \mathsf{P}_n) \geqslant cn^d.$$

Hence the *linear cost* (proportionnal to n^d), is the least one can expect. Then the two following are equivalent:

(i) There is a constant c_{match} such that $\mathbf{E}\mathsf{W}_p^p(\mathsf{P}_n, \mathsf{Z}_n) \leqslant c_{\text{match}} n^d$,

(ii) There is a constant c_{alloc} such that $\mathbf{E}\mathsf{W}_p^p(\mathsf{P}_n, \mathscr{L}_n^d) \leqslant c_{\text{alloc}} n^d$.

Proof. The lower bound is a consequence of the isoperimetric inequality, which gives for any domain Ω with volume 1, and any $y \in \mathbb{R}^d$

$$\int_\Omega \|x - y\|^p dx \geqslant c_{d,p} := \int_{B(y, \kappa_d^{-1/d})} \|x - y\|^p dx.$$

Then just make the summation over the cells $\Omega_i := T^{-1}(x_i)$.

The equivalence comes from the general triangular inequality between finite measures: to prove that (i) implies (ii), use

$$W_p(\mathsf{P}_n, \mathscr{L}_n^d) \leqslant W_p(\mathsf{P}_n, \mathsf{Z}_n) + W_p(\mathsf{Z}_n, \mathscr{L}_n^d).$$

This inequality is proved in all textbooks about optimal transport, see for instance [70, 86]. One can obtain the estimate

$$W_p^p(\mathsf{Z}_n, \mathscr{L}_n^d) \leqslant cn^d$$

using the following elementary allocation: cut $\mathbf{C}_n$ into n^d small cubes with sidelength 1 and define $T_0 : \mathbf{C}_n \to \mathsf{Z}_n$ by

$$T_0(\mathbf{k} + [0,1)^d) = \{\mathbf{k}\}; \mathbf{k} \in \mathsf{Z}_n.$$

It gives $W_p^p(\mathsf{Z}_n, \mathscr{L}_n^d) \leqslant C_p(T_0) \leqslant cn^d$. Then use the inequality $(a+b)^p \leqslant c_p(a^p + b^p), a, b \geqslant 0$ to have (i) implies (ii). The same inequality after switching the roles of $\mathsf{Z}_n, \mathscr{L}_n^d$ gives (ii) implies (i).

□

An example of paramount imporance in geometric probability is when P_n consists in i.i.d. points uniformly distributed over $\mathbf{C}_n$.

Theorem 4.5 (AKT Theorem [4])**.** Let P_n made up of n^d i.i.d. points uniform in $\mathbf{C}_n$, and $p \geqslant 1$. Then (i) and (ii) hold if and only if $d \geqslant 3$

This result admits many generalisations, it is in particular valid for processes with asymptotically integrable covariance, see for instance [52, 13, 35]. Hence in dimension $d \geqslant 3$, hyperuniform or not, a standard point process has a satisfying behaviour in terms of transport / allocation / matching.

In dimension $d = 2$, on the other hand, one cannot find a nice allocation from $\mathbf{C}_n$ to P_n, in the sense that, by the AKT theorem

$$\lim_n \frac{\mathbf{E}\mathsf{W}_{2,\mathrm{alloc}}^2(\mathsf{P}_n, \mathscr{L}_n^d)}{n^d} \to \infty.$$

This can be observed on Figure 4.1, the middle sample is formed by independent points, and the cells are elongated, generating a large transport cost. Left and right samples, resp. Ginibre eigenvalues and zeros of a GAF, have more spherical cells, yielding a lower cost. We will see now that the reason is that such samples are asymptotically hyperuniform, as we saw previously.

Pioneer works in this direction are [38, 68], giving results for the finite Ginibre ensembles. Let us give first the result of Butez, Dallaporta & Garcia-Zelada, bearing directly on the fluctuations of the discrepancy, in the spirit of the original geometric definition of hyperuniformity in terms of number variance (1.1), valid for any $p \geqslant 1$.

Theorem 4.6 ([13])**.** Let $p \geqslant 1, \mathsf{P}_n$ with n^d points such that for some $c, \varepsilon > 0$, for any square $B \subset \mathbf{C}_n$ with $\mathscr{L}^d(B) > 1$,

$$\sup_n \mathbf{E}\left|\frac{\mathsf{P}_n(B) - \mathscr{L}^d(B)}{\sqrt{\mathscr{L}^d(B)}}\right|^p \leqslant \frac{c}{\ln(\mathscr{L}^d(B))^{p+\varepsilon}}. \tag{4.2}$$

Then

$$\mathsf{W}_p^p(\mathscr{L}_n^d, \mathsf{P}_n) \leqslant c' n^d$$

for some constant c' not depending on n.

Hyperuniformity for finite samples corresponds to the left hand member going to 0 for $p = 2$ as B grows to infinity, hence it almost automatically yields linear bounds for the 2-Wasserstein distance. If one has moments of higher order on the discrepancy, it gives a linear bound for the Wasserstein distance of corresponding order. It unfortunately does not apply to systems where the bulk behaviour is different from the behaviour on the boundary, as in Theorem 3.2.

We saw in Theorem 2.1-(iii) that hyperuniformity is equivalently characterised by the vanishing of the spectrum in 0. The equivalent concept for a finite sample is the *scattering intensity*. For a point process P_n in $\mathbf{C}_n$, denote by $\mathscr{F}\mathsf{P}_n(u) = \int e^{-iu\cdot t}\mathsf{P}_n(dt), u \in \mathbb{R}^d$, the Stieljes-Fourier transform. Then the expectation of the scattering intensity is

$$\hat{\mathsf{S}}_{\mathsf{P}_n}(u) := \frac{1}{\mathbf{E}\mathsf{P}_n(\mathbf{C}_n)}\mathbf{E}\left|\mathscr{F}\mathsf{P}_n(u)\right|^2.$$

For P a L^2_{loc} stationary point process, and $\mathsf{P}_n = \mathsf{P} \cap \mathbf{C}_n$, $\hat{\mathsf{S}}_{\mathsf{P}_n}(u)$ provides a simple and natural approximation of the structure factors $\mathsf{S}(u)$ of P, see for instance [17, Th. 5.1] for a weak convergence result. Actual estimators are built from multiscale tapered versions, see Chapter **??**. We give here for completeness the following convergence result:

Proposition 4.3. Let P a wide sense stationary point process which covariance measure C is integrable. Then S is continuous with respect to $\mathscr{L}^d$ by standard Fourier considerations, and for $u \in \mathbb{R}^d$ the density $\mathsf{s}(u)$ satisfies

$$\hat{\mathsf{S}}_{\mathsf{P}_n}(u) \to \mathsf{S}(u)$$

if and only if $u \in 2\pi\mathbb{Z}^d\backslash\{0\}$ or more than half of u's coordinates do not vanish.

Proof. Assume without loss of generality that P has unit intensity, hence $\mathbf{E}\mathsf{P}_n(\mathbf{C}_n) = n^d$. Define $f_{u,n}(x) = 1_{\mathbf{C}_n}(x)e^{-iu\cdot x}$ so that $n^d\hat{\mathsf{S}}_{\mathsf{P}_n}(u) = \mathbf{E}|\mathsf{P}(f_{u,n})|^2$. Let us use (2.1) and Lemma 2.1:

$$\begin{aligned}
\mathbf{E}|\mathsf{P}(f_{u,n})|^2 =& \mathrm{Var}\left(\mathsf{P}(f_{u,n})\right) + \left|\mathbf{E}\mathsf{P}(f_{u,n})\right|^2\\
=& \int f_{u,n}\star\bar{f}_{u,n}(x)\mathsf{C}(dx) + \left|\widehat{1_{\mathbf{C}_n}}(u)\right|^2\\
=& \int e^{iu\cdot x}\underbrace{1_{\mathbf{C}_n}\star 1_{\mathbf{C}_n}}_{\sim n^d}(x)\mathsf{C}(dx) + \left|n^d\widehat{1_{\mathbf{C}_1}}(un)\right|^2\\
=& n^d\mathsf{S}(u)(1+o(1)) + n^{2d}\left|\prod_{i:u_i\neq 0}\frac{e^{iu_in}-1}{u_in}\right|^2,
\end{aligned}$$

and the last term vanishes if $u \in 2\pi\mathbb{Z}^d\backslash\{0\}$, and otherwise is in $n^{2(d-m)}$ where m is the number of i such that $u_i \neq 0$, this concludes the proof. □

We see that the bias is minimal at non-zero multiples of $2\pi/n$, hence the best approximations are the $\hat{\mathsf{S}}_{\mathsf{P}}(2\pi\mathbf{k}/n), \mathbf{k} \in \mathsf{Z}_n\backslash\{0\}$. See [33] for a more precise result and variants of the scattering intensity more useful in practical situations.

In the physics litterature, it is common to estimate the structure factor at such lattice points. In conclusion, if P is hyperuniform, $\mathsf{S}(u)$ should be small for u close to 0, especially at the $\hat{\mathsf{S}}_{\mathsf{P}_n}(2\pi\mathbf{k}/n)$.

A recent result of Bobkov and Ledoux [12] allows to derive transport bounds in terms of the Fourier-Stieljes transform. The drawback is that this bound is valid for the toric Wasserstein distance $\widetilde{W_p^p}$, i.e. when the distance $\|x-y\|^p$ is replaced by the toric distance $d_{\mathbf{C}_n}(x,y)^p = \inf_{\mathbf{k}\in\mathbb{Z}^d}\|x-y+\mathbf{k}n\|^p$ in (4.1). When n goes to ∞, the influence of boundaries in the toric distance will vanish under some additional assumptions (see the next section).

Theorem 4.7. Let P_n consists of n^d distinct points. Assume that for some $c < \infty$, for $\mathbf{k} \in \mathsf{Z}_n \backslash \{0\}$,

$$\mathbf{E}\hat{\mathsf{S}}_{\mathsf{P}_n}(2\pi\mathbf{k}/n) \leqslant c \times \begin{cases} 1 \text{ if } d \geqslant 3 \\ \frac{1}{|\ln(2\pi\mathbf{k}/n)|^{1+\varepsilon}} \text{ if } d = 2. \end{cases} \tag{4.3}$$

Then

$$\tilde{\mathsf{W}}_2^2(\mathscr{L}_n^d, \mathsf{P}_n) \leqslant cn^d. \tag{4.4}$$

The relation between hyperuniformity and the Bobkov-Ledoux bound has been explored in [52], combine (27) and (18) from [52] with $t_0 = n$ to show the bound above. If now $\mathsf{P}_n = \mathsf{P} \cap \mathbf{C}_n$ for P stationary, $\#\mathsf{P}_n$ is random and the previous bound cannot apply directly, we give in the next section the results of [52, 21] where this problem is handled.

As noted at Section 2.3, the number variance on squares and decay of the structure factor cannot be formally deduced from one another. Using a computation similar to the implication (iii)⇒ (i) in the proof of Theorem 2.1, a decay of the structure factor in $|\ln(\|u\|)|^{-1-\varepsilon}$ as $u \to 0$ yields a number variance on balls of order

$$\mathrm{Var}\left(\mathsf{P}(B_n)\right) \asymp n^4 \mathsf{S}(B_{1/n}) \asymp n^4 \int_0^{1/n} \frac{r}{|\ln(r)|^{1+\varepsilon}} dr \asymp n^2 \ln(n)^{-1-\varepsilon},$$

hence comparing with (4.2), for $p = 2$ the bound on the structure factor seems slighlty less restrictive.

4.2.3 Infinite samples

We investigate here how to translate the previous results to an infinite stationary hyperuniform model P with unit intensity, to obtain the representation as a perturbed lattice, i.e. $\mathsf{P} = \mathsf{P}^{\mathsf{U}}$ for some stationary $\mathsf{U} = \{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$, as announced in the chapter introduction. Previous results yield that for a Poisson (or *Poisson-like*) process in dimension $d \geqslant 3$, or a hyperuniform process in dimension 2, on a large window $\mathbf{C}_n$ where P has approximately n^d points, there should be a nice matching between $\mathsf{P} \cap \mathbf{C}_n$ and Z_n, or a nice allocation from $\mathbf{C}_n$ to $\mathsf{P} \cap \mathbf{C}_n$. Remember that the matching can equivalently be written as a bijection σ between $\mathbb{Z}^d$ and P, with $\sigma(\mathbf{k}) := \mathbf{k} + U_{\mathbf{k}}$. The length $\|U_0\|$ can be interpreted as the law of the distance between a typical point of P and the point of $\mathbb{Z}^d$ it has been assigned to; a good matching is such that $\mathbf{E}\|U_0\|^p < \infty$ for sufficiently high p, we shall typically investigate this question for $p = 2$.

In analogy with the finite case, we call invariant allocation to P a random measurable mapping $T : \mathbb{R}^d \to \mathsf{P}$ which law is invariant under shifts, i.e. $\tau_y T \overset{(d)}{=} T, y \in \mathbb{R}^d$, and such that a.s., the cells $T^{-1}(\{x\}); x \in \mathsf{P}$ form disjoint measurable sets of volume 1. As before, the law of $\|T(0)\|$ and its moments are a good indicator of how good is the allocation. It is possible to show under mild assumptions that there is a stationary allocation T with $\mathbf{E}\|T(0)\|^p < \infty$ for some $p > 0$ if and only if P is a p-perturbed lattice, i.e. $\mathsf{P} = \mathsf{Z}^{\mathsf{U}}$ where U is an invariant perturbation U with $\mathbf{E}\|U_0\|^p < \infty$. We shall state the next results in terms of matchings, but it could equivalently be stated in terms of allocations, recalling Proposition 4.2.

As a preliminary remark, with or without hyperuniformity, there always exist such a perturbation U or allocation T under the minimal assumption that P is ergodic, see for instance [35] where a matching is built with the Gale-Shapley stable marriage algorithm. This procedure might actually yield suboptimal matchings. It is proved in [35] that this stable marriage yields $\mathbf{E}\|U_0\| = \infty$ in dimension $d \geqslant 3$ whereas other matching procedures admit exponential moments.

We require below that the covariance measure is integrable, i.e. has finite total mass over $\mathbb{R}^d$ (this is assumed equivalently for the reduced pair correlation measure in [52]). This is typically the case for disordered point processes. Let us start by the standard case in dimension $d \geqslant 3$.

Theorem 4.8. Let P a L^2_{loc} stationary point process in $\mathbb{R}^d, d \geqslant 3$.

(i) Assume $\mathrm{Var}\,(\mathsf{P}(B_R)) \leqslant cR^d$ for some $c < \infty$. Then P is a p-perturbed lattice for $p < d/2$, in the sense that $\mathsf{P} = \mathsf{Z}^{\mathsf{U}}$ for some perturbation U with $\mathbf{E}\|U_0\|^p < \infty$.

(ii) Assume C is integrable. Then P is a 2-perturbed lattice

Point (i) is not formally present in the literature, but can be shown with the same method than in [35], see [52, Remark 1]. Point (ii) is [52, Theorem 1]. The fact that C is integrable yields with (2.1) a linear/sublinear variance $\mathrm{Var}\,(\mathsf{P}(B_R)) = O(R^d)$, hence (i) is stronger in dimension $d \geqslant 5$, but (ii) is preferable in lower dimensions when one can prove the integrability of C.

Let us now give results in the two dimensional case, where we need a little bit more than mere hyperuniformity to obtain a linear matching cost.

Theorem 4.9 ([37, 52])**.** Let P a stationary hyperuniform point process of $\mathbb{R}^2$. Assume that either (i) we have the integrability

$$\int_{\mathbb{R}^2\setminus B_1} \ln(\|x\|)dx|\mathsf{C}(dx)| < \infty$$

or that (ii) for some $c, \varepsilon > 0$

$$\mathrm{Var}\,(\mathsf{P}(B_R)) \leqslant c\frac{R^d}{\ln(R)^{1+\varepsilon}}.$$

Then P is a L^2-perturbed lattice.

The relation between (i) and Theorem 4.7 about finite samples is the following: one can prove Theorem 4.9 by first showing that the finite samples $\mathsf{P}_n := \mathsf{P} \cap C_n$ indeed satisfy a linear transport cost as in (4.4). One of the difficulties is that $N := \#\mathsf{P}_n$ is not n^d a.s., hence one applies Theorem 4.7 to a renormalised lattice $\tilde{\mathsf{Z}}_n$ of Z_n with random mass N, and shows it has a good matching with P_n. Then one uses the compacity of the space of matchings between atomic measures with same intensity to show that asymptotically, we have a L^2 matching between Z^d and P.

The result (ii) is proved in [37]. The authors actually show that the conclusion holds under a more general condition, the *finite Coulomb energy* for P, implied by both (i) and (ii) (see their Proposition 1.7). When C has a density (at least at ∞), it seems that all these conditions are equivalent. [37] give also an example of a 2D hyperuniform point processes that does not have finite Coulomb energy, see also Example 3.1.

In conclusion, the dichotomy between perturbed lattices and disordered hyperuniform processes seems rather to be a continuum. In dimension 1, hyperuniformity is not enough to guarantee linear W_2^2 cost, but we do not investigate this question as 1D optimal transport tools are pretty specific.

A stationary allocation T naturally induces a *fair partition*, i.e. a partition of the space into cells of equal volumes. There are further links between hyperuniformity and fair partitions that we shall explore in a future version.

Chapter 5

Rigidity

Another intriguing phenomenon has been noticed for some seemingly disordered hyperuniform processes, that of rigidity. Ghosh and Peres [31] have shown that, for either $\mathsf{P} = \mathsf{P}_{\mathsf{GAF}}$ or $\mathsf{P} = \mathsf{P}_{\mathrm{Gin}}$, we have *number rigidity*: the number of points in a ball can be a.s. guessed from the outside configuration, i.e. for $R > 0$,

$$\mathsf{P}(B_R) \in \sigma(\mathsf{P} \cap B_R^c). \tag{5.1}$$

Note that this property is irrealistic for a standard process, or a process that is considered to be asymptotically independent. The proptotypical example is the Poisson process, where the independence of $\mathsf{P}1_{B_R}$ and $\mathsf{P}1_{B_R^c}$ makes (5.1) impossible. This number rigidity is once again reminiscent of lattices, in the sense that for "small enough" perturbations $U_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d$ with law μ, one expects that $\mathsf{Z}^{d,\mu}$ is number rigid. The shifted lattice (obtained for $\mu = \delta_0$) is an extreme example, as not only the number of points, but in fact all of $\mathsf{P}1_{B_R}$ can be inferred from $\mathsf{P}1^c_{B_R}$.

5.1 Heuristics and linear rigidity

Let us see how we can guess $\mathsf{P}(B_R)$ for a point process with low variance for linear statistics. Let f smooth with compact support and $f(0) = 1$. Then as $R \to \infty$, with Bienaymé-Tchebyshev inequality

$$\mathsf{P}(B_1) \approx \mathsf{P}(1_{B_1} f_R) = \mathsf{P}(f_R) - \mathsf{P}(f_R 1^c_{B_1}) \approx \mathbf{E}\mathsf{P}(f_R) \pm \sqrt{\mathrm{Var}\left(\mathsf{P}(f_R)\right)} - \mathsf{P}(f_R 1^c_{B_1}).$$

For P sufficiently hyperuniform (see Proposition 2.4), $\mathrm{Var}\left(\mathsf{P}(f_R)\right)$ goes to 0 and the right hand side gets arbitrarily close to an element of $\sigma(\mathsf{P}1_{B_1^c})$. This method has been initiated in [31] and generalised in [30].

For $\mathsf{P}_{\mathsf{GAF}}$, the hyperuniformity exponent is larger (see Lemma 3.2), Ghosh and Peres [31] show that one can furthermore guess the first moment of $\mathsf{P}1_{B_R}$ with a similar reasoning and $\tilde{f}(x) = xf(x)$:

$$\int x d\mathsf{P}_{\mathsf{GAF}} \approx R\mathbf{E}[\mathsf{P}(\tilde{f}_R)] \pm R\sqrt{\mathrm{Var}\left(\mathsf{P}(\tilde{f}_R)\right)} - R\mathsf{P}(\tilde{f}_R 1_{B_1^c}).$$

This phenomenon is called 1-rigidity, since one can infer the moment of order 1 based on the outside configuration. From this perspective, number rigidity can also be called 0-rigidity.

In both examples, the rigidity can be described as *linear*, in the sense that the quantity of interest (the moment of order 0 or 1 on the unit ball) can be approximated by external linear statistics $\mathsf{P}(f_R 1_{B_1^c})$. Generalising to higher moments leads to the following definitions.

Definition 5.1. Say that a wide sense stationary random measure M is k-rigid for $k \in \mathbb{N}$ iff for every $\mathbf{k} = (k_i) \in \mathbb{N}^d$ with $\sum_i k_i = k$,

$$\int \prod_{i=1}^{d} x_i^{k_i} d\mathsf{M} \in \sigma(\mathsf{M} 1_{B_1}^c).$$

Say that it is *linearly* k-rigid if for some kernels $h_n \in \mathscr{C}_c^\infty(B_1^c)$, we have the convergence in L^2 and a.s.

$$\mathsf{M}(h_n) \to \int \prod_{i=1}^{d} x_i^{k_i} d\mathsf{M}.$$

5.1.1 Necessary and sufficient conditions for linear k-rigidity

It turns out that linear rigidity on a compact subset A of $\mathbb{R}^d$ involves the tempered distributions which spectrum is supported by A, and those are by the Paley-Wiener theorem a subclass of the analytic functions. Exploiting this theory, the following results are derived in [50].

Let us first give a condition easy to state in some structurally more simple cases. We consider hereafter a wide sense stationary random measure M. Decompose its spectral measure S according to the Radon-Nykodym theorem: there is a non-negative measurable symmetric function $\mathsf{s} : \mathbb{R}^d \to \mathbb{R}_+$, with $\mathsf{S} = \mathsf{s}\mathscr{L}^d + \mathsf{S}_s$ and S_s is singular with respect to $\mathscr{L}^d$, we call s the *spectral density* of M.

Theorem 5.1. Let $k \in \mathbb{N}$. Assume that one of the following holds:

- (i) $k = 0$
- (ii) s is invariant under rotations
- (iii) s is *separable*, i.e. $\mathsf{s}(u) = \mathsf{s}_1(u_1) \dots \mathsf{s}_d(u_d), u \in \mathbb{R}^d$, for some measurable symmetric $\mathsf{s}_i \geqslant 0$.

Then M is linearly k-rigid iff

$$\int \frac{\|u\|^{2k}}{\mathsf{s}(u)} du = \infty. \tag{5.2}$$

Remark 5.1.
- This condition only bears on the continuous part of the spectral measure. It is reminiscent of Szegö and Kolmogorov's theorems on time series [79, 45], and there is indeed a relation, as we shall see at Section 5.2.
- This theorem shows that linear rigidity obeys some rules of monotonicity: for $\mathsf{s}' \leqslant \mathsf{s}$, k-rigidity for s' implies k-rigidity for s', and for $k' \leqslant k$, k-rigidity for s implies k'-rigidity for s.

Example 5.1. We have built at Section 4.1.1 a class of models P_p, for p a prime number, which can have an arbitrarily high hyperuniformity exponent. In particular, Theorems 4.3 and 5.1 yield that P_p is $(p-1)$-rigid.

When none of (i),(ii),(iii) holds, one can still state a more algebraic necessary condition, see [50]. We shall state some general necessary condition.

Proposition 5.1 ([50], Proposition 2)**.** Assume the spectral density s satisfies for some $c, p, \varepsilon > 0$:

- s has finitely many zeros $u_1, \dots, u_m \in \mathbb{R}^d$ and they have finite order, i.e.

$$\int_{B(u_i,\varepsilon)} \frac{\|u - u_i\|^p}{\mathsf{s}(u)} du < \infty$$

- s does not vanish too fast at ∞: for $u \notin \bigcup_i B(u_i, \varepsilon)$, $\mathsf{s}(u) \geqslant c(1 + \|u\|)^{-p}$

Then for $k \in \mathbb{N}$, M is not linearly k-rigid if (5.2) does not hold.

This yields the following characterisation of rigidity for DPPs, using (3.13) and Lemma 2.3:

Theorem 5.2 ([50], Theorem 5)**.** Let P a stationary DPP with locally square integrable Hermitian kernel K. Then P is number rigid in dimension $d = 1, 2$ if P is hyperuniform and for all $\varepsilon > 0$,

$$\int_{B_\varepsilon} \frac{1}{\mathsf{s}(u)} du = \infty,$$

and is not linearly k-rigid in any other situation.

As seen at Sections 3.3.3 and 3.3.4, the Sine_1 process is a DPP with kernel $K(x-y) = \frac{\sin(x-y)}{x-y}$ on $\mathbb{R}$, and the Ginibre processes with kernel $K(z,w) = \pi^{-1} e^{z\bar{w} - |z|^2/2 - |w|^2/2}$ on $\mathbb{C}$. Hence one can show with (3.13) that they are both number rigid. An example of a non-rigid hyperuniform DPP is given by the tensor product of the sine kernel

$$K(x,y) = \mathrm{sinc}(x_1 - y_1)\mathrm{sinc}(x_2 - y_2), (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2,$$

because the spectral measure is continuous with density

$$\mathsf{s}(u_1, u_2) = |u_1| + |u_2| + o(u_1) + o(u_2) \text{ as } u \to 0,$$

hence $\int_{B_\varepsilon} \|u\|^{-1} du < \infty$.

The main reason why DPPs cannot be more then number rigid, and only in dimension 1 or 2, and why it is in general difficult to find such processes in general, is Lemma 2.3, that yields $\mathsf{s}(u) \geqslant \sigma \|u\|^2$ as $u \to 0$; a similar phenomenon occurs for independently perturbed lattices.

5.1.2 Linear and non-linear rigidity of perturbed lattices

We can deduce from Example 2.4, Lemma 2.3 and Theorem 5.1 the following. For μ a non-negative measure on $\mathbb{R}^d$, call μ_c its component continuous with respect to Lebesgue measure and say μ has a continuous part if μ_c is not the null measure.

Proposition 5.2. Let μ a symmetric probability measure on $\mathbb{R}^d$ with a continuous part. Assume furthermore that μ is symmetric or has a finite second moment. Let $\mathsf{Z}^{d,\mu}$ the corresponding perturbed lattice. Then $\mathsf{Z}^{d,\mu}$ is not linearly k-rigid if $k \geqslant 1$ or if $d \geqslant 3$.

For $k = 0, d \in \{1, 2\}$, let $\psi(u) = \int_{\mathbb{R}^d} e^{-iu\cdot t} \mu(dt)$. Then $\mathsf{Z}^{d,\mu}$ is linearly number rigid iff for all $\varepsilon > 0$

$$\int_{B_\varepsilon} \frac{1}{1 - |\psi(u)|^2} du = \infty.$$

Proof. We use the spectral measure expression at (2.5) with Theorem 5.1 and Proposition 5.1. Let f the density of μ_c, i.e. $\mu = f\mathscr{L}^d + \mu_s$ for some singular measure μ_s, and $p := \mu_s(\mathbb{R}^d) < 1$. The Rieman-Lebesgue lemma yields that

$$\int e^{-iu\cdot x} f(x) dx \to 0$$

as $\|u\| \to \infty$, hence for $\|u\|$ sufficiently large, $|\psi(u)|^2 \leqslant p + \varepsilon < 1$ for $\varepsilon > 0$ sufficiently small. Furthermore, $|\psi(u)| \neq 1$ for $u \in \mathbb{R}^d$ because μ cannot be concentrated on $\{x : \langle x, u\rangle \in 2\pi\mathbb{Z}\}$ since the continuous part of μ has a support with positive Lebesgue measure. Therefore, the spectral density s satisfies the two first conditions in Proposition 5.1.

Lemma 2.3 yields that for $\|u\|$ sufficiently small, $|\psi(u)|^2 \leqslant 1 - \sigma\|u\|^2$ for some $\sigma > 0$. Hence

$$\int \frac{\|u\|^{2k}}{1 - |\psi(u)|^2} du < \infty$$

if $k \geqslant 1$ or $d \geqslant 3$. Then for $k = 0$, Theorem 5.1-(i) yields number rigidity iff the integral is infinite. □

The symmetry assumption on μ can be replaced by a moment assumption.

In disordered models, most rigidity proofs actually yield linear rigidity [31, 16, 29], but other arguments use a different route. In [22], the authors prove number rigidity for Coulomb gases using DLR equations, and [16] proved that this rigidity is actually linear. Proving that a model does not experience rigidity actually gives information on its covariance decay [50].

For perturbed lattices, conversely, many models seem to experience non-linear rigidity. Consider for instance in any dimension a probability measure μ supported by $B_{1/4}$. Then clearly the resulting process $\mathsf{Z}^{d,\mu}$ is number rigid, as one can unambiguously recover the underlying structure and find the $\mathbb{Z}^d$-neighbours of each particle, and in dimension $d \geqslant 3$, by the results above, this number rigidity is not linear.

There is also the notable example of [65] that proves with an ad-hoc argument that in dimension $d = 3$, a lattice perturbed by Gaussian variables with sufficiently small variance is number rigid, and this rigidity is also necessarily non-linear by Proposition 5.2.

Theorem 5.3 ([31])**.** Let $\mu = \mathscr{N}(0, \sigma^2 I_d)$ on $\mathbb{R}^d$. Then $\mathsf{Z}^{d,\mu}$ is number rigid if and only if $d \in \{1, 2\}$ or if $d = 3$ and $\sigma < \sigma_c$ for some $\sigma_c \in (0, \infty)$.

Question 5.1. What are the number rigid lattices in dimension $d \geqslant 3$?

5.2 Stealthy processes and maximal rigidity

We saw that the speed of decay in 0 of the structure factor determines the degree of rigidity of the random structure. An extreme case is when the spectrum vanishes identically on a neighbourhood of the origin.

Definition 5.2. A wide sense stationary random measureis *stealthy* if its spectral density vanishes on a non-empty open set.

We dropped the assumption that the gap contains 0 as it might be non-relevant mathematically, but is often present in the physics litterature. Stealthy point processes have attracted considerable interest in physics, in particular for their peculiar optical properties in condensed matter physics, see the non-exhaustive bibliographic sample [83, 90, 89, 80, 82, 61, 72] . It bears relations with the concept of blue noise in image analysis [88], quantization in numerical probability [44], or numerical integration, since they yield superpolynomial decay for the variance of smooth linear statistics by Proposition 2.1.

To the author's knowledge, the only known mathematical examples of stealthy point processes on $\mathbb{R}^d$ are finite unions of shifted lattices as in Example 3.1 (with only finitely many a_m). The interest of physicists is to put in evidence stealthy models of point processes which are also disordered, e.g. isotropic and mixing, through some simulation procedures.

Stealthy random measures exhibit a very strong form of rigidity, called maximal rigidity, where the restriction of M on a subset $B \subset \mathbb{R}^d$ can be completely inferred from values of M outside B [30], it is also a consequence of Theorem 5.1.

Definition 5.3. Say that a wide sense stationary random measure M is *maximally rigid* on $B \subset \mathbb{R}^d$ if $\mathsf{M}1_B \in \sigma(\mathsf{M}1_B^c)$.

Stealthy measures are actually even more rigid than that, in that the set B can be non-compact. Call strictly convex cone a closed cone $C \subset \mathbb{R}^d$ which does not contain both x and $-x$ for some $x \neq 0$, or equivalently such that for some $x_0 \in C$,

$$\inf_{x \in C, \|x\|=1} x_0 \cdot x > 0.$$

Proposition 5.3. A stealthy wide sense stationary random measureis maximally rigid on any strictly convex cone C, i.e. $\mathsf{M}1_C \in \sigma(\mathsf{M}1_{C^c})$.

This result is proved in the forthcoming paper [51], where it is also shown that maximal rigidity on a bounded set persists when the stealthiness hypothesis is relaxed to that of a deep zero. It is argued

there that maximal rigidity comes from an instance of the uncertainty principle in harmonic analysis called *annihilating pairs*, putting in relation a set B with a set $\hat{B} \subset \mathbb{R}^d$ such that if a test function f is supported by B and $\hat{f}$ is supported by $\hat{B}$, then $f \equiv 0$.

Stealthy processes have other striking features that distinguish them from the common point processes, even the perturbated lattices. The bounded holes property [90], giving somehow an a.s. lower bound on the density, was proved in [30, Theorem 1.1]. They also give an upper bound on the density with their Theorem 1.3. Let us give a simple statement and proof.

Proposition 5.4. Let M a stealthy non-negative wide sense stationary random measure. Then there exist finite $A, R, \eta > 0$ such that a.s., for every $x \in \mathbb{R}^d$, $\mathsf{M}(B(x,1)) \leqslant A$ and $\mathsf{M}(B(x,R)) \geqslant \eta$.

In the case where M is a point process, $\mathsf{M}(B(x,R)) \in \mathbb{N}$, hence the second statement gives the bounded holes property: there is no ball with radius R without point.

Proof. Assume without loss of generality unit intensity. The stealthiness yields $\varepsilon > 0$ such that, with (2.2), for φ a non-negative Schwarz function supported by B_ε, $\mathrm{Var}\,(\mathsf{M}(\hat{\varphi})) = 0$. Hence a.s.

$$\mathsf{M}(\hat{\varphi}) = \mathbf{E}\mathsf{M}(\hat{\varphi}) = \int \hat{\varphi}.$$

To make sure $\hat{\varphi} \geqslant 0$, take $\varphi = \varphi_0 \otimes \varphi_0$ for some smooth φ_0 supported by $B_{\varepsilon/2}$. Assume also $\varphi_0 \geqslant 0$ and $\varphi(0) > 0$. Also, $\hat{\varphi}(0) = \int \varphi > 0$ and there is $\kappa, a > 0$ such that

$$\hat{\varphi} \geqslant \kappa 1_{B_a}.$$

If $a < 1$, one can still shrink φ by factor a, hence $\hat{\varphi}$ expands by a factor $1/a$, meaning we can assume without losing the previous sign properties that $a = 1$. By positivity of M and $\hat{\varphi} = |\hat{\varphi}_0|^2$,

$$\kappa \mathsf{M}(B_1) \leqslant \mathsf{M}(\hat{\varphi}) = \int \hat{\varphi} < \infty \tag{5.3}$$

which yields the first part of the statement. Since φ is non-negative, $\sup \hat{\varphi} = \hat{\varphi}(0)$. We have for $R > 0$

$$0 < \int \hat{\varphi} = \mathsf{M}(\hat{\varphi}) \leqslant \hat{\varphi}(0)\mathsf{M}(B_R) + \mathsf{M}(\hat{\varphi} 1_{B_R^c}). \tag{5.4}$$

Since $\hat{\varphi}$ has fast decay and the $B(\mathbf{k},1), \mathbf{k} \in d^{-1/2}\mathbb{Z}^d$, cover the space, we have for some $c > 0$

$$\hat{\varphi} 1_{B_R^c}(x) \leqslant \sum_{\mathbf{k} \in d^{-1/2}\mathbb{Z}^d \setminus B_R} c\|\mathbf{k}\|^{-d-1} 1_{B(\mathbf{k},1)}(x), x \in \mathbb{R}^d.$$

It finally yields with stationarity, (5.3) and (5.4)

$$\int \hat{\varphi} \leqslant \hat{\varphi}(0)\mathsf{M}(B_R) + \sum_{\mathbf{k} \in d^{-1/2}\mathbb{Z}^d \setminus B_R} c\|\mathbf{k}\|^{-d-1} \mathsf{M}(B(\mathbf{k},1)) \leqslant \hat{\varphi}(0)\mathsf{M}(B_R) + cA \sum_{\mathbf{k} \in d^{-1/2}\mathbb{Z}^d \setminus B_R} \|\mathbf{k}\|^{-d-1}.$$

The series converges, hence the rest goes to 0, and $\mathsf{M}(B_R) \geqslant \eta := \int \hat{\varphi}/2\hat{\varphi}(0)$ for R large enough. □

The proof can be refined to optimise the values of η, R, A, see [30, 17].

5.2.1 Further questions on stealthy point processes

Spectral considerations therefore give the main properties of stealthy random measures. If on the other hand one focuses on the class of point processes, which is the prominent class studied in the literature, deeper questions arise.

Experimenters in physics and image analysis have generated very large point samples that seem to be disordered and exhibit a stealthy behaviour, i.e. a flat spectrum at the origin [44, 61, 72]. The existence of such models has not been established mathematically, except some toy models such as unions of shifted lattices, see Example 3.1. These examples are actually non hardcore, i.e. there are arbitrarily close pairs of particles, contrary to simulated samples.

Question 5.2. Are there disordered stealthy hyperuniform processes in dimension $d \geqslant 2$?.

Regarding the level of *disorder*, mixing might already be good, even if of course Brillinger mixing would be ideal (see Section 2.5). Here again, this question is trivial if one considers random measures in general. A Gaussian field which spectral measure is a non-negative Schwarz function supported by, say, $B_2 \backslash B_1$, is stealthy and as mixing as possible (it is actually 1-dependent). In passing, an interesting phenomenon can occur for such continuous m-dependent models despite their mixing properties: they can exhibit maximal rigidity on a ball, see [51], with a phase transition according to the ball radius. The question is much more intricate for point processes, and largely open on the mathematical side.

Another question concerns the transport properties of stealthy processes. Using Section 4.2, it might be possile to show that for each $p \geqslant 1$, stealthy point processes are L^p perturbed lattices, i.e. of the form Z^{U} where $\mathsf{U} = \{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^d\}$ is stationary and $\mathbf{E}\|U_0\|^p < \infty$.

Question 5.3. Can a stealthy point process always be represented as a L^∞-perturbed lattice, i.e. with the $U_{\mathbf{k}}$ a.s. bounded by some finite $A > 0$?

This would corroborate the upper and lower density bounds of Proposition 5.4, and it is not contradicted by Example 3.1 with a finite number of a_i, which is actually the more complex examples we have of a stealthy point process to date.

A final question is about the class of sets where stealthy processes are maximally rigid. We saw that they are maximally rigid on closed strictly convex cones, but they might be interpolable on larger classes, such as complements of convex cones, called major cones, like for quasicrystals (see below). For instance, stealthy processes from Example 3.1 are maximally rigid on the complement of an infinite large strip, i.e. on $(\mathbb{R}\backslash[-A, A]) \times \mathbb{R}^{d-1}$, for A sufficiently large.

Question 5.4. Are stealthy point processes maximally rigid on major cones? Or on larger sets?

5.3 Quasicrystals

We saw at Section 3.5 that quasicrystals provide a great deal of hyperuniform point processes, more on the ordered side. The cut-and-project models are in general not stealthy, as their spectral measure has a dense support. Nevertheless, as we saw before, it is often the spectral density s that bears informations on the rigidity behaviour. In this respect, quasicrystals are extremely rigid since their spectral density is 0. We give the following result, which is applicable to random measures with zero spectral density. Note that such examples need not be hyperuniform.

Theorem 5.4 ([51])**.** Let M a wide sense stationary random measure. If s is purely atomic, M is maximally rigid on the complement of any cone with non-empty interior.

Hence in our current knowledge, for quasicrystals, the maximal rigidity situation is even more extreme than for stealthy processes: it is enough to know the process on an arbitrarily small convex cone with non-empty interior to uniquely determine its values on the whole space.

Chapter 6

Quantitative aspects and the `structure-factor` package

In the forthcoming long version of this survey, we shall explore quantitative aspects of hyperuniformity, such as the problem of estimating the structure factor and detecting mere hyperuniformity, or simulating large samples. See the online slides for a preview `https://helios2.mi.parisdescartes.fr/~rlachiez/recherche/talks/slides-hu.pdf`.

We shall also give a longer introduction to the package `structure-factor`, that aims at generating large samples and estimate the covariance and spectral measures, with a focus on hyperuniformity detection. See the illustrating paper [33] and the documentation at `https://pypi.org/project/structure-factor/`.

We will describe the existing methods used to generate hyperuniform samples, either exact or non-exact. There does not seem to exist a procedure to obtain large disordered samples in time $n\ln(n)$, which raises a final question.

Question 6.1. Find a stationary disordered hyperuniform point process P such that $\mathsf{P} \cap \mathbf{C}_n$ can be generated in time $n\ln(n)$, or more generally a procedure to generate samples P_n with complexity $n\ln(n)$ such that the $\mathsf{P}_n, n \geqslant 1$ are asymptotically hyperuniform (as in Theorem 3.2).

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