Quantum groups I : Basic definitions

10 avril 2012

1 Algebraic side

1.1 Introduction : braid group and Yang-Baxter equation

Why ? Objects that we will meet again, and also part of the initial motivations.

Braid group The braid group with n strands is defined non rigourously as the group whose elements are ...braids with n numbered strands, and composition law is just gluing the strands of two braids according to their numbers. Generators of this group are given by the braids with only one crossing of two adjacent strands. One can see that two such generators commute whenever they do not affect a same brand, by drawing a picture one can see a relation between two generators sharing one brand. Algebraically, the group B_n is given by n generators $s_1, \dots, s_{n-1}$ subject to the relations :

1. $s_i s_j = s_j s_i$ for $|i-j| > 1$ (commutativity of two "disjoint" generators)
2. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (relation bewteen two "adjacent" generators) [YB1]

It is a (hard) theorem from Artin (1925) that the intuitive description (which can be turned on a rigourous topological-minded definition of a group) gives the same result as this algebraically-minded description. There is a natural surjection of B_n onto S_n (permutation group), by sending s_i on the transposition $(i, i+1)$. In fact, S_n admits a group presentation similar to the preceeding one for B_n, by simply adding the relation $s_i^2 = 1$ for each $i = 1 \dots n-1$. This difference induces two major differences : B_n is torsion-free and (thus) infinite.

Let us now consider the one-dimensional representations ρ of B_n. The condition [YB1] implies that ρ is constant on the generators. Let $e^{i\theta}$ be this constant value, then ρ_θ is a variation on the cases $\theta = 0$ and $\theta = -\pi$, which pass to the quotient S_n and correspond to the trivial representation (bosonic statistics) and the signature (fermionic statistics).

Yang-Baxter equation A way of constructing higher-dimensional representations is to start with a vector space V, an automorphism R of $V \otimes V$, and to let the generator s_i act on $V^{\otimes n}$ by :

$$s_i(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \dots R(v_i \otimes v_{i+1}) \cdots \otimes v_n) \tag{1}$$

This gives a representation if and only if R is compatible with the relation [YB1], which translates into the following condition (expressed in $V^{\otimes 3}$) :

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \tag{2}$$

We will use the notation $R_{i,i+1}$ to designate the action of R on the $i, i+1$-th coordinate, so we can rewrite 2 as $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$. This will be the first occurence of the Yang-Baxter equation. Every solution to this equation gives rise to a representation of B_n for all n. Observe that a trivial solution to 2 is given by the flip $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$, which corresponds to the signature representation of B_n. A less trivial example, which will also be the first occurence of a *deformation* is given by : pick an invertible scalar q in the base field of a f.d. vector space V, and define $R(e_i \otimes e_j)$ as : $qe_i \otimes e_i$ if i$i = j$, $e_j \otimes e_i$ if $i > j$ and $e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j$ for $j > i$.

1.2 Hopf algebras

Let F be a (covariant) functor from the category of (commutative) $\mathbb{C}$-algebra to the category of groups defined by $F = \mathrm{Hom}(A_F, .)$ (an affine groups scheme). The algebra A_F carries an extra-structure : since

$F(\mathbb{C})$ is a group, $\mathrm{Hom}(A_F, \mathbb{C})$ possesses a neutral element ϵ which is a distinguished morphism $A_F \longrightarrow \mathbb{C}$. There is also a map

$$\mathrm{Hom}(A_F, \mathbb{C}) \times \mathrm{Hom}(A_F, \mathbb{C})(\cong \mathrm{Hom}(A_F \otimes A_F, \mathbb{C})) \longrightarrow \mathrm{Hom}(A_F, \mathbb{C})$$

which gives raise to a map $\Delta : A_F \longrightarrow A_F \otimes A_F$. Moreover, the inversion rule in the group corresponds to a map

$$\mathrm{Hom}(A_F, \mathbb{C}) \longrightarrow \mathrm{Hom}(A_F, \mathbb{C})$$

which corresponds to a map $S : A_F \longrightarrow A_F$. The axioms obtained are the one of a Hopf algebra : an algebraic object which has both an algebra structure (in the usual sense), and a co-algebra structure (obtained by reversing the arrows in the definition of an algebra), and also (which is the difference between a "bialgebra" and a "Hopf algebra") this inversion morphism S (called antipode) which recalls the "groupic" origin of the coalgebra structure. Note that, conversely, a (commutative) Hopf algebra gives raise to a group affine scheme.

Definition An Hopf algebra (over $\mathbb{C}$) $(A, m, \eta, \Delta, \epsilon, S)$ is a $\mathbb{C}$-module A such that :

1. (A, m, η) is a (unital) $\mathbb{C}$-algebra ($\eta : k \longrightarrow A$ gives the unit).
2. (A, δ, ϵ) is a (unital) $\mathbb{C}$-coalgebra. The map $\Delta : A \longrightarrow A \otimes A$ and the map $\epsilon : A \longrightarrow k$ satisfy
   (a) Coassociativity : $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$
   (b) Neutrality of ϵ : $(\epsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id}$
   (c) Compatibility of the structures : Δ, ϵ and m, η are morphisms of algebra and coalgebra respectively.
   (d) Antipode : the antipode S is an anti-morphism (for both the product and coproduct), and $m \circ (S \otimes \mathrm{Id}) \circ \Delta = m \circ (\mathrm{Id} \otimes S) \circ \Delta = \eta \circ \epsilon$.

Elementary properties of Hopf algebras Fix an Hopf algebra H. Let τ be the flip $\tau : H \otimes H \longrightarrow H \otimes H, v \otimes w \mapsto w \otimes v$. The Hopf algebra H is *cocommutative* if $\tau \circ \Delta = \Delta$. Let us a show a simple general proposition on Hopf algebras to use the definitions.

Proposition 1. *If H is either commutative or cocommutative, then the antipode S is involutive.*

Démonstration. The axioms of the antipod can be restated by saying that S is the (two-sided) inverse of Id for the convolution product on $End(H)$ (as a vector space) given by :

$$f \star g = m \circ (f \otimes g) \circ \Delta \tag{3}$$

whose neutral element is $\eta \circ \epsilon$. Such an inverse is unique. But by applying S to the left (resp. to the right) to $m \circ (S \otimes \mathrm{Id}) \circ \Delta = \eta \circ \epsilon$, we get - S being an anti-morphism of bialgebra - that

$$\tau \circ m \circ (S^2 \otimes S) \circ \Delta = S \circ \eta \circ \epsilon$$

resp.

$$m \circ (S^2 \otimes S) \circ \tau \circ \Delta = \eta \circ \epsilon \circ S$$

The right-hand side is always equal to $\eta \circ \epsilon$, and if H is commutative or co-commutative one of the two former equations expresses the fact that S^2 is a left inverse to S for the convolution product, and thus must be equal to Id. □

Some "classical" examples We quote some classical examples of Hopf algebras. They are classical in the sense that they are all commutative and/or cocommutative, whereas a quantum group will be seen as a non-commutative, non-cocommutative Hopf algebra.

1. For a group G, the group algebra $\mathbb{C}G$, spanned by the elements of g as a $\mathbb{C}$-vector space, with the multiplicative law extending the one on G ($\delta_g \delta_h := \delta_{gh}$). In this case the coproduct is given by $\Delta(g) = g \otimes g$. This comes from the group structure on $\mathrm{Hom}(\mathbb{C}G, \mathbb{C})$, given by pointwise multiplication.
2. If G is finite, the dual $\mathcal{F}G$ of $\mathbb{C}G$ (the $\mathbb{C}$-valued functions on G) is an Hopf algebra with coproduct $\Delta f(g \otimes h) = f(gh)$. This comes from the group structure on $\mathrm{Hom}(\mathcal{F}G, \mathbb{C})$.

3. Let g be a finite-dimensional Lie algebra over $\mathbb{C}$, the universal enveloping Lie algebra is defined as the algebra generated by 1 and the elements of a basis of g modulo the relations $xy - yx = [x, y]$ (it is then a Lie algebra with the commutator bracket). We can endow $U(g)$ with an Hopf algebra structure, with coproduct $\Delta x = x \otimes 1 + 1 \otimes x$, $\epsilon x = 0$ (for x non-scalar) and $Sx = -x$. Note that since this Hopf algebra is not commutative, it does not come from an affine group scheme. The map Δ is rather the composition of the diagonal morphism $x \mapsto (x, x)$ and the isomorphism $U(g \oplus g) \cong U(g) \otimes U(g)$. Nonetheless, $U(g)$ is co-commutative.

We will have at hearth to deform these examples to obtain truly "quantum" objects, but not too badly so that the Hopf algebra remains "almost co-commutative" in the following sense.

Almost co-commutative Hopf algebra An Hopf algebra A is called "almost co-commutative" if the following is true : there exists an invertible element $R \in A \otimes A$ such that $\tau \circ \Delta(a) = R\Delta(a)R^{-1}$ (of course, a commutative, almost co-commutative Hopf algebra is co-commutative). This assumption has good consequences on the representation theory of H (cf. later), e.g. the tensor product $V \otimes W$ of two representations V and W is then isomorphic to $W \otimes V$ (as H-module). The antipode is, in general, not involutive, but satisfies the following property :

Proposition 2. *If (A, R) is almost co-commutative, then "S^2 is almost the identity" in the sense that $u = m(S \otimes Id)(\tau(R))$ is an invertible element such that $S^2(a) = uau^{-1}$ for all element a.*

The proof is a (non-trivial) computation.

1.3 Quantum double and deformation

Morally, here is what we do : given an Hopf algebra and its dual, we define an Hopf algebra structure on their tensor product, with a coproduct governed by the duality bracket. Then, we observe that one can replace this bracket by any "pairing" (in a to-be-precised sense) and this allows us to define non-cocommutative structures on the "quantum double". We then apply this construction to the universal enveloping algebra of sl_2.

Looking for a twist At the end of the last section, we defined the notion of almost co-commutative Hopf algebra. The search for an almost co-commutative structure is equivalent to the one for an invertible element R that conjugates the coproduct and the flipped coproduct. More generally, we may try to construct coproducts by looking at $\Delta_F(a) = F\Delta(a)F^{-1}$ for an invertible element F in $A \otimes A$ (where A is an Hopf algebra). Such an element must satisfies some properties to insure the coassociativity of Δ_F and the other axioms.

Lemme 1. *We give a sufficient condition :*

1. *A sufficient condition for Δ_F to be coassociative is $F_{12}(\Delta \otimes Id)(F) = F_{23}(id \otimes \Delta)(F)$*
2. *If we ask furthermore that $(\Delta \otimes Id)(F) = F_{13}F_{23}$ and $(id \otimes \Delta)(F) = F_{13}F_{12}$ then the preceeding condition is the Yang-Baxter equation for F*

Preuve. Simply check. □

We are now left with looking to such elements. It turns out that there is a canonical way to construct a valid F when dealing with the tensor product of an Hopf algebra with its dual.

The dual quantum double construction Let A be a finite-dimensional (as vector space) Hopf algebra (with invertible antipode). Let A^* be the dual Hopf algebra (defined the natural way, with more or less permuting the roles of m and Δ by transposition) and A^{op} the opposite algebra (with flipped multiplication), and let $\tilde{A} = A^* \otimes A^{op}$. Pick a basis (e_i), its dual basis (e_i^*) and define (indepently of this choice) :

$$\tilde{F} := \sum (1_{A^*} \otimes e_i) \otimes (e_i^* \otimes 1_A) \tag{4}$$

Lemme 2. *$\tilde{F}$ is invertible and $\tilde{F}^{-1} = \sum (1_{A^*} \otimes S^{-1}e_i) \otimes (e_i^* \otimes 1_A)$ satisfies the condition of the preceeding lemma.*

Preuve. Computation with F, and taking the inverse. To find $\tilde{F}^{-1}$, rewrite $\tilde{F}$ as $1_{A^*} \otimes (\sum e_i \otimes e_i^*) \otimes 1_A$ in $A^* \otimes (A^{op} \otimes A^*) \otimes A^{op}$. Under the identification $A^{op} \otimes A^* \cong End(A^{op})$, the product on $A^{op} \otimes A^*$ is the convolution product on $End(A^{op})$ given by the coalgebra structure on A. Inverting $\tilde{F}$ is then equivalent to invert $\sum e_i \otimes e_i^* = \mathrm{Id}$, which can be done since the antipode is supposed to be invertible (if S is the invertible antipode for an Hopf algebra H, then H^{op} is an Hopf algebra with antipode S^{-1}). □

Proposition 3. *We then get an Hopf algebra H on $\tilde{A}$ by conjugating the tensor coproduct by $\tilde{F}^{-1}$.*

Preuve. Something is to be checked concerning the antipode (which in fact needs to be defined). $\square$

The quantum double construction We consider now the dual Hopf algebra $H^* = A \otimes A^{*,cop}$ of the Hopf algebra we just obtained. Let us describe it :

1. As a coalgebra, it is the tensor product of the coalgebra A and the co-opposite coalgebra A^*.
2. It admits A and $A^{*,cop}$ as Hopf subalgebras
3. The product is given by : for all l in A^* and a in A, we have
   (a) $(a \otimes 1)(1 \otimes l) = a \otimes l$, whereas :
   (b) $(1 \otimes l)(a \otimes 1) = \sum \langle l_1, S^{-1}a_1\rangle \ \langle l_3, a_3 \ \rangle a_2 \otimes l_2$ with the duality bracket $\langle , \rangle$.

The multiplication may seem weird. Remember that we twisted the co-algebra structure on H, than have taken the dual : now it is the algebra structure which is unusual.

Example : if G is a group and $A = \mathcal{F}(G)$ is the Hopf algebra of functions on G. Then $A^{*,cop}$ is simply the group algebra $\mathbb{C}[G]$, and the quantum double $D(A)$ is isomorphic (as an algebra) to the crossed product of $\mathcal{F}(G)$ by G acting by conjugation. We can convince ourselves of this fact : in the crossed product $\mathcal{F}(G) \rtimes G$, we have indeed $(F, 1) \times (1, g) = (F, g)$ and $(1, g) \times (F, 1) = (g.F, g)$ where $g.F(x) = F(g^{-1}xg)$. The expression given for the product is, on the other hand, $\sum \langle g_1, S^{-1}F_1\rangle \ \langle g_3, F_3 \ \rangle F_2 \otimes g_2$, with $g_1 = g_2 = g_3 = g$, hence $\sum \langle g^{-1}, F_1\rangle \ \langle g, F_3 \ \rangle F_2 \otimes g$ and the left component of this tensor is indeed $F(g.g^{-1})$.

Generalized double The duality bracket in the former description can be replaced by any bilinear application $\phi : A \otimes A^*$, provided that φ satisfies appropriate conditions. This leads us to the following definition :

Définition 1 (Hopf pairing)**.** *Let A and B be Hopf algebras with invertible antipodes. A Hopf pairing between A and B is a bilinear form $\varphi : A \times B \longrightarrow \mathbb{C}$ such that :*

1. $\varphi(a, bb') = \sum \varphi(a_1, b)\varphi(a_2, b')$ *idem pour* $\varphi(aa', b)$.
2. $\varphi(Sa, b) = \varphi(a, S^{-1}b)$
3. $\varphi(a, 1_B) = \epsilon(a)$ *and idem for* b.

These conditions express the "adjoint" relation between the algebra structure on B and the coalgebra structure on A, and vice-versa.

Re-writing the proof we have not done of the quantum double construction, with the duality bracket replaced by a Hopf pairing as defined, we get :

Théorème 1. *Let A and B be Hopf algebras and $\varphi : A \times B \longrightarrow \mathbb{C}$ a Hopf pairing. Then there is a Hopf algebra structure on $A \otimes B$ such that :*

1. *As a coalgebra, it is the tensor product of the coalgebra A and B.*
2. *It admits A and B as Hopf subalgebras*
3. *The product is given by : for all l in A and a in B, we have :*
   (a) $(a \otimes 1)(1 \otimes l) = a \otimes l$, *whereas :*
   (b) $(1 \otimes l)(a \otimes 1) = \sum \varphi(l_1, S^{-1}a_1) \ \varphi(l_3, a_3)a_2 \otimes l_2$.

We would like to know : how easy is it to construct such an Hopf pairing ? Because we would like to deform existing Hopf algebras. The following lemma provides us a general result for existence of non-trivial Hopf pairing on a free Hopf algebra (the work will then be to check when such an Hopf pairing agrees with the relations we put) :

Lemme 3. *Let A be a free algebra generated by $a_1, \ldots, a_p$ such that $\Delta(a_i)$ is a linear combination of $a_r \otimes a_s$ for each i. Take B with the same assumptions $(b_1, \ldots, b_q)$. Fix pq scalars λ_{ij}. Then there is a (unique) Hopf pairing $\varphi : A \times B \longrightarrow \mathbb{C}$ given by $\varphi(a_i, b_j) = \lambda_{ij}$.*

Preuve. Simply define $\varphi(a_{i_1} \ldots a_{i_r}, b)$ as $\sum \varphi(a_{i_1}, b_{(r)}) \ldots \varphi(a_{i_r}, b_{(1)})$ (and vice-versa). Then extend the definition on all A and B. We crucially use the freeness and the fact that the terms that appear in the iterated coproducts are themselves basis elements. $\square$

1.4 The quantized enveloping algebra

We now apply the quantum double construction (with the freedom stated by the last lemma) to the quantization of $U(sl(2))$.

Reminder on $sl(2)$ Let $sl(2)$ be the Lie algebra of complex 2×2 matrices with trace 0. A basis of $sl(2)$ is given by $E = E_{1,2}$, $F = E_{2,1}$ and $H = E_{11} - E_{22}$. The relations are $[E, F] = H$, $[H, E] = 2E$ and $[H, F] = -2F$. The situation is identical for the universal enveloping algebra $U(sl(2))$.

Construction of $U_q(sl(2))$ Let q be an indeterminate, and choose as ground field the field of fractions $\mathbb{C}(q)$ (observe that so far, we can replace $\mathbb{C}$ by an arbitrary field k in the definitions and results). Let U_+ be the $\mathbb{C}(q)$-algebra generated by three elements E, K and K^{-1} subject to the relations expressing that K is the inverse of K^{-1} and the following commutation relation :

$$KE = q^2 EK \tag{5}$$

Define similarly U_- with F, K', K'^{-1} and $K'F = q^-2FK'$. The co-algebras structures put on U_+ and U_- are : $\Delta(K) = K \otimes K$ (idem for K') and $\Delta(E) = E \otimes 1 + K \otimes E$, $\Delta(F) = F \otimes K'^{-1} + 1 \otimes F$. Explain doubling the Cartan subalgebra etc ? We can define a pairing if we stay careful about the relations put on U_+ and U_-. We choose :

1. $\varphi(E, F) = -\frac{1}{q-q^{-1}}$
2. $\varphi(E, K') = \varphi(F, K) = 0$
3. $\varphi(K, K') = q^{-2}$

The first choice is arbitary, and merely a convention. The second choice helps φ to be consistent with the relations on the algebras. The third choice will guarantee the following : in the double quantum construction, we have $KF = q^{-2}FK$, $K'E = q^2EK'$ and $[E, F] = \frac{K-K'^{-1}}{q-q^{-1}}$. To obtain the "real" deformed universal enveloping algebra, one then quotients by the annihilator of φ, and - more important - by the ideal $(K - K')$, because one wants to identify these two copies of K. Note that one can extend the construction to construct $U_q(sl(n + 1))$ for any $n \geq 1$.

2 Analytic side

2.1 Motivation

A constant idea in "non-commutative geometry" is to obtain non-commutative analoguous of classical objects by considering the algebras of functions defined on these objects, and allow them to be non-commutative. A celebrated example is the one of the $C^* - algebras$. If X is a compact (Hausdorff) topological space, its algebra of function $C(X)$ is endowed with an involution $f \mapsto \bar{f}$ and a (complete) norm $||.|| = sup_{x \in X}|f(x)|$ such that $||f\bar{f}|| = ||f||^2$. These are the axioms for a C^*-algebra : an algebra with an involution, complete for a norm such that $||xx^*|| = ||x||^2$. The Gelfand-Naimark theorem states that every commutative C^*-algebra A is the algebra of functions on a topological space $Spec(A)$, which is locally compact in general, and compact if and only if A is unital. The points of $Spec(A)$ are the characters of A, i.e. the algebra morphisms $A \longrightarrow \mathbb{C}$ (which can be thought of as "evaluation" morphisms). A general, non necessarily commutative, C^*-algebra will be then considered as a "non-commutative" topological space.

The definition of a compact quantum group will be following the same idea : it is a compact topological space, hence a unital C^*-algebra, but it carries an extra structure, which reflects on its algebra of function (the C^*-algebra) through a bialgebra structure. Some problems of topological nature though arise in order to define a somehow general topological quantum group (fortunately it turns out that these problems are relatively mild in the compact case).

2.2 Quantum group

Let $G = (A, \Phi)$ with A a (separable) unital C^*-algebra and $\Phi : A \longrightarrow A \otimes A$ a unital *-homomorphism. We ask that Φ satisfies the axiom of co-associativity. We replace the antipods axioms by the following requirement :

1. The sets $\{(b \otimes I)\Phi(c) : b, c \in A\}$ and $\{(I \otimes b)\Phi(c) : b, c \in A\}$ are dense in $A \otimes A$.

Why this hypothesis ? The existence of a coproduct corresponds morally to the existence of an associativ Te composition law on the underlying "quantum space", which hence is *a priori* only a semi-group. However, we have :

Proposition 4. *Let G be a compact (classical) semi-group, $A = C(G)$ and Φ the natural comultiplication. If the sets $\{(b \otimes I)\Phi(c) : b, c \in A\}$ and $\{(I \otimes b)\Phi(c) : b, c \in A\}$ are dense in $A \otimes A$, then G has the cancellation property.*

Preuve. Note that by applying the involution, we have a symmetric assumption on $\Phi(c)(I\otimes b)$ and $\Phi(c)(b\otimes I)$. Take p, q, r in G and assume $pr = qr$. Then for all f, g in A, we have by definition $(\Phi(f)(1\otimes g))(p, r) = f(pr)g(r)$ and $\Phi(f)(1 \otimes g)(q, r) = f(qr)g(r)$, and these numbers are equal. By density, this is true for all $h \in A$ we have : $h(q, r) = h(q, r)$, hence $p = q$ (such functions separate points). $\square$

Proposition 5. *A compact semi-group G with cancellation is a group.*

Preuve. First, we look for an identity e of G. Intuitively, e is the single element in the intersection of all subgroups of G, but there are no "subgroups" yet, only ideals (sub-semi-groups, i.e. subsets I such that $gI \subset I$ for all g in G). Pick an element s and consider the closed-sub-semi-group H generated by s (which is still compact). Then take the intersection I of all (closed) ideals of H, this is a non-empty closed ideal of H. For all p in I, $pI \subset I$ is again a closed ideal, but I is the smallest such, hence $pI \subset I$. Hence there exists $e \in I$ such that $pe = p$. Multiplying by any q on the right and using cancellation, we obtain $eq = q$ for any $q \in G$, then the same thing on the left (again by cancellation), hence e is the identity. Moreover $sI = I$, hence (by taking $sq = e$) the existence of an inverse. $\square$

These two general results justify the replacement of the antipode assumption by these assumptions. We will see that this implies the existence of a true Hopf (*-)algebra structure (with an antipode !) on a dense subset.

2.3 Haar measure

First, one has to ask : what is the good definition for a Haar measure in the quantum case ? The classical Haar measure on a compact group is a probability measure invariant by the left and right actions of the group. In the unital C^*-algebra setting, a probability measure corresponds to a state, i.e. a linear function $varphi : A \longrightarrow A$ such that $\phi(1) = 1$, $\phi(ff^*) \geq 0$. Indeed, we think of the integration against a probability measure dp as a linear functional $f \mapsto \int f dp$, which is positive (dp is a positive measure) and normalized ($\int dp = 1$). In the classical setting, the Haar measure is such that for all function f :

$$\int f(g_1 g)dp(g) = \int f(g)dp(g) \tag{6}$$

Which amounts, when written as a coproduct relation : $(\mathrm{Id} \otimes h)\varphi(\Phi(f)) = h(f)1$. Similarly for the right invariance. We will use this remark as a definition for the quantum Haar measure, we ask that this relation holds for any $f \in A$ (together with its companion for right invariance). We now turn to the existence of a Haar measure. Note that the coproduct on A induces a product on the linear functional A^*

Lemme 4. *For $h \in A^*$ positive normalized to be a Haar measure, it is enough to show that h satisfies $hw = wh = h$ for one faithful state w.*

Preuve. Sketch of the proof. Define the map $\Psi(c) = h \star c - h(c)I$ on A. Our goal is to show that $\Psi(c)$ is identically 0, wich by faithfulness is equivalent to $(\rho)(\Psi * \Psi(c))$ is always 0. Consider $L_{h\otimes\rho}$ the closed left ideal related to the state $h \otimes \rho$ on $A \otimes A$. We want to show $(\mathrm{Id} \otimes \Psi(c)) \in L_{h\otimes\rho}$. For that, first show that the elements of the type $(\mathrm{Id} \otimes \Psi)(\Phi(c))$ belong to this ideal : this is a short computation. Then, since $L_{h\otimes\rho}$ is an ideal, it contains all the elements of the form $(b \otimes \Psi)(\Phi(c))$, but by density and closedness this gives the result. $\square$

Lemme 5. *Let w be a state on A, then there is a state h such that $hw = wh = h$*

Preuve. Take an accumulation point of $\frac{1}{n}(w + \cdots + w^n)$. $\square$

Conclusion : take a faithful state, apply the two last lemmas, this gives a Haar measure.

2.4 Compact matrix quantum groups

We defined abstract quantum groups, but there are many concrete examples as "matrix quantum groups" (and in fact the representation theory connects closely a compact quantum group with its concrete realizations).

Définition 2. *A compact matrix quantum group is a unital C^*-alebra A generated by N^2 elements U_{ij} with an $*$-homomorphism $\Phi : A \longrightarrow A \otimes A$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j.*

Quantum classic compact groups Following the motto of "consider the algebra of functions and let it be noncommutative", we define the quantum orthogonal, unitary, and symmetric group of rank n by the following universal C^*-algebra :

$$\begin{aligned} A_o(n) &= C^*(u_{ij}|u \text{ orthogonal}) \\ A_u(n) &= C^*(u_{ij}|\ u \text{ unitary}) \\ A_s(n) &= C^*(u_{ij}|\ u \text{ magic unitary}) \end{aligned}$$

A magic matrix has projection-valued coefficients, such that the sum on each line and column is equal to 1. The other definitions are standard. Here A is a C^*-algebra.

Exemple 1 (Quantum permutations groups)**.** *The algebra $A_s(n)$ is a quantum analogue of $C(S_n)$, and the underlying quantum group is indeed "quantum permutations". Consider $X = \{1, \ldots, n\}$ and the action of S_n on X translates into a map $\alpha_{com} : C(X) \longrightarrow C(X) \otimes C(S_n)$ defined by $\alpha_{com}(\delta_i) = \sum \delta_j \otimes u_{ji}$. We extend this definition for $\alpha : C(X) \longrightarrow C(X) \otimes A_s(n)$ and the obvious diagram commutes. Question : is $A_s(n)$ really bigger than $C(S_n)$. The answer is : yes for $n \geq 4$ (then $A_s(n)$ is non commutative, infinite dimensional), no for $n = 1, 2, 3$ where the canonical map $A_s(n) \longrightarrow C(S_n)$ is an isomorphism (the entries of a $n \times n$ permutation matrix are commuting for small values). An example for $n = 4$ is given by*

$$\begin{matrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{matrix}$$

with p, q two free projections.