CLT FOR FLUCTUATIONS OF β-ENSEMBLES WITH GENERAL POTENTIAL

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ABSTRACT. We prove a central limit theorem for the linear statistics of one-dimensional log-gases, or β-ensembles. We use a method based on a change of variables which allows to treat fairly general situations, including multi-cut and, for the first time, critical cases, and generalizes the previously known results of Johansson, Borot-Guionnet and Shcherbina. In the one-cut regular case, our approach also allows to retrieve a rate of convergence as well as previously known expansions of the free energy to arbitrary order.

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1. INTRODUCTION

Let $\beta > 0$ be fixed. For $N \geq 1$, we are interested in the N-point canonical Gibbs measure[1] for a one-dimensional log-gas at the *inverse temperature* β, defined by

$$d\mathbb{P}^V_{N,\beta}(\vec{X}_N) = \frac{1}{Z^V_{N,\beta}} \exp\left(-\frac{\beta}{2}\mathcal{H}^V_N(\vec{X}_N)\right) d\vec{X}_N, \tag{1.1}$$

where $\vec{X}_N = (x_1, \dots, x_N)$ is an N-tuple of points in $\mathbb{R}$, and $\mathcal{H}^V_N(\vec{X}_N)$, defined by

$$\mathcal{H}^V_N(\vec{X}_N) := \sum_{1 \leq i \neq j \leq N} -\log|x_i - x_j| + \sum_{i=1}^N NV(x_i), \tag{1.2}$$

is the energy of the system in the state $\vec{X}_N$, given by the sum of the pairwise repulsive logarithmic interaction between all particles plus the effect on each particle of an external field or confining potential NV whose intensity is proportional to N. We will use $d\vec{X}_N$ to denote the Lebesgue measure on $\mathbb{R}^N$. The constant $Z^V_{N,\beta}$ in the definition (1.1) is the normalizing constant, called the *partition function*, and is equal to

$$Z^V_{N,\beta} := \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\mathcal{H}^V_N(\vec{X}_N)\right) d\vec{X}_N.$$

Such systems of particles with logarithmic repulsive interaction on the line have been extensively studied, in particular because of their connection with random matrix theory, see [For10] for a survey.

Under mild assumptions on V, it is known that the empirical measure of the particles converges almost surely to some deterministic probability measure on $\mathbb{R}$ called the *equilibrium*

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[1] We use $\frac{\beta}{2}$ instead of β in order to match the existing literature. The first sum in (1.2), over indices $i \neq j$, is twice the physical one, but is more convenient for our analysis.

measure μ_V, with no simple expression in terms of V. For any $N \geq 1$, let us define the *fluctuation measure*

$$\mathrm{fluct}_N := \sum_{i=1}^{N} \delta_{x_i} - N\mu_V, \tag{1.3}$$

which is a random signed measure. For any test function ξ regular enough we define the *fluctuations of the linear statistics associated to* ξ as the random real variable

$$\mathrm{Fluct}_N(\xi) := \int_{\mathbb{R}} \xi \, d\mathrm{fluct}_N. \tag{1.4}$$

The goal of this paper is to prove a Central Limit Theorem (CLT) for $\mathrm{Fluct}_N(\xi)$, under some regularity assumptions on V and ξ.

1.1. **Assumptions.**

(H1) - Regularity and growth of V: The potential V is in $C^{\mathsf{p}}(\mathbb{R})$ and satisfies the growth condition

$$\liminf_{|x|\to\infty} \frac{V(x)}{2\log|x|} > 1. \tag{1.5}$$

It is well-known, see e.g. [ST13], that if V satisfies (H1) with $\mathsf{p} \geq 0$, then the *logarithmic potential energy* functional defined on the space of probability measures by

$$\mathcal{I}_V(\mu) = \int_{\mathbb{R}\times\mathbb{R}} -\log|x-y| \, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}} V(x)\, d\mu(x) \tag{1.6}$$

has a unique global minimizer μ_V, the aforementioned *equilibrium measure*. This measure has a compact support that we will denote by Σ_V, and μ_V is characterized by the fact that there exists a constant c_V such that the function ζ_V defined by

$$\zeta_V(x) := \int -\log|x-y| d\mu_V(y) + \frac{V(x)}{2} - c_V \tag{1.7}$$

satisfies the Euler-Lagrange conditions

$$\zeta_V \geq 0 \text{ in } \mathbb{R}, \quad \zeta_V = 0 \text{ on } \Sigma_V. \tag{1.8}$$

We will work under two additional assumptions: one deals with the possible form of μ_V and the other one is a non-criticality hypothesis concerning ζ_V.

(H2) - Form of the equilibrium measure: The support Σ_V of μ_V is a finite union of $\mathsf{n}+1$ non-degenerate intervals

$$\Sigma_V = \bigcup_{0\leq l\leq \mathsf{n}} [\alpha_{l,-};\alpha_{l,+}], \text{ with } \alpha_{l,-} < \alpha_{l,+}.$$

The points $\alpha_{l,\pm}$ are called the *endpoints* of the support Σ_V. For x in Σ_V, we let

$$\sigma(x) := \prod_{l=0}^{\mathsf{n}} \sqrt{|x-\alpha_{l,-}||x-\alpha_{l,+}|}. \tag{1.9}$$

We assume that the equilibrium measure has a density with respect to the Lebesgue measure on Σ_V given by

$$\mu_V(x) = S(x)\sigma(x), \tag{1.10}$$

where S can be written as

$$S(x) = S_0(x) \prod_{i=1}^{\mathsf{m}} (x - s_i)^{2\mathsf{k}_i}, \quad S_0 > 0 \text{ on } \Sigma_V, \tag{1.11}$$

where $\mathsf{m} \geq 0$, all the points s_i, called *singular points*[2], belong to Σ_V and the k_i are natural integers.

(H3) - Non-criticality of ζ_V: The function ζ_V is positive on $\mathbb{R} \setminus \Sigma_V$.

We introduce the operator Ξ_V, which acts on C^1 functions by

$$\Xi_V[\psi] := -\frac{1}{2}\psi V' + \int \frac{\psi(\cdot) - \psi(y)}{\cdot - y}\, d\mu_V(y). \tag{1.12}$$

1.2. Main result.

Theorem 1 (Central limit theorem for fluctuations of linear statistics)**.** *Let ξ be a function in $C^{\mathsf{r}}(\mathbb{R})$, assume that (H1)-(H3) hold. We let*

$$\mathsf{k} = \max_{i=1,\dots,\mathsf{m}} 2\mathsf{k}_i,$$

where the k_i's are as in (1.11), and assume that, p (resp. r) denoting the regularity of V (resp. ξ)

$$\mathsf{p} \geq (3\mathsf{k} + 5), \quad \mathsf{r} \geq (2\mathsf{k} + 3). \tag{1.13}$$

If $\mathsf{n} \geq 1$, assume that ξ satisfies the n following conditions

$$\int_{\Sigma_V} \frac{\xi(y) y^d}{\sigma(y)} dy = 0 \qquad \text{for } d = 0, \dots, \mathsf{n} - 1. \tag{1.14}$$

Moreover, if $\mathsf{m} \geq 1$, assume that for all $i = 1, \dots, \mathsf{m}$

$$\int_{\Sigma_V} \frac{\xi(y) - R_{s_i,d}\xi(y)}{\sigma(y)(y - s_i)^d} dy = 0 \qquad \text{for } d = 1, \dots, 2\mathsf{k}_i, \tag{1.15}$$

where $R_{x,d}\xi$ is the Taylor expansion of ξ to order $d-1$ around x given by

$$R_{x,d}\xi(y) = \xi(x) + (y - x)\xi'(x) + \dots + \frac{(y - x)^{d-1}}{(d-1)!}\xi^{(d-1)}(x).$$

Then there exists a constant c_ξ and a function ψ of class C^2 in some open neighborhood U of Σ_V such that $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$ on U, and the fluctuation $\mathrm{Fluct}_N(\xi)$ converges in law as $N \to \infty$ to a Gaussian distribution with mean

$$m_\xi = \left(1 - \frac{2}{\beta}\right) \int \psi'\, d\mu_V,$$

and variance

$$v_\xi = -\frac{2}{\beta} \int \psi \xi' d\mu_V.$$

It is proven in (B.32) that the variance v_ξ has the equivalent expression

$$v_\xi := \frac{2}{\beta}\left(\iint \left(\frac{\psi(x) - \psi(y)}{x - y} \right)^2 d\mu_V(x) d\mu_V(y) + \int V'' \psi^2 d\mu_V \right). \tag{1.16}$$

Let us note that ψ, hence also m_ξ and v_ξ, can be explicitly written in terms of ξ.

[2]Let us emphasize that a singular point s_i can be equal to an endpoint $\alpha_{l,\pm}$.

1.3. **Comments on the assumptions.** The growth condition (1.5) is standard and expresses the fact that the logarithmic repulsion is beaten at long distance by the confinement, thus ensuring that μ_V has a compact support. Together with the non-criticality assumption (H3) on ζ_V, it implies that the particles of the log-gas effectively stay within some neighborhood of Σ_V, up to very rare events.

The case $\mathsf{n} = 0$, where the support has a single connected component, is called *one-cut*, whereas $\mathsf{n} \geq 1$ is a *multi-cut* situation. If $\mathsf{m} \geq 1$, we are in a *critical case*.

The relationship between V and μ_V is complicated in general, and we mention some examples where μ_V is known to satisfy our assumptions.

- If V is real-analytic, then the assumptions are satisfied with n finite, m finite and S analytic on Σ_V, see [DKM98, Theorem 1.38], [DKM+99, Sec.1].
- If V is real-analytic, then for a "generic" V the assumptions are satisfied with n finite, $\mathsf{m} = 0$ and S analytic on Σ_V, see [KM00].
- If V is uniformly convex and smooth, then the assumptions are satisfied with $\mathsf{n} = 0$, $\mathsf{m} = 0$, and S smooth on Σ_V, see e.g. [BdMPS95, Example 1].
- Examples of multi-cut, non-critical situations with $\mathsf{n} = 0, 1, 2$ and $\mathsf{m} = 0$, are mentioned in [BdMPS95, Examples 3-4].
- An example of criticality at the edge of the support is given by $V(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x$, for which the equilibrium measure, as computed in [CKI10, Example 1.2], is given by
$$\mu_V(x) = \frac{1}{10\pi}\sqrt{|x-(-2)||x-2|}(x-2)^2\mathbf{1}_{[-2,2]}(x).$$
- An example of criticality in the bulk of the support is given by $V(x) = \frac{x^4}{4} - x^2$, for which the equilibrium measure, as computed in [CK06], is
$$\mu_V(x) = \frac{1}{2\pi}\sqrt{|x-(-2)||x-2|}(x-0)^2\mathbf{1}_{[-2,2]}(x).$$

Following the terminology used in the literature [DKM+99, KM00, CK06], we may say that our assumptions allow the existence of singular points of type II (the density vanishes in the bulk) and III (the density vanishes at the edge faster than a square root). Assumption (H3) rules out the possibility of singular points of type I, also called "birth of a new cut", for which the behavior might be quite different, see [Cla08, Mo08].

1.4. **Existing literature, strategy and perspectives.**

1.4.1. *Connection to previous results.* The CLT for fluctuations of linear statistics in the context of β-ensembles was proven in the pioneering paper [Joh98] for polynomial potentials in the case $\mathsf{n} = 0, \mathsf{m} = 0$, and generalized in [Shc13] to real-analytic potentials in the possibly multi-cut, non-critical cases ($\mathsf{n} \geq 0, \mathsf{m} = 0$), where a set of n necessary and sufficient conditions on a given test function in order to satisfy the CLT is derived. If these conditions are not fulfilled, the fluctuations are shown to exhibit oscillatory behaviour. Such results are also a by-product of the all-orders expansion of the partition function obtained in [BG13b] ($\mathsf{n} = 0, \mathsf{m} = 0$) and [BG13a] ($\mathsf{n} \geq 0, \mathsf{m} = 0$). A CLT for the fluctuations of linear statistics for test functions living at mesoscopic scales was recently obtained in [BL16].

1.4.2. *Motivation and strategy.* Our goal is twofold: on the one hand, we provide a simple proof of the CLT using a change of variables argument, retrieving the results cited above. On the other hand, our method allows to substantially relax the assumptions on V, in particular for the first time we are able to treat critical situations where $\mathsf{m} \geq 1$.

Our method, which is adapted from the one introduced in [LS16] for two-dimensional log-gases, can be summarized as follows

(1) We prove the CLT by showing that the Laplace transform of the fluctuations converges to the Laplace transform of the correct Gaussian law. This idea is already present in [Joh98] and many further works. Computing the Laplace transform of $\mathrm{Fluct}_N(\xi)$ leads to working with a new potential $V + t\xi$ (with t small), and thus to considering the associated perturbed equilibrium measure.
(2) Following [LS16], our method then consists in finding a change of variables (or a transport map) that pushes μ_V onto the perturbed equilibrium measure. In fact we do not exactly achieve this, but rather we construct a transport map $I + t\psi$, which is a perturbation of identity, and consider the *approximate* perturbed equilibrium measure $(I + t\psi)\#\mu_V$. The map ψ is found by inverting the operator (1.12), which is well-known in this context, it appears e.g. in [BG13b, BG13a, Shc13, BFG13]. A CLT will hold if the function ξ is (up to constants) in the image of Ξ_V, leading to the conditions (1.14)–(1.15). The change of variables approach for one-dimensional log-gases was already used e.g. in [Shc14, BFG13], see also [GMS07, GS14] which deal with the non-commutative context.
(3) The proof then leverages on the expansion of $\log Z^V_{N,\beta}$ up to order N proven in [LS15], valid in the multi-cut and critical case, and whose dependency in V is explicit enough. This step replaces the a priori bound on the commutators used e.g. in [BG13b].

1.4.3. *More comments and perspectives.* Using the Cramér-Wold theorem, the result of Theorem 1 extends readily to any finite family of test functions satisfying the conditions ((1.14), (1.15)): the joint law of their fluctuations converges to a Gaussian vector, using the bilinear form associated to (1.16) in order to determine the covariance.

In the multi-cut case, the CLT results of [Shc13] or [BG13a] are stated under n necessary and sufficient conditions on the test function, and the non-Gaussian nature of the fluctuations if these conditions are not satisfied is explicitly described. In the critical cases, we only state sufficient conditions (1.15) under which the CLT holds. It would be interesting to prove that these conditions are necessary, and to characterize the behavior of the fluctuations for functions which do not satisfy (1.15).

Finally, we expect Theorem 1 to hold also at mesoscopic scales.

1.5. **The one-cut noncritical case.** In the case $\mathsf{n} = 0$ and $\mathsf{m} = 0$, following the transport approach, we can obtain the convergence of the Laplace transform of fluctuations with an explicit rate, under the assumption that ξ is very regular (we have not tried to optimize in the regularity):

Theorem 2 (Rate of convergence in the one-cut noncritical case)**.** *Under the assumptions of Theorem 1, if in addition* $\mathsf{n} = 0$, $\mathsf{m} = 0$, $\mathsf{p} \geq 6$ *and* $\mathsf{r} \geq 17$, *then we also have*

$$\left|\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(s\mathrm{Fluct}_N(\xi)) - sm_\xi - s^2 v_\xi\right| \\ \leq C \left(\frac{s}{N} \|\xi\|_{C^{17}(\mathbb{R})} + \frac{s^3}{N} \|\xi\|_{C^2(\mathbb{R})} + \frac{s^4}{N^2} \|\xi\|^4_{C^3(\mathbb{R})} \right), \tag{1.17}$$

where the constant C *depends only on* V.

The assumed regularity on ξ allows to avoid using the result of [LS15] on the expansion of $\log Z^V_{N,\beta}$. Our transport approach also provides a functional relation on the expectation of fluctuations which allows by a boostrap procedure to recover an expansion of $\log Z^V_{N,\beta}$ (relative to a reference potential) to arbitrary powers of $1/N$ in very regular cases, i.e the result of [BG13b] but without the analyticity assumption. All these results are presented in Appendix A.

1.6. **Some notation.** We denote by $P.V.$ the principal value of an integral having a singularity at x_0, i.e.

$$P.V. \int f = \lim_{\varepsilon \to 0} \int_{-\infty}^{x_0 - \varepsilon} f + \int_{x_0+\varepsilon}^{+\infty} f. \tag{1.18}$$

If Φ is a C^1-diffeomorphism and μ a probability measure, we denote by $\Phi\#\mu$ the push-forward of μ by Φ, which is by definition such that for $A \subset \mathbb{R}$ Borel,

$$(\Phi\#\mu)(A) := \mu(\Phi^{-1}(A)).$$

If $A \subset \mathbb{R}$ we denote by $\mathring{A}$ its interior.

For $k \geq 0$, and U some bounded domain in $\mathbb{R}$, we endow the spaces $C^k(U)$ with the usual norm

$$\|\psi\|_{C^k(U)} := \sum_{j=0}^{k} \sup_{x \in U} |\psi^{(j)}(x)|.$$

If z is a complex number, we denote by $\mathcal{R}(z)$ (resp. $\mathcal{I}(z)$) its real (resp. imaginary) part.

For any probability measure μ on $\mathbb{R}$ we denote by h^μ the logarithmic potential generated by μ, defined as the map

$$x \in \mathbb{R}^2 \mapsto h^\mu(x) = \int -\log|x-y| d\mu(y). \tag{1.19}$$

2. Expressing the Laplace transform of the fluctuations

We start by the standard approach of reexpressing the Laplace transform of the fluctuations in terms the ratio of partition functions of a perturbed log-gas by that of the original one. This is combined with the energy splitting formula of [SS15] that separates fixed leading order terms from variable next order ones.

2.1. **The next-order energy.** For any probability measure μ, let us define,

$$F_N(\vec{X}_N,\mu) = -\iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log|x-y|\Big(\sum_{i=1}^N \delta_{x_i}-\mu\Big)(x)\Big(\sum_{i=1}^N \delta_{x_i}-\mu\Big)(y), \tag{2.1}$$

where $\triangle$ denotes the diagonal in $\mathbb{R}\times\mathbb{R}$.

We have the following splitting formula for the energy, as introduced in [SS15] (we recall the proof in Section B.1).

Lemma 2.1. *For any $\vec{X}_N \in \mathbb{R}^N$, it holds that*

$$\mathcal{H}_N^V(\vec{X}_N) = N^2\mathcal{I}_V(\mu_V) + 2N\sum_{i=1}^N \zeta_V(x_i) + F_N(\vec{X}_N,\mu_V)\ . \tag{2.2}$$

Using this splitting formula (2.2), we may re-write $\mathbb{P}^V_{N,\beta}$ as

$$d\mathbb{P}^V_{N,\beta}(\vec{X}_N) = \frac{1}{K_{N,\beta}(\mu_V,\zeta_V)}\exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N,\mu_V)+2N\sum_{i=1}^N\zeta_V(x_i)\right)\right)d\vec{X}_N, \tag{2.3}$$

with a next-order partition function $K_{N,\beta}(\mu_V,\zeta_V)$ defined by

$$K_{N,\beta}(\mu_V,\zeta_V) := \int_{\mathbb{R}^N}\exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N,\mu_V)+2N\sum_{i=1}^N\zeta_V(x_i)\right)\right)d\vec{X}_N. \tag{2.4}$$

We extend this notation to $K_{N,\beta}(\mu,\zeta)$ where μ is a probability density and ζ is a confinement potential.

2.2. **Perturbed potential and equilibrium measure.** Let ξ be in $C^0(\mathbb{R})$ with compact support.

Definition 2.2. *For any $t\in\mathbb{R}$, we define*

- *The perturbed potential V_t as $V_t := V + t\xi$.*
- *The perturbed equilibrium measure μ_t as the equilibrium measure associated to V_t. Since ξ has compact support, V_t satisfies the growth assumption* (1.5) *and thus μ_t is well-defined. In particular, μ_0 coincides with μ_V.*
- *The next-order confinement term $\zeta_t := \zeta_{V_t}$, as in* (1.7).
- *The next-order energy $F_N(\vec{X}_N,\mu_t)$ as in* (2.1).
- *The next-order partition function $K_{N,\beta}(\mu_t,\zeta_t)$ as in* (2.4).

2.3. **The Laplace transform of fluctuations as ratio of partition functions.**

Lemma 2.3. *For any $s\in\mathbb{R}$ we have, letting $t := \frac{-2s}{\beta N}$,*

$$\mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left[\exp\left(s\mathrm{Fluct}_N(\xi)\right)\right] = \frac{K_{N,\beta}(\mu_t,\zeta_t)}{K_{N,\beta}(\mu_0,\zeta_0)}\exp\left(-\frac{\beta}{2}N^2\left(\mathcal{I}_{V_t}(\mu_t)-\mathcal{I}_V(\mu_0)-t\int\xi d\mu_0\right)\right). \tag{2.5}$$

Proof. First, we notice that, for any s in $\mathbb{R}$

$$\mathbf{E}_{\mathbb{P}^V_{N,\beta}}[\exp(s\mathrm{Fluct}_N(\xi))] = \frac{Z^{V_t}_{N,\beta}}{Z^V_{N,\beta}}\exp\left(-Ns\int\xi\,d\mu_V\right) \tag{2.6}$$

Using the splitting formula (2.2) and the definition of $K_{N,\beta}$ as in (2.4) we see that for any t

$$K_{N,\beta}(\mu_t, \zeta_t) = Z_{N,\beta}^{V_t} \exp\left(\frac{\beta}{2} N^2 \mathcal{I}_{V_t}(\mu_t)\right), \tag{2.7}$$

thus combining (2.6) and (2.7), with $t := \frac{-2s}{\beta N}$ we obtain (2.5). □

2.4. **Comparison of partition functions.** If μ is a probability density, we denote by $\mathrm{Ent}(\mu)$ the entropy function given by $\mathrm{Ent}(\mu) := \int_{\mathbb{R}} \mu \log \mu$. The following asymptotic expansion is proven [LS15, Corollary 1.5] (cf. [LS15, Remark 4.3]) and valid in a general multi-cut critical situation.

Lemma 2.4. *Let μ be a probability density on $\mathbb{R}$. Assume that μ has the form (1.10), (1.11) with S_0 in $C^2(\Sigma)$, and that ζ is some Lipschitz function on $\mathbb{R}$ satisfying*

$$\zeta = 0 \text{ on } \Sigma, \quad \zeta > 0 \text{ on } \mathbb{R} \setminus \Sigma, \quad \int_{\mathbb{R}} e^{-\beta N \zeta(x)} dx < \infty \text{ for } N \text{ large enough.}$$

Then, with the notation of (2.4) and for some C_β depending only on β, we have

$$\log K_{N,\beta}(\mu, \zeta) = \frac{\beta}{2} N \log N + C_\beta N - N\left(1 - \frac{\beta}{2}\right) \mathrm{Ent}(\mu) + N o_N(1). \tag{2.8}$$

2.5. **Additional bounds.**

2.5.1. *Exponential moments of the next-order energy.*

Lemma 2.5. *We have, for some constant C depending on β and V*

$$\left| \log \mathbf{E}_{\mathbb{P}_{N,\beta}^V}\left[\exp\left(\frac{\beta}{4}\left(F_N(\vec{X}_N, \mu_V) + N \log N\right)\right)\right]\right| \le CN. \tag{2.9}$$

Proof. This follows e.g. from [SS15, Theorem 6], but we can also deduce it from Lemma 2.4. We may write

$$\mathbf{E}_{\mathbb{P}_{N,\beta}^V}\left[\exp\left(\frac{\beta}{4} F_N(\vec{X}_N, \mu_V)\right)\right] = \frac{1}{K_{N,\beta}(\mu_V, \zeta_V)} \int \exp\left(-\frac{\beta}{4}\left(F_N(\vec{X}_N, \mu_V) - 2N \sum_{i=1}^N 2\zeta_V(x_i)\right)\right) d\vec{X}_N = \frac{K_{N,\frac{\beta}{2}}(\mu_V, 2\zeta_V)}{K_{N,\beta}(\mu_V, \zeta_V)}.$$

Taking the log and using (2.8) to expand both terms up to order N yields the result. □

2.5.2. *The next-order energy controls the fluctuations.* The following result is a consequence of the analysis of [SS15, PS14], we give the proof in Section B.2 for completeness. It shows that F_N controls fluct_N. Here $|\mathrm{Supp}\,\xi|$ denotes the diameter of the support of ξ.

Proposition 2.6. *If ξ is compactly supported and Lipschitz, we have, for some universal constant C*

$$\left|\int \xi \, d\mathrm{fluct}_N\right| \le C |\mathrm{Supp}\,\xi|^{\frac{1}{2}} \|\nabla \xi\|_{L^\infty} \left(F_N(\vec{X}_N, \mu_V) + N \log N + C(\|\mu_V\|_{L^\infty} + 1) N\right)^{1/2}. \tag{2.10}$$

2.5.3. *Confinement bound.* We will also need the following bound on the confinement. The proof is very simple and identical to the proof of Lemma 3.3 of [LS16].

Lemma 2.7. *For any fixed open neighborhood U of Σ,*

$$\mathbb{P}^V_{N,\beta}\left(\vec{X}_N \in U^N\right) \geq 1 - \exp(-cN)$$

where $c > 0$ depends on U and β.

Lemma 2.7 is the only place where we use the non-degeneracy assumption (H3) on the next-order confinement term ζ_V.

3. Inverting the operator and defining the approximate transport

The goal of this section is to find transport maps ϕ_t for t small enough such that the transported measure $\phi_t \# \mu_0$ approximates the equilibrium measures μ_t. Since the equilibrium measures are characterized by (1.7) with equality on the support, it is natural to seek ϕ_t such that the quantity

$$\int -\log|\phi_t(x) - \phi_t(y)| d\mu_0(y) + \frac{1}{2} V_t(\phi_t(x))$$

is close to a constant.

3.1. **Preliminaries.**

Lemma 3.1. *We have the following*

- *The non-vanishing function S_0 in (1.11) is in $C^{\mathsf{p}-3-2\mathsf{k}}(\Sigma_V)$.*
- *There exists an open neighborhood U of Σ_V and a non-vanishing function M in $C^{\mathsf{p}-3-2\mathsf{k}}(U \setminus \mathring{\Sigma}_V)$ such that*

$$\zeta_V'(x) = M(x)\sigma(x) \prod_{i=1}^{m} (x - s_i)^{2\mathsf{k}_i}. \tag{3.1}$$

In particular, (3.1) quantifies how fast ζ_V' vanishes near an endpoint of the support. We postpone the proof to Section B.3.

3.2. **The approximate equilibrium measure equation.** In the following, we let

- U be an open neighborhood of Σ_V such that (3.1) holds.
- B be the open ball of radius $\frac{1}{2}$ in $C^2(U)$.

We define a map $\mathcal{F}$ from $[-1,1] \times B$ to $C^1(U)$ by setting $\phi := \mathrm{Id} + \psi$ and

$$\mathcal{F}(t, \psi) := \int -\log|\phi(\cdot) - \phi(y)| d\mu_V(y) + \frac{1}{2} V_t \circ \phi(\cdot) \ , \tag{3.2}$$

Lemma 3.2. *The map $\mathcal{F}$ takes values in $C^1(U)$ and has continuous partial derivatives in both variables. Moreover there exists C depending only on V such that for all (t, ψ) in $[-1,1] \times B$ we have*

$$\left\| \mathcal{F}(t,\psi) - \mathcal{F}(0,0) - \frac{t}{2}\xi + \Xi_V[\psi] \right\|_{C^1(U)} \leq C t^2 \|\psi\|^2_{C^2(U)}. \tag{3.3}$$

The proof is postponed to Section B.4.

3.3. **Inverting the operator.**

Lemma 3.3. *Let ψ be defined by*

$$\psi(x) = -\frac{1}{2\pi^2 S(x)}\left(\int_\Sigma \frac{\xi(y)-\xi(x)}{\sigma(y)(y-x)}dy\right) \quad \text{for } x \text{ in } \Sigma_V, \tag{3.4}$$

$$\psi(x) = \frac{\displaystyle\int \frac{\psi(y)}{x-y}d\mu_V(y) + \frac{\xi}{2} + c_\xi}{\displaystyle\int \frac{1}{x-y}d\mu_V(y) - \frac{1}{2}V'(x)} \quad \text{for } x \in U\backslash\Sigma_V, \tag{3.5}$$

then ψ is in $C^{\mathsf{l}}(U)$ with $\mathsf{l} = (\mathsf{p}-3-3\mathsf{k}) \wedge (\mathsf{r}-1-2\mathsf{k})$ and

$$\|\psi\|_{C^{\mathsf{l}}(U)} \leq C\|\xi\|_{C^{\mathsf{r}}(\mathbb{R})} \tag{3.6}$$

for some constant C depending only on V, and there exists a constant c_ξ such that

$$\Xi_V[\psi] = \frac{\xi}{2} + c_\xi \text{ in } U,$$

with Ξ_V as in (1.12).

The proof of Lemma 3.3 is postponed to Section B.5. We may extend ψ to $\mathbb{R}$ in such a way that ψ is in $C^{\mathsf{l}}(\mathbb{R})$ with compact support.

3.4. **Approximate transport and equilibrium measure.** We let ψ be the function defined in Lemma 3.3, and c_ξ be such that

$$\Xi_V[\psi] = \frac{\xi}{2} + c_\xi \text{ on } U.$$

Definition 3.4. *For $t \in [-t_{\max}, t_{\max}]$, where $t_{\max} = \left(2\|\psi\|_{C^1(U)}\right)^{-1}$,*

- *We let ψ_t be given by $\psi_t := t\psi$.*
- *We let ϕ_t be the approximate transport, defined by $\phi_t := \mathrm{Id} + \psi_t$.*
- *We let $\tilde{\mu}_t$ be the approximate equilibrium measure, defined by $\tilde{\mu}_t := \phi_t \# \mu_V$.*
- *We let $\tilde{\zeta}_t$ be the approximate confining term $\tilde{\zeta}_t := \zeta_V \circ \phi_t^{-1}$*
- *We let $\mathbb{P}^{(t)}_{N,\beta}$ be the probability measure*

$$d\mathbb{P}^{(t)}_{N,\beta}(\vec{X}_N) = \frac{1}{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)} \exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N, \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right)\right) d\vec{X}_N, \tag{3.7}$$

where $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$ is as in (2.4).

Finally, we let τ_t be defined by

$$\tau_t := \mathcal{F}(t, \psi_t) - \mathcal{F}(0,0) - \tilde{c}_t. \tag{3.8}$$

This quantifies how close $\tilde{\mu}_t$ is from satisfying the Euler-Lagrange equation for V_t and thus how well $\tilde{\mu}_t$ approximates the real equilibrium measure μ_t. We also define the extension $\hat{\tau}_t$ of $\tau_t \circ \phi_t^{-1}$ to $\mathbb{R}^2$ by

$$\hat{\tau}_t(x,y) = \chi(x,y)\ \tau_t \circ \phi_t^{-1}(x), \tag{3.9}$$

where χ is equal to one in a fixed neighborhood of $\mathrm{supp}(\mu_V)$ included in U and is in $C^\infty_c(\mathbb{R}^2)$.

Lemma 3.5. *The following holds*

- *The map* ψ_t *satisfies*
$$\Xi_V[\psi_t] = \frac{t}{2}\xi + \tilde{c}_t, \text{ for } \tilde{c}_t := tc_\xi.$$
- *The map* ϕ_t *is a* C^1*-diffeomorphism which coincides with the identity outside a compact support independent of* $t \in [-t_{\max}, t_{\max}]$.
- *The error* τ_t *is a* $O(t^2)$*, more precisely*

$$\|\tau_t\|_{C^1(U)} \leq Ct^2 \|\psi\|^2_{C^2(U)} \tag{3.10}$$
$$\|\hat{\tau}_t\|_{C^1(\mathbb{R}^2)} \leq Ct^2 \|\psi\|^2_{C^2(U)}. \tag{3.11}$$

- *On* $\phi_t(\Sigma_V)$*, we have*

$$\tilde{\zeta}_t = h^{\tilde{\mu}_t} + \frac{V_t}{2} - \tilde{c}_t - c_V - \tau_t \circ \phi_t^{-1}. \tag{3.12}$$

Proof. The first two points are straightforward, the bound (3.10) follows from combining (3.3) with the conclusions of Lemma 3.2, and then (3.11) is an easy consequence.

For (3.12), let us first recall that
$$\mathcal{F}(t, \psi_t) = \int -\log|\phi_t(\cdot) - \phi_t(y)| d\mu_0(y) + \frac{1}{2} V_t \circ \phi_t,$$
which, with the notation of (1.19), yields
$$\mathcal{F}(t, \psi_t) = h^{\tilde{\mu}_t} \circ \phi_t + \frac{1}{2} V_t \circ \phi_t.$$
On the other hand, by definition of τ_t as in (3.8), we have
$$\mathcal{F}(t, \psi_t) = \mathcal{F}(0, 0) + \tilde{c}_t + \tau_t.$$
Finally, we know that, on Σ_V
$$\mathcal{F}(0, 0) = \zeta_V + c_V.$$
We thus see that
$$\zeta_V + c_V + \tilde{c}_t + \tau_t = h^{\tilde{\mu}_t} \circ \phi_t + \frac{1}{2} V_t \circ \phi_t.$$
Since, by definition, $\tilde{\zeta}_t = \zeta_V \circ \phi_t^{-1}$, we get (3.12). □

4. Study of the Laplace transform

The next goal is to compare the partition functions associated to μ_t and $\mu_0 = \mu_V$. We split the comparison into two steps: first, we compare $K_{N,\beta}(\mu_t, \zeta_t)$ with $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$ using the bounds, obtained in the previous section, showing that $\tilde{\mu}_t$ is a good approximation to $\overline{\mu}_t$, and then we compare $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$ and $K_{N,\beta}(\mu_0, \zeta_0)$ using the transport ϕ_t, as in [LS16].

4.1. Energy comparison: from μ_t to $\tilde{\mu}_t$.

Lemma 4.1. *We have*

$$\int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 \leq Ct^4 \|\psi\|^4_{C^2(U)}, \tag{4.1}$$
$$\int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + \int_{\mathbb{R}} \tilde{\zeta}_t d\mu_t \leq Ct^4 \|\psi\|^4_{C^2(U)}, \tag{4.2}$$

where C *is universal.*

Proof. For t small enough, $\phi_t(U)$ contains some fixed open neighborhood of Σ_V, which itself contains the support of μ_t. Integrating by parts we thus get

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla h^{\mu_t-\tilde\mu_t}|^2 = 2\pi\int h^{\mu_t-\tilde\mu_t}d(\mu_t-\tilde\mu_t) \tag{4.3}$$
$$= \int\left(\zeta_t-\tilde\zeta_t-\tau_t\circ\phi_t^{-1}\right)d(\mu_t-\tilde\mu_t)$$
$$= -\int\zeta_t d\tilde\mu_t - \int\tilde\zeta_t d\mu_t - \int_{\mathbb{R}}\tau_t\circ\phi_t^{-1}d(\mu_t-\tilde\mu_t)$$
$$\le -\int\tau_t\circ\phi_t^{-1}d(\mu_t-\tilde\mu_t).$$

In the first equality, we have re-written h^{μ_t} and $h^{\tilde\mu_t}$ using the confining terms ζ_t and $\tilde\zeta_t$, see (1.7) and (3.12), discarding the constants which disappear when integrated against $d(\mu_t-\tilde\mu_t)$. In the second equality, we have used the fact that ζ_t vanishes on the support of μ_t and $\tilde\zeta_t$ on the support of $\tilde\mu_t$. Finally, the last inequality is due to the fact that ζ_t and $\tilde\zeta_t$ are nonnegative on $\mathbb{R}$. Using (3.9) and (3.11), we may thus write

$$\frac{1}{2\pi}\|\nabla h^{\mu_t-\tilde\mu_t}\|^2_{L^2(\mathbb{R}^2)} \le \left|\int_{\mathbb{R}^2}\tau_t\circ\phi_t^{-1}d(\mu_t-\tilde\mu_t)\delta_{\mathbb{R}}\right| \le \|\nabla\hat\tau_t\|_{L^2(\mathbb{R}^2)}\|\nabla h^{\mu_t-\tilde\mu_t}\|_{L^2(\mathbb{R}^2)}$$
$$\le Ct^2\|\psi\|^2_{C^2(U)}\|\nabla h^{\mu_t-\tilde\mu_t}\|_{L^2(\mathbb{R}^2)},$$

which proves (4.1). Coming back to (4.3), we also obtain

$$0 \le -\int\zeta_t d\tilde\mu_t - \int\tilde\zeta_t d\mu_t + O\left(t^4\|\psi\|^4_{C^2(U)}\right),$$

which in turn implies (4.2). □

Lemma 4.2 (Energy comparison : from μ_t to $\tilde\mu_t$)**.** *For any $\vec X_N \in (\phi_t(U))^N$, we have*

$$\left|\left(F_N(\vec X_N,\mu_t)+2N\sum_{i=1}^N\zeta_t(x_i)\right)-\left(F_N(\vec X_N,\tilde\mu_t)+2N\sum_{i=1}^N\tilde\zeta_t(x_i)\right)\right| \tag{4.4}$$
$$\le C\left(Nt^2\|\psi\|^2_{C^2(U)}(F_N(\vec X_N,\tilde\mu_t)+N\log N)^{1/2}+N^2t^4\|\psi\|^4_{C^2(U)}\right).$$

Proof. By the definition (2.1) of the next-order energy, we may write

$$F_N(\vec X_N,\mu_t)-F_N(\vec X_N,\tilde\mu_t) = N^2\int_{\mathbb{R}\times\mathbb{R}}-\log|x-y|\,d\left(\tilde\mu_t-\mu_t\right)(x)d\left(\tilde\mu_t-\mu_t\right)(y) \tag{4.5}$$
$$+2N\int_{\mathbb{R}\times\mathbb{R}}-\log|x-y|d(\tilde\mu_t-\mu_t)(x)\left(\sum_{i=1}^N\delta_{x_i}-N\tilde\mu_t\right)(y)$$
$$= N^2\int_{\mathbb{R}^2}|\nabla h^{\mu_t-\tilde\mu_t}|^2+2N\int_{\mathbb{R}}h^{\tilde\mu_t-\mu_t}\Big(\sum_{i=1}^N\delta_{x_i}-N\tilde\mu_t\Big).$$

On the other hand, using that $\tilde{\zeta}_t$ vanishes on the support of $\tilde{\mu}_t$, we get

$$\sum_{i=1}^N \left(\zeta_t(x_i) - \tilde{\zeta}_t(x_i)\right) = N \int_{\mathbb{R}} (\zeta_t - \tilde{\zeta}_t) d\tilde{\mu}_t + \int_{\mathbb{R}} (\zeta_t - \tilde{\zeta}_t) \left(\sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t\right) \tag{4.6}$$
$$= N \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + \int_{\mathbb{R}} (\zeta_t - \tilde{\zeta}_t) \left(\sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t\right).$$

Combining (4.5) and (4.6), we obtain

$$\left(F_N(\vec{X}_N, \mu_t) + 2N\sum_{i=1}^N \zeta_t(x_i)\right) - \left(F_N(\vec{X}_N, \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right)$$
$$= N^2 \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 + 2N^2 \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + 2N \int_{\mathbb{R}} (h^{\tilde{\mu}_t - \mu_t} + \zeta_t - \tilde{\zeta}_t) \left(\sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t\right).$$

From (1.7), (3.12) (see also the notation (1.19)), we have

$$h^{\tilde{\mu}_t - \mu_t} + \zeta_t - \tilde{\zeta}_t = \tau_t \circ \phi_t^{-1} + \text{constant},$$

hence we find

$$\left(F_N(\vec{X}_N, \mu_t) + 2N\sum_{i=1}^N \zeta_t(x_i)\right) - \left(F_N(\vec{X}_N, \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right) \tag{4.7}$$
$$= N^2 \int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 + 2N^2 \int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + 2N \int \tau_t \circ \phi_t^{-1} \left(\sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t\right).$$

By the results of Lemma 4.1, the first two terms in the right-hand side of (4.7) are $O(N^2 t^4)$, while the last term is bounded, using (3.10) and Proposition 2.6, by

$$N \int \tau_t \circ \phi_t^{-1} \left(\sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t\right) = O\left(N t^2 (F_N(\vec{X}_N, \tilde{\mu}_t) + N\log N)^{1/2}\right),$$

which concludes the proof. □

Lemma 4.3. *We have, for any fixed $s \in \mathbb{R}$, with $t = \frac{-2s}{\beta N}$*

$$\left|\log \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_t, \zeta_t)}\right| \le C N t^2 \sqrt{N} \|\psi\|^2_{C^2(U)} + C t^4 N^2 \|\psi\|^4_{C^2(U)} \tag{4.8}$$
$$= O\left(s^2 N^{-1/2} \|\psi\|^2_{C^2} + s^4 N^{-2} \|\psi\|^4_{C^2}\right).$$

Proof. By definition of the next-order partition functions we may write

$$\frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_t, \zeta_t)} = \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\left(\left(F_N(\vec{X}_N, \mu_t) + 2N\sum_{i=1}^N \zeta_t(x_i)\right)\right.\right.$$
$$\left.\left. - \left(F_N(\vec{X}_N, \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right)\right)\right) d\vec{X}_N.$$

The result follows from combining (2.9) and (4.4), and using Lemma 2.7 to argue that the particles $\vec{X}_N$ may be assumed to all belong to the neighborhood U for t small enough, except for an event of exponentially small probability. □

4.2. **Energy comparison: from $\tilde{\mu}_t$ to μ_0.** Let us define

$$\mathrm{fluct}_N^{(t)} = \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \qquad \mathrm{Fluct}_N^{(t)}(\xi) = \int \xi \, d\mathrm{fluct}_N^{(t)}.$$

For any ψ, let us define the following quantity (that may be called *anisotropy* by analogy with [LS16])

$$\mathsf{A}^{(t)}[\vec{X}_N, \psi] = \iint_{\mathbb{R}\times\mathbb{R}} \frac{\psi(x)-\psi(y)}{x-y} \, d\mathrm{fluct}_N^{(t)}(x) \, d\mathrm{fluct}_N^{(t)}(y). \tag{4.9}$$

Lemma 4.4. *Assume $\psi \in C^2(\mathbb{R})$. For any $\vec{X}_N \in U^N$, letting $\Phi_t(\vec{X}_N) = (\phi_t(x_1), \cdots, \phi_t(x_N))$, we have*

$$\left| F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) - F_N(\vec{X}_N, \mu_0) - \sum_{i=1}^N \log \phi_t'(x_i) + \frac{t}{2} \mathsf{A}^{(0)}[\vec{X}_N, \psi] \right| \leq Ct^2 \left(F_N(\vec{X}_N, \mu_0) + N \log N \right). \tag{4.10}$$

Proof. Since by definition $\tilde{\mu}_t = \phi_t \# \mu_0$ we may write

$$\begin{aligned}
&F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) - F_N(\vec{X}_N, \mu_0) \\
&= -\iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log|x-y| \Big(\sum_{i=1}^N \delta_{\phi_t(x_i)} - N\tilde{\mu}_t \Big)(x) \Big(\sum_{i=1}^N \delta_{\phi_t(x_i)} - N\tilde{\mu}_t \Big)(y) \\
&\qquad + \iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log|x-y| d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y) \\
&= -\iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log \frac{|\phi_t(x)-\phi_t(y)|}{|x-y|} d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y) \\
&= -\iint_{\mathbb{R}\times\mathbb{R}} \log \frac{|\phi_t(x)-\phi_t(y)|}{|x-y|} d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y) + \sum_{i=1}^N \log \phi_t'(x_i).
\end{aligned}$$

Using that by definition $\phi_t = \mathrm{Id} + t\psi$ where ψ is in $C_c^2(\mathbb{R})$, we get by the chain rule

$$\log \frac{|\phi_t(x)-\phi_t(y)|}{|x-y|} = t \, \frac{\psi(x)-\psi(y)}{x-y} + t^2 \, \varepsilon_t(x,y),$$

with $\|\varepsilon_t\|_{C^2(\mathbb{R}\times\mathbb{R})}$ uniformly bounded in t. Applying Proposition 2.6 twice, we get that

$$\left| \iint \varepsilon_t(x,y) d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y) \right| \leq Ct^2 \left(F_N(\vec{X}_N, \mu_0) + N \log N \right),$$

which yields the result. □

4.3. **Comparison of partition functions I: using the transport.** In this section and the following one, we will write A instead of $\mathsf{A}^{(0)}[\vec{X}_N, \psi]$

Proposition 4.5. *We have, for any t small enough*

$$\frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} = \exp\left(N\left(1 - \frac{\beta}{2}\right)(\mathrm{Ent}(\mu_0) - \mathrm{Ent}(\tilde{\mu}_t))\right) \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left(\exp\left(\frac{\beta}{2} t\mathsf{A} + t^2 \mathsf{Error}_1(\vec{X}_N) + t\mathsf{Error}_2(\vec{X}_N)\right)\right), \tag{4.11}$$

with error terms bounded by

$$|\mathsf{Error}_1(\vec{X}_N)| \le C\left(F_N(\vec{X}_N, \mu_0) + N \log N\right), \tag{4.12}$$

$$|\mathsf{Error}_2(\vec{X}_N)| \le C\left(F_N(\vec{X}_N, \mu_0) + N \log N\right)^{1/2}. \tag{4.13}$$

Proof. By a change of variables and in view of (4.10), we may write

$$\begin{aligned} K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) &= \int \exp\left(-\frac{\beta}{2}\Big(F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t \circ \phi_t(x_i)\Big) + \sum_{i=1}^N \log \phi_t'(x_i)\right) d\vec{X}_N \\ &= \int \exp\left(-\frac{\beta}{2}\left(F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) + 2N\sum_{i=1}^N \zeta_0(x_i)\right) + \sum_{i=1}^N \log \phi_t'(x_i)\right) d\vec{X}_N, \end{aligned} \tag{4.14}$$

since $\zeta_0 = \tilde{\zeta}_t \circ \phi_t$ by definition. Using Lemma 4.4 we may write

$$\begin{aligned} \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} &= \frac{1}{K_{N,\beta}(\mu_0, \zeta_0)} \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N, \mu_0) + 2N\sum_{i=1}^N \zeta(x_i)\right)\right. \\ &\quad \left. + \left(1 - \frac{\beta}{2}\right)\sum_{i=1}^N \log \phi_t'(x_i) + \frac{\beta}{2} t\,\mathsf{A} + t^2 \mathsf{Error}_1(\vec{X}_N)\right) d\vec{X}_N \\ &= \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left(\exp\left(\left(1 - \frac{\beta}{2}\right)\sum_{i=1}^N \log \phi_t'(x_i) + \frac{\beta}{2} t\,\mathsf{A} + t^2 \mathsf{Error}_1(\vec{X}_N)\right)\right), \end{aligned} \tag{4.15}$$

where the Error_1 term is bounded as in (4.12). On the other hand, since ϕ_t is regular enough, using Proposition 2.6 we may write

$$\sum_{i=1}^N \log \phi_t'(x_i) = N \int_{\mathbb{R}} \log \phi_t' \, d\mu_0 + t\mathsf{Error}_2(\vec{X}_N)$$

with an Error_2 term as in (4.13). Finally, since by definition $\phi_t \# \mu_0 = \tilde{\mu}_t$ we may observe that $\phi_t' = \frac{\mu_0}{\tilde{\mu}_t \circ \phi_t}$ and thus

$$\int_{\mathbb{R}} \log \phi_t' \, d\mu_0 = \int_{\mathbb{R}} \log \mu_0 \, d\mu_0 - \int_{\mathbb{R}} \log \mu_t \circ \phi_t \, d\mu_0 = \mathrm{Ent}(\mu_0) - \mathrm{Ent}(\tilde{\mu}_t). \tag{4.16}$$

This yields (4.11). □

4.4. Comparison of partition functions II: the anisotropy is small.

Proposition 4.6. *For any s, we have*

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{-s}{N} \mathsf{A} \right) \right) = o_N(1). \tag{4.17}$$

Proof. Applying Cauchy-Schwarz to (4.11) we may write

$$\begin{aligned} \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{\beta}{4} t\mathsf{A} \right) \right)^2 & \\ \leq \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{\beta}{2} t\mathsf{A} + t^2 \mathsf{Error}_1 + t\mathsf{Error}_2 \right) \right) & \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(-t^2 \mathsf{Error}_1 - t\mathsf{Error}_2 \right) \right) \\ \leq \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} \exp\left(\left(1 - \frac{\beta}{2} \right) N \left(\mathrm{Ent}(\tilde{\mu}_t) - \mathrm{Ent}(\mu_0) \right) \right) & \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp(-t^2 \mathsf{Error}_1 - t\mathsf{Error}_2) \right). \end{aligned} \tag{4.18}$$

In view of (2.9) we get, for t small enough,

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(t\mathsf{Error}_1)) \leq CtN, \qquad \log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(t\mathsf{Error}_2)) \leq CtN. \tag{4.19}$$

Inserting (2.8) into (4.18) we obtain that for t small enough,

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{\beta}{4} t\mathsf{A} \right) \right) \leq C(Nt^2 + N^{1/2} t) + \delta_N, \tag{4.20}$$

for some sequence $\{\delta_N\}_N$ with $\lim_{N\to\infty} \delta_N = 0$. Applying this to $t = 4\varepsilon/\beta$ with ε small and using Hölder's inequality, we deduce

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{-s}{N} \mathsf{A} \right) \right) \leq \frac{|s|}{N\varepsilon} \log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(\varepsilon \mathsf{A})) \leq C|s|\varepsilon + \frac{|s|}{\varepsilon} \delta_N.$$

In particular, choosing $\varepsilon = \sqrt{\delta_N}$, we get (4.17).

□

4.5. Conclusion: proof of Theorem 1.

Proof. Combining (4.11) for $t = -\frac{2s}{\beta N}$ (where s is independent of N) and (4.17) we find

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = \left(1 - \frac{\beta}{2} \right) N \left(\mathrm{Ent}(\mu_0) - \mathrm{Ent}(\tilde{\mu}_{\frac{-2s}{\beta N}}) \right) + o_N(1). \tag{4.21}$$

Using again (4.16) and $\phi_t' = 1 + t\psi'$, we may rewrite this as

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = -\left(1 - \frac{\beta}{2} \right) \frac{2s}{\beta} \int \psi' d\mu_0 + o_N(1). \tag{4.22}$$

Combining (4.8) and (4.22) and sending N to $+\infty$ we obtain,

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = -\left(1 - \frac{\beta}{2} \right) \frac{2s}{\beta} \int \psi' d\mu_0 + o_N(1), \tag{4.23}$$

with an error $o_N(1)$ uniform for s in a compact set of $\mathbb{R}$.

To conclude, we need the following relation, whose proof is given in Section B.6.

Lemma 4.7.

$$\mathcal{I}_{V_t}(\mu_t) - \mathcal{I}_V(\mu_0) = t\int \xi d\mu_0 + \frac{t^2}{2}\int \xi'\psi d\mu_0 + O(t^3\|\xi\|_{C^2(U)} + t^4\|\psi\|^4_{C^2(U)}), \tag{4.24}$$

where the O only depends on V.

Combining (2.5) with (4.23) and (4.24) we obtain,

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}}(\exp(s\mathrm{Fluct}_N(\xi)) = -\left(1-\frac{\beta}{2}\right)\frac{2s}{\beta}\int \psi' d\mu_V - \frac{s^2}{\beta}\int_{\mathbb{R}} \xi'\psi\, d\mu_V + o_N(1),$$

with an error $o_N(1)$ uniform for s in a compact set of $\mathbb{R}$.

Thus the Laplace transform of $\mathrm{Fluct}_N(\xi)$ converges (uniformly on compact sets) to that of a Gaussian of mean m_ξ and variance v_ξ, which implies convergence in law and proves the main theorem. □

Appendix A. The one-cut regular case

In the one-cut noncritical case, every regular enough function is in the range of the operator Ξ, so that the map ψ can always be built. This allows to bootstrap the approach used for proving Theorem 1. In this appendix, we expand on how we can proceed in this simpler setting without refering to the result of [LS15] but assuming more regularity of ξ, and retrieve the findings of [BG13b] (but without assuming analyticity), as well as a rate of convergence for the Laplace transform of the fluctuations.

A.1. **The bootstrap argument.** Let us first explain the main computational point for the bootstrap argument: by (4.14) and in view of Lemma 4.4, we may write

$$\frac{d}{dt}_{|t=0} \log K_{N,\beta}(\tilde\mu_t, \tilde\zeta_t) = \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(0)}[\vec{X}_N, \psi] + (1-\frac{\beta}{2})\frac{d}{dt}_{|t=0}\sum_{i=1}^N \log\phi_t'(x_i)\right]. \tag{A.1}$$

Differentiating (2.5) with respect to t and using Lemma 4.7 we thus obtain

$$\frac{-\beta N}{2}\mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}[\mathrm{Fluct}^{(0)}_N(\xi)] = \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(0)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\frac{d}{dt}_{|t=0}\sum_{i=1}^N \log\phi_t'(x_i)\right].$$

This is true as well for all $t\in[-t_{\max}, t_{\max}]$, i.e.

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}[\mathrm{Fluct}^{(t)}_N(\xi)] = -\frac{2}{\beta N}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\frac{d}{dt}\sum_{i=1}^N \log\phi_t'(x_i)\right]. \tag{A.2}$$

We may in addition write that

$$\frac{d}{dt}\sum_{i=1}^N \log\phi_t'(x_i) = N\int \frac{d}{dt}\log\phi_t'\, d\tilde\mu_t + \mathrm{Fluct}^{(t)}_N\left(\frac{d}{dt}\log\phi_t'\right) \tag{A.3}$$

so that

$$\begin{aligned}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}[\mathrm{Fluct}^{(t)}_N(\xi)] = &-\frac{2}{\beta}\left(1-\frac{\beta}{2}\right)\int \frac{d}{dt}\log\phi_t'\, d\tilde\mu_t \\ &- \frac{2}{\beta N}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\mathrm{Fluct}^{(t)}_N\left(\frac{d}{dt}\log\phi_t'\right)\right].\end{aligned} \tag{A.4}$$

This provides a functional equation which gives the expectation of the fluctuation in terms of a constant term plus a lower order expectation of another fluctuation and the A term (which itself can be written as a fluctuation, as noted below), allowing to expand it in powers of $1/N$ recursively.

A.2. **Improved control on the fluctuations.**

Lemma A.1. *Under the assumptions of Theorem 1 and assuming in addition*

$$\mathsf{p} \geq 3\mathsf{k}+6 \qquad \mathsf{r} \geq 2\mathsf{k}+4 \tag{A.5}$$

we have for any t in $(-t_{\max}, t_{\max})$ and[3] *s in $\mathbb{R}$*

$$\log \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}} \left[\exp\left(s \mathrm{Fluct}_N^{(t)}(\xi)\right)\right] \tag{A.6}$$
$$\leq C\left(s\|\xi\|_{C^{2\mathsf{k}+4}(U)} + s^2 \|\xi\|^2_{C^{2\mathsf{k}+3}(U)} + \frac{s^3}{N}\|\xi\|_{C^2(U)} + \frac{s^4}{N^2}\|\xi\|_{C^{2\mathsf{k}+3}(U)} + \frac{s^4}{N^2}\|\xi\|^4_{C^{2\mathsf{k}+3}(U)}\right)$$

where C depends only on V.

Proof. Note that in view of Lemma 3.3, the assumption (A.5) ensures that the transport map ψ is in $C^3(U)$. By (4.14) and in view of Lemma 4.4, we may write

$$\frac{d}{dt}_{|t=0} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(0)}[\vec{X}_N, \psi] + (1-\frac{\beta}{2})\frac{d}{dt}_{|t=0}\sum_{i=1}^N \log \phi_t'(x_i)\right]. \tag{A.7}$$

Similarly, we have for all t,

$$\frac{d}{dt} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + (1-\frac{\beta}{2})\frac{d}{dt}\sum_{i=1}^N \log \phi_t'(x_i)\right]. \tag{A.8}$$

Indeed, V_t has the same regularity as V and $\tilde{\mu}_t$ the same as μ_0.

Next, we express the anisotropy term as a fluctuation, by writing

$$\mathsf{A}^{(t)}[\vec{X}_N, \psi] = \int g(x) d\mathrm{fluct}_N^{(t)}(x), \tag{A.9}$$

where we let

$$g(x) := \int \hat{\psi}(x,y) d\mathrm{fluct}_N^{(t)}(y), \quad \hat{\psi}(x,y) := \frac{\psi(x)-\psi(y)}{x-y}. \tag{A.10}$$

It is clear that

$$\|\hat{\psi}\|_{C^2(U\times U)} \leq \|\psi\|_{C^3(U)}. \tag{A.11}$$

Using Proposition 2.6 twice, we can thus write

$$\|\nabla g\|_{L^\infty} \leq \left|\int \nabla_x \hat{\psi}(x,y) d\mathrm{fluct}_N^{(t)}(y)\right| \leq C\|\nabla_x\nabla_y \hat{\psi}\|_{L^\infty}\left(F_N(\vec{X}_N, \tilde{\mu}_t) + N\log N + CN\right)^{\frac{1}{2}}$$

and

$$|\mathsf{A}^{(t)}[\vec{X}_N, \psi]| = \left|\int g(x) d\mathrm{fluct}_N^{(t)}(x)\right| \leq C\|\nabla g\|_{L^\infty}\left(F_N(\vec{X}_N, \tilde{\mu}_t) + N\log N + CN\right)^{\frac{1}{2}}$$
$$\leq C\|\hat{\psi}\|_{C^2(U\times U)}\left(F_N(\vec{X}_N, \tilde{\mu}_t) + N\log N + CN\right).$$

[3]In this statement, s and t are not related.

In view of (2.9) and (A.11), we deduce that

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N,\psi]\right]\right| \le CN\|\psi\|_{C^3(U)}. \tag{A.12}$$

For the term $\log \phi_t'$ we use (A.3) and in view of Proposition 2.6, since $\phi_t = \mathrm{Id} + t\psi$ is regular enough, we may write

$$\left|\int \frac{d}{dt}\log\phi_t'\, d\mathrm{fluct}_N^{(t)}\right| \le C\|\psi\|_{C^2(U)}\left(F_N(\vec{X}_N,\tilde{\mu}_t) + N\log N + CN\right)^{\frac{1}{2}}. \tag{A.13}$$

We conclude from (A.8), using again (2.9) that

$$\left|\frac{d}{dt}\log K_{N,\beta}(\tilde{\mu}_t,\tilde{\zeta}_t)\right| \le CN\|\psi\|_{C^3(U)}. \tag{A.14}$$

Integrating this relation between 0 and $-\frac{2s}{\beta N}$, and combining with (4.8), we find that, for $t = \frac{-2s}{\beta N}$,

$$\left|\log\frac{K_{N,\beta}(\mu_t,\zeta_t)}{K_{N,\beta}(\mu_0,\zeta_0)}\right| \le Cs\|\psi\|_{C^3(U)}. \tag{A.15}$$

Inserting this, (4.8) and (4.24) into (2.5), we deduce that

$$\begin{aligned}\left|\log \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}[\exp(s\mathrm{Fluct}_N(\xi))]\right| \\ \le C\Bigg(s\|\psi\|_{C^3(U)} + s^2\|\psi\|_{C^0(U)}\|\xi\|_{C^1(U)} + \frac{s^3}{N}\|\xi\|_{C^2(U)} + \frac{s^4}{N^2}\|\psi\|_{C^2(U)} \\ + \frac{s^2}{\sqrt{N}}\|\psi\|^2_{C^2(U)} + \frac{s^4}{N^2}\|\psi\|^4_{C^2(U)}\Bigg).\end{aligned} \tag{A.16}$$

In view of (3.6), it yields the result for the expectation under $\mathbb{P}^{(0)}_{N,\beta}$, and then this can be generalized from $\mathbb{P}^{(0)}_{N,\beta}$ to $\mathbb{P}^{(t)}_{N,\beta}$ for t in $(-t_{\max}, t_{\max})$ because $\tilde{\mu}_t$ has the same regularity as μ_0. □

Assuming from now on that $\mathsf{n} = 0$ and $\mathsf{m} = 0$ (so that every regular function is in the range of Ξ) we can upgrade this control of exponential moments into the control of a weak norm of $\mathrm{Fluct}_N^{(t)}$. Here we use the Sobolev spaces $H^\alpha(\mathbb{R})$.

Lemma A.2. *Under the same assumptions, for $\alpha \ge 8$ we have*

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}_N^{(t)}\|^2_{H^{-\alpha}}\right]\right| \le C, \tag{A.17}$$

where C depends only on V.

Proof. The proof is inspired by [AKM17], in particular we start from [AKM17, Prop. D.1] which states that

$$\|u\|^2_{H^{-\alpha}(\mathbb{R})} \le C\int_0^1 r^{\alpha-1}\|u * \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\, dr \tag{A.18}$$

where $\Phi(r,\cdot)$ is the standard heat kernel, i.e. $\Phi(r,x) = \frac{1}{\sqrt{4\pi r}} e^{-\frac{|x|^2}{4r}}$. It follows that

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}^{(t)}_N\|^2_{H^{-\alpha}(\mathbb{R})}\|\right] \leq C\int_0^1 r^{\alpha-1}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}^{(t)}_N * \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\right] dr. \tag{A.19}$$

On the other hand we may easily check that, letting $\xi_{x,r} := \Phi(r, x-\cdot)$, we have

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}^{(t)}_N * \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\right] = \int \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\left(\mathrm{Fluct}^{(t)}_N(\xi_{x,r})\right)^2\right] dx. \tag{A.20}$$

Applying the result of Lemma A.1 to $\xi_{x,r}$ gives us a control on the second moment of $\mathrm{Fluct}^{(t)}_N[\xi_{x,r}]$ of the form

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[(\mathrm{Fluct}^{(t)}_N(\xi_{x,r}))^2\right] \leq C\left(\|\xi_{x,r}\|_{C^4(U)} + \|\xi_{x,r}\|^2_{C^3(U)}\right).$$

Inserting into (A.19) and (A.20), we are led to

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}^{(t)}_N\|^2_{H^{-\alpha}(\mathbb{R})}\right] \leq C\int_0^1\int r^{\alpha-1}\left(\|\xi_{x,r}\|_{C^4(U)} + \|\xi_{x,r}\|^2_{C^3(U)}\right) dx\, dr.$$

Since U is bounded, we may check that this right-hand side can be bounded by $C\int_0^1 r^{\alpha-1}r^{-7}\,dr$, which converges if $\alpha > 7$. □

A.3. **Proof of Theorem 2.** For any test function $\phi(x,y)$ we may write

$$\int \phi(x,y) d\mathrm{fluct}^{(t)}_N(x)\, d\mathrm{fluct}^{(t)}_N(y) \leq \|\phi\|_{C^{2\alpha}(U\times U)}\|\mathrm{fluct}^{(t)}_N\|^2_{H^{-\alpha}(\mathbb{R})}$$

and so by the result of Lemma A.2, we find

$$\left|\mathbf{E}_{\mathbb{P}_{N,\beta}}\left(\int \phi(x,y) d\mathrm{fluct}^{(t)}_N(x)\, d\mathrm{fluct}^{(t)}_N(y)\right)\right| \leq C\|\phi\|_{C^{2\alpha}(U\times U)}. \tag{A.21}$$

We may now bootstrap the result of Lemma A.1 by returning to (A.9) and, using (A.21), writing that

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathsf{A}^{(t)}[\vec{X}_N,\psi]\right]\right| \leq C\|\psi\|_{C^{2\alpha+1}(U)}. \tag{A.22}$$

On the other hand, by differentiating (A.6) applied with $\xi = \frac{d}{dt}\log\phi'_t$, we have

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\int \frac{d}{dt}\log\phi'_t d\mathrm{fluct}^{(t)}_N\right]\right| \leq C\|\psi\|_{C^5(U)} \tag{A.23}$$

Inserting (4.16) and (A.22) and (A.23), (A.3) into (A.8), and integrating between 0 and $t = -2s/N\beta$, we obtain

$$\log\frac{K_{N,\beta}(\tilde{\mu}_t,\tilde{\zeta}_t)}{K_{N,\beta}(\mu_0,\zeta_0)} = \left(1-\frac{\beta}{2}\right)N\left(\mathrm{Ent}(\tilde{\mu}_t) - \mathrm{Ent}(\mu_0)\right) + \frac{s}{N}O(\|\xi\|_{C^{2\alpha+1}(U)}).$$

Using again (4.16) and $\phi'_t = 1 + t\psi'$, we may rewrite this as

$$\log\frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}},\tilde{\zeta}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0,\zeta_0)} = -\left(1-\frac{\beta}{2}\right)\frac{2s}{\beta}\int \psi' d\mu_0 + O(\frac{s}{N}\|\xi\|_{C^{2\alpha+1}(U)})$$

Combining this with (4.8), (2.5) with (4.23) and (4.24) we obtain

$$\left|\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left(\exp(s\mathrm{Fluct}_N(\xi)\right) + \left(1-\frac{\beta}{2}\right)\frac{2s}{\beta}\int \psi' d\mu_V + \frac{s^2}{\beta}\int_{\mathbb{R}} \xi'\psi\, d\mu_V\right| \leq C\left(\frac{s}{N}\|\xi\|_{C^{2\alpha+1}} + \frac{s^3}{N}\|\xi\|_{C^2} + \frac{s^4}{N^2}\|\xi\|^4_{C^3}\right). \tag{A.24}$$

with C depending only on V. This proves Theorem 2.

A.4. **Iteration and expansion of the partition function to arbitrary order.** Let V, W be two C^∞ potentials, such that the associated equilibrium measures μ_V, μ_W satisfy our assumptions with $\mathsf{n} = 0, \mathsf{m} = 0$. In this section, we explain how to iterate the procedure described above to obtain a relative expansion of the partition function, namely an expansion of $\log Z^W_{N,\beta} - \log Z^V_{N,\beta}$ to any order of $1/N$. Up to applying an affine transformation to one of the gases, whose effect on the partition function is easy to compute, we may assume that μ_V and μ_W have the same support Σ, which is a line segment.

Since V, W are C^∞ and μ_V, μ_W have the same support and a density of the same form (1.10) which is C^∞ on the interior of Σ, the optimal transportation map (or monotone rearrangement) ϕ from μ_V to μ_W is C^∞ on Σ and can be extended as a C^∞ function with compact support on $\mathbb{R}$. We let $\psi := \phi - \mathrm{Id}$, which is smooth, and for $t \in [0,1]$ the map $\phi_t := \mathrm{Id} + t\psi$ is a C^∞-diffeomorphism, by the properties of optimal transport. We let $\tilde{\mu}_t := \phi_t \# \mu_V$ as before.

We can integrate (A.8) to obtain

$$\begin{aligned}
&\log \frac{K_{N,\beta}(\mu_W, \zeta_W)}{K_{N,\beta}(\mu_V, \zeta_V)} \\
&= \int_0^1 \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)N\int \frac{d}{dt}\log \phi_t'\, d\tilde{\mu}_t + \left(1-\frac{\beta}{2}\right)\int \frac{d}{dt}\log \phi_t' d\mathrm{fluct}_N^{(t)}\right] dt \\
&= N\left(1-\frac{\beta}{2}\right)\left(\mathrm{Ent}(\mu_W) - \mathrm{Ent}(\mu_V)\right) \\
&\quad + \int_0^1 \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\mathrm{Fluct}_N\left[\int \frac{d}{dt}\log \phi_t' d\mathrm{fluct}_N^{(t)}\right]\right] dt.
\end{aligned}$$

The integral on the right-hand side is of order 1, and we claim that the terms in the integral can actually be computed and expanded up to an error $O(1/N)$ using the previous lemma. This is clear for the term $\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(\frac{d}{dt}\log \phi_t')\right]$ which can be computed up to an error $O(1/N)$ by the result of Theorem 2. The term $\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi]\right]$ can on the other hand be deduced from the knowledge of the covariance structure of the fluctuations. Let $\mathcal{F}$ denote the Fourier transform. In view of (A.9), using the identity

$$\frac{\psi(x) - \psi(y)}{x - y} = \int_0^1 \psi'(sx + (1-s)y)ds$$

and the Fourier inversion formula we may write

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathsf{A}^{(t)}[\vec{X}_N,\psi]\right] = \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\iint_{\mathbb{R}\times\mathbb{R}}\int_0^1 \psi'(sx+(1-s)y)ds\, d\mathrm{fluct}_N^{(t)}(x)d\mathrm{fluct}_N^{(t)}(y)\right]$$
$$= \int\int_0^1 \lambda\mathcal{F}(\psi)(\lambda)\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\mathrm{Fluct}_N^{(t)}(e^{i(1-s)\lambda\cdot})\right] ds\, d\lambda. \tag{A.25}$$

On the other hand, let $\varphi_{s,\lambda}$ be the map associated to $e^{is\lambda\cdot}$ by Lemma 3.3. Separating the real part and the imaginary part we may use the results of the previous subsection to $e^{is\lambda\cdot}$ and obtain

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\right] = \left(1-\frac{2}{\beta}\right)\int \varphi'_{s,\lambda}d\tilde{\mu}_t + O(\frac{1}{N})\ .$$

By polarization of the expression for the variance (see (1.16)) and linearity

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\mathrm{Fluct}_N^{(t)}(e^{i(1-s)\lambda\cdot})\right] = \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\right]\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{i(1-s)\lambda\cdot})\right]$$
$$+\frac{2}{\beta}\Big(\iint\left(\frac{\varphi_{s,\lambda}(u)-\varphi_{s,\lambda}(v)}{u-v}\right)\left(\frac{\varphi_{(1-s),\lambda}(u)-\varphi_{(1-s),\lambda}(v)}{u-v}\right)d\tilde{\mu}_t(u)d\tilde{\mu}_t(v)$$
$$+\int V_t''\varphi_{s,\lambda}\varphi_{(1-s),\lambda}d\tilde{\mu}_t\Big) + O(\frac{1}{N}).$$

Letting $N\to\infty$, we may then find the expansion up to $O(1/N)$ of $\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N,\psi]\right]$. Inserting it into the integral gives a relative expansion to order $1/N$ of the (logarithm of the) partition function $\log K_{N,\beta}$. This procedure can then be iterated to yield a relative expansion to arbitrary order of $1/N$ as desired.

Appendix B. Auxiliary proofs

B.1. **Proof of Lemma 2.1.**

Proof. Denoting $\triangle$ the diagonal in $\mathbb{R}\times\mathbb{R}$ we may write

$$\mathcal{H}_N^V(\vec{X}_N) = \sum_{i\neq j} -\log|x_i-x_j| + N\sum_{i=1}^N V(x_i)$$
$$= \iint_{\triangle^c} -\log|x-y|\Big(\sum_{i=1}^N\delta_{x_i}\Big)(x)\Big(\sum_{i=1}^N\delta_{x_i}\Big)(y) + N\int_{\mathbb{R}} V(x)\Big(\sum_{i=1}^N\delta_{x_i}\Big)(x).$$

Writing $\sum_{i=1}^N\delta_{x_i}$ as $N\mu_V + \mathrm{fluct}_N$ we get

$$\mathcal{H}_N^V(\vec{X}_N) = N^2\iint_{\triangle^c} -\log|x-y|d\mu_V(x)d\mu_V(y) + N^2\int_{\mathbb{R}} Vd\mu_V$$
$$+2N\iint_{\triangle^c} -\log|x-y|d\mu_V(x)d\mathrm{fluct}_N(y) + N\int_{\mathbb{R}} Vd\mathrm{fluct}_N$$
$$+\iint_{\triangle^c} -\log|x-y|d\mathrm{fluct}_N(x)d\mathrm{fluct}_N(y). \tag{B.1}$$

We now recall that ζ_V was defined in (1.7), and that $\zeta_V = 0$ in Σ_V. With the help of this we may rewrite the medium line in the right-hand side of (B.1) as

$$\begin{aligned}2N \iint_{\triangle^c} &- \log|x-y| d\mu_V(x) d\mathrm{fluct}_N(y) + N \int_{\mathbb{R}} V d\mathrm{fluct}_N \\ &= 2N \int_{\mathbb{R}} \left(-\log|\cdot| * d\mu_V)(x) + \frac{V}{2}\right) d\mathrm{fluct}_N = 2N \int_{\mathbb{R}} (\zeta_V + c) d\mathrm{fluct}_N \\ &= 2N \int_{\mathbb{R}} \zeta_V d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big) = 2N \sum_{i=1}^N \zeta_V(x_i).\end{aligned}$$

The last equalities are due to the facts that ζ_V vanishes on the support of μ_V and that fluct_N has a total mass 0 since μ_V is a probability measure. We may also notice that since μ_V is absolutely continuous with respect to the Lebesgue measure, we may include the diagonal back into the domain of integration. By that same argument, one may recognize in the first line of the right-hand side of (B.1) the quantity $N^2 \mathcal{I}_V(\mu_V)$. ☐

B.2. **Proof of Proposition 2.6.** We follow the energy approach introduced in [SS15, PS14], which views the energy as a Coulomb interaction in the plane, after embedding the real line in the plane. We view $\mathbb{R}$ as identified with $\mathbb{R} \times \{0\} \subset \mathbb{R}^2 = \{(x,y), x \in \mathbb{R}, y \in \mathbb{R}\}$. Let us denote by $\delta_{\mathbb{R}}$ the uniform measure on $\mathbb{R} \times \{0\}$, i.e. such that for any smooth $\varphi(x,y)$ (with $x \in \mathbb{R}, y \in \mathbb{R}$) we have

$$\int_{\mathbb{R}^2} \varphi \delta_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(x,0)\, dx.$$

Given $(x_1, \dots, x_N)$ in $\mathbb{R}^N$, we identify them with the points $(x_1,0), \dots, (x_N,0)$ in $\mathbb{R}^2$. For a fixed $\vec{X}_N$ and a given probability density μ we introduce the electric potential H_N^μ by

$$H_N^\mu = (-\log|\cdot|) * \left(\sum_{i=1}^N \delta_{(x_i,0)} - N\mu\delta_{\mathbb{R}} \right). \tag{B.2}$$

Next, we define versions of this potential which are truncated hence regular near the point charges. For that let $\delta_x^{(\eta)}$ denote the uniform measure of mass 1 on $\partial B(x,\eta)$ (where B denotes an Euclidean ball in $\mathbb{R}^2$). We define $H_{N,\eta}^\mu$ in $\mathbb{R}^2$ by

$$H_{N,\eta}^\mu = (-\log|\cdot|) * \left(\sum_{i=1}^N \delta_{(x_i,0)}^{(\eta)} - N\mu\delta_{\mathbb{R}} \right). \tag{B.3}$$

These potentials make sense as functions in $\mathbb{R}^2$ and are harmonic outside of the real axis. Moreover, $H_{N,\eta}^\mu$ solves

$$-\Delta H_{N,\eta}^\mu = 2\pi \left(\sum_{i=1}^N \delta_{(x_i,0)}^{(\eta)} - N\mu\delta_{\mathbb{R}} \right). \tag{B.4}$$

Lemma B.1. *For any probability density μ, $\vec{X}_N$ in $\mathbb{R}^N$ and η in $(0,1)$, we have*

$$F_N(\vec{X}_N, \mu) \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla H_{N,\eta}^\mu|^2 + N\log\eta - 2N^2 \|\mu\|_{L^\infty} \eta. \tag{B.5}$$

Proof. First we notice that $\int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2$ is a convergent integral and that

$$\int_{\mathbb{R}^2} |\nabla H_{N,\vec{\eta}}|^2 = 2\pi \iint -\log|x-y| d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(y). \tag{B.6}$$

Indeed, we may choose R large enough so that all the points of $\vec{X}_N$ are contained in the ball $B_R = B(0,R)$. By Green's formula and (B.4), we have

$$\int_{B_R} |\nabla H_{N,\eta}|^2 = \int_{\partial B_R} H_{N,\eta} \frac{\partial H_N}{\partial \nu} + 2\pi \int_{B_R} H_{N,\eta} \left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right). \tag{B.7}$$

In view of the decay of H_N and ∇H_N, the boundary integral tends to 0 as $R \to \infty$, and so we may write

$$\int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2 = 2\pi \int_{\mathbb{R}^2} H_{N,\eta} \left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\right)$$

and thus (B.6) holds. We may next write

$$\begin{aligned}\iint -\log|x-y| d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(y) \\ - \iint_{\triangle^c} -\log|x-y|\, d\mathrm{fluct}_N(x)\, d\mathrm{fluct}_N(y)\end{aligned} \tag{B.8}$$

$$= -\sum_{i=1}^N \log\eta + \sum_{i\neq j} \iint -\log|x-y| \left(\delta_{x_i}^{(\eta)}\delta_{x_j}^{(\eta)} - \delta_{x_i}\delta_{x_j}\right) + 2N\sum_{i=1}^N \iint -\log|x-y| \left(\delta_{x_i} - \delta_{x_i}^{(\eta)}\right)\mu.$$

Let us now observe that $\int -\log|x-y|\delta_{x_i}^{(\eta)}(y)$, the potential generated by $\delta_{x_i}^{(\eta)}$ is equal to $\int -\log|x-y|\delta_{x_i}$ outside of $B(x_i,\eta)$, and smaller otherwise. Since its Laplacian is $-2\pi\delta_{x_i}^{(\eta)}$, a negative measure, this is also a superharmonic function, so by the maximum principle, its value at a point x_j is larger or equal to its average on a sphere centered at x_j. Moreover, outside $B(x_i,\eta)$ it is a harmonic function, so its values are equal to its averages. We deduce from these considerations, and reversing the roles of i and j, that for each $i \neq j$,

$$-\int \log|x-y|\delta_{x_i}^{(\eta)}\delta_{x_j}^{(\eta)} \leq -\int \log|x-y|\delta_{x_i}\delta_{x_j}^{(\eta)} \leq -\int \log|x-y|\delta_{x_i}\delta_{x_j}.$$

We may also obviously write

$$\int -\log|x-y|\delta_{x_i}\delta_{x_j} - \int -\log|x-y|\delta_{x_i}^{(\eta)}\delta_{x_j}^{(\eta)} \leq -\log|x_i - x_j|\mathbf{1}_{|x_i-x_j|\leq 2\eta}.$$

We conclude that the second term in the right-hand side of (B.8) is nonpositive, equal to 0 if all the balls are disjoint, and bounded below by $\sum_{i\neq j}\log|x_i - x_j|\mathbf{1}_{|x_i-x_j|\leq 2\eta}$. Finally, by the above considerations, since $\int -\log|x-y|\delta_{x_i}^{(\eta)}$ coincides with $\int -\log|x-y|\delta_{x_i}$ outside $B(x_i,\eta)$, we may rewrite the last term in the right-hand side of (B.8) as

$$2N\sum_{i=1}^N \int_{B(x_i,\eta)} (-\log|x-x_i| + \log\eta))d\mu\delta_{\mathbb{R}}.$$

But we have that

$$\int_{B(0,\eta)} (-\log|x| + \log \eta)\delta_{\mathbb{R}} = \eta \tag{B.9}$$

so if $\mu \in L^\infty$, this last term is bounded by $2\|\mu\|_{L^\infty} N^2 \eta$. Combining with all the above results yields the proof. □

Proof of Proposition 2.6. We now apply Lemma B.1 for μ_V with $\eta = \frac{1}{2N}$. We obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla H^{\mu}_{N,\eta}|^2 \le F_N(\vec{X}_N, \mu_V) + N\log N + C(\|\mu_V\|_{L^\infty} + 1)N. \tag{B.10}$$

Let ξ be a smooth compactly supported test function in $\mathbb{R}$. We may extend it to a smooth compactly supported test function in $\mathbb{R}^2$ coinciding with $\xi(x)$ for any (x, y) such that $|y| \le 1$ and equal to 0 for $|y| \ge 2$. Letting $\#I$ denote the number of balls $B(x_i, \eta)$ intersecting the support of ξ, we have

$$\begin{aligned}\left|\int \left(\mathrm{fluct}_N - \left(\sum_{i=1}^N \delta^{(\eta)}_{x_i} - N\mu_V\right)\right)\xi\right| &= \left|\int \left(\sum_{i=1}^N (\delta_{x_i} - \delta^{(\eta)}_{x_i})\right)\xi\right| \\ &\le \#I\eta\|\nabla\xi\|_{L^\infty} = \frac{1}{2}\frac{\#I}{N}\|\nabla\xi\|_{L^\infty}.\end{aligned} \tag{B.11}$$

But in view of (B.4), we also have

$$\begin{aligned}\left|\int \left(\sum_{i=1}^N \delta^{(\eta)}_{x_i} - N\mu_V\right)\xi\right| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}^2} \nabla H^{\mu_V}_{N,\eta}\cdot\nabla\xi\right| \\ &\le |\mathrm{Supp}\,\xi|^{\frac{1}{2}}\|\nabla\xi\|_{L^\infty}\|\nabla H^{\mu_V}_{N,\eta}\|_{L^2(\mathrm{Supp}\,\xi)}.\end{aligned} \tag{B.12}$$

Combining (B.10), (B.11) and (B.12), we obtain

$$\begin{aligned}&\left|\int \xi\,\mathrm{fluct}_N\right| \\ &\le C\|\nabla\xi\|_{L^\infty}\left(\frac{\#I}{N} + |\mathrm{Supp}\,\xi|^{\frac{1}{2}}\left(F_N(\vec{X}_N, \mu_V) + N\log N + C(\|\mu_V\|_{L^\infty} + 1)N\right)^{\frac{1}{2}}\right).\end{aligned} \tag{B.13}$$

Bounding $\#I$ by N yields the result. □

B.3. **Proof of Lemma 3.1.**

Proof. Since μ_V minimizes the logarithmic potential energy (1.6), for any bounded continuous function h we have

$$\iint \frac{h(x) - h(y)}{x - y} d\mu_V(x) d\mu_V(y) = \int V'(x) h(x) d\mu_V(x). \tag{B.14}$$

Of course, an identity like (B.14) extends to complex-valued functions, and applying it to $h = \frac{1}{z-\cdot}$ for some fixed $z \in \mathbb{C} \setminus \Sigma_V$ leads to

$$G(z)^2 - G(z)V'(\mathcal{R}(z)) + L(z) = 0, \tag{B.15}$$

where G is the usual Stieltjes transform of μ_V

$$G(z) = \int \frac{1}{z - y} d\mu_V(y), \tag{B.16}$$

and L is defined by

$$L(z) = \int \frac{V'(\mathcal{R}(z)) - V'(y)}{z-y} d\mu_V(y). \tag{B.17}$$

Solving (B.15) for G yields

$$G(z) = \frac{1}{2}\left(V'(\mathcal{R}(z)) - \sqrt{V'(\mathcal{R}(z))^2 - 4L(z)}\right). \tag{B.18}$$

As is well-known, $-\frac{1}{\pi}\mathcal{I}(G(x+i\varepsilon))$ converges towards the density $\mu_V(x)$ as $\varepsilon \to 0^+$, hence we have for x in Σ_V

$$\mu_V(x)^2 = S(x)^2\sigma^2(x) = -\frac{1}{(2\pi)^2}(V'(x)^2 - 4L(x)). \tag{B.19}$$

This proves that μ_V has regularity $C^{\mathsf{p}-2}$ at any point where it does not vanish. Assuming the form (1.11) for S, we also deduce that the function S_0 has regularity at least $C^{\mathsf{p}-3-2\mathsf{k}}$ on Σ_V.

Applying (B.18) on $\mathbb{R} \setminus \Sigma$, we obtain

$$\frac{1}{2}V'(x) - \int \frac{1}{x-y} d\mu_V(y) = \frac{1}{2}\sqrt{V'(x)^2 - 4L(x)},$$

and the left-hand side is equal to $\zeta'(x)$.

Using (1.11), (B.19) and the fact that V is regular, we may find a neighborhood U small enough such that ζ' does not vanish on $U \setminus \Sigma_V$ and on which we can write ζ' as in (3.1). $\square$

B.4. **Proof of Lemma 3.2.**

Proof. We first prove that the image of F is indeed contained in $C^1(U)$.

For $(t,\psi) = (0,0)$, we have indeed $\mathcal{F}(0,0) = \zeta_V + c$ and ζ_V is in $C^1(\mathbb{R})$ by the regularity assumptions on V. We may also write

$$\mathcal{F}(t,\psi) = \mathcal{F}(0,0) - \int \log \frac{|\phi(\cdot) - \phi(y)|}{|\cdot - y|} d\mu_V(y) + \frac{1}{2}(V_t \circ \phi - V \circ \phi),$$

and since $\|\psi\|_{C^2(U)} \le 1/2$, the second and third terms are also in $C^1(U)$.

Next, we compute the partial derivatives of $\mathcal{F}$ at a fixed point $(t_0,\psi_0) \in [-1,1] \times B$. It is easy to see that

$$\frac{\partial \mathcal{F}}{\partial t}\Big|_{(t_0,\psi_0)} = \frac{1}{2}\xi \circ \phi_0,$$

and the map $(t_0,\psi_0) \mapsto \xi \circ \phi_0$ is indeed continuous.

The Fréchet derivative of F with respect to the second variable can be computed as follows

$$\begin{aligned}
\mathcal{F}(t_0,\psi_0+\psi_1) &= -\int \log\Big|(\phi_0(\cdot) - \phi_0(y)) + (\psi_1(\cdot) - \psi_1(y))\Big| d\mu_V(y) + \frac{1}{2}V_{t_0} \circ (\phi_0 + \psi_1) \\
&= \mathcal{F}(t_0,\psi_0) - \int \log\Big|1 + \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)}\Big| d\mu_V(y) + \frac{1}{2}(V_{t_0} \circ (\phi_0+\psi_1) - V_{t_0} \circ \phi_0) \\
&= \mathcal{F}(t_0,\psi_0) - \int \frac{\psi_1(\cdot) - \psi_1(y)}{\phi_0(\cdot) - \phi_0(y)} d\mu_V(y) + \frac{1}{2}\psi_1 V'_{t_0} \circ \phi_0 + \varepsilon_{t_0,\psi_0}(\psi_1),
\end{aligned}$$

where $\varepsilon_{t_0,\psi_0}(\psi_1)$ is given by

$$\varepsilon_{t_0,\psi_0}(\psi_1) = -\int \Big[\log\Big|1+\frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}\Big| - \frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}\Big] d\mu_V(y) + \frac{1}{2}\big(V_{t_0}\circ(\phi_0+\psi_1) - V_{t_0}\circ\phi_0 - \psi_1 V'_{t_0}\circ\phi_0\big)\ .$$

By differentiating twice inside the integral we get the bound

$$\|\varepsilon_{t_0,\psi_0}(\psi_1)\|_{C^1(U)} \le C(t_0,\psi_0)\|\psi_1\|^2_{C^2(U)},$$

with a constant depending on V. It implies that

$$\frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(t_0,\psi_0)}[\psi_1] = -\int \frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)} d\mu_V(y) + \frac{1}{2}\psi_1 V'_{t_0}\circ\phi_0\ ,$$

and we can check that this expression is also continuous in (t_0,ψ_0). In particular, we may observe that

$$\frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(0,0)}[\psi] = -\Xi_V[\psi]. \tag{B.20}$$

Finally, we prove the bound (3.3). For any fixed $(t,\psi)\in[-1,1]\times B$, we write

$$\mathcal{F}(t,\psi)-\mathcal{F}(0,0) = \int_0^1 \frac{d\mathcal{F}(st,s\psi)}{ds} ds = \int_0^1 \Big(t\frac{\partial \mathcal{F}}{\partial t}\Big|_{(st,s\psi)} + \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(st,s\psi)}[\psi]\Big) ds\ ,$$

we get

$$\|\mathcal{F}(t,\psi)-\mathcal{F}(0,0)-\frac{t}{2}\xi+\Xi_V[\psi]\|_{C^1(U)} \le \int_0^1 \Big(\frac{t}{2}\|\xi\circ\phi_s-\xi\|_{C^1(U)} + \Big\|\frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(st,s\psi)}[\psi] - \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(0,0)}[\psi]\Big\|_{C^1(U)}\Big) ds, \tag{B.21}$$

with $\phi_s = \mathrm{Id}+s\psi$. It is straightforward to check that

$$\|\xi\circ\phi_s-\xi\|_{C^1(U)} \le C\|\xi\|_{C^2(U)}\|\psi\|_{C^1(U)}\ .$$

To control the second term inside the integral we write

$$\frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(st,s\psi)}[\psi] - \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(0,0)}[\psi] = -\int \Big(\frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)} - \frac{\psi(\cdot)-\psi(y)}{\cdot - y}\Big) d\mu_V(y) + \frac{1}{2}\left(V'_{st}\circ\phi_s - V'\right)\psi$$

and we obtain

$$\Big\|\frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(st,s\psi)}[\psi] - \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(0,0)}[\psi]\Big\|_{C^1(U)} \le \int \Big\|\frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)} - \frac{\psi(\cdot)-\psi(y)}{\cdot - y}\Big\|_{C^1(U)} d\mu_V(y) + \|(V'_{st}\circ\phi_s - V')\psi\|_{C^1(U)}$$

We now use that

$$\begin{aligned}\left\|\Big(\frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)}-\frac{\psi(\cdot)-\psi(y)}{\cdot - y}\Big)\right\|_{C^1(U)} &= \left\|\Big(\frac{\psi(\cdot)-\psi(y)}{\cdot - y}\Big)\Big(\frac{\cdot - y}{\phi_s(\cdot)-\phi_s(y)}-1\Big)\right\|_{C^1(U)} \\ &\le C\|\psi\|_{C^2(U)}\left\|\frac{\cdot - y}{\phi_s(\cdot)-\phi_s(y)}-1\right\|_{C^1(U)} \\ &= Cs\|\psi\|_{C^2(U)}\left\|\frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)}\right\|_{C^1(U)} \\ &\le C\|\psi\|_{C^2(U)}^2\left\|\frac{\cdot - y}{\phi_s(\cdot)-\phi_s(y)}\right\|_{C^1(U)} \\ &\le C\|\psi\|_{C^2(U)}^2 .\end{aligned}$$

In the second and the fourth line, we used Leibniz formula . In the last line we used that $s(\psi(\cdot)-\psi(y))/(\cdot - y)$ is uniformely bounded by $1/2$ in $C^2(U)$ so its composition with the function $x \to 1/(1+x)$ is bounded in $C^2(U)$. We conclude by checking that

$$\|(V'_{st}\circ\phi_s - V')\psi\|_{C^1(U)} \le C\Big(\|V\|_{C^3(U)}\|\psi\|_{C^1(U)} + t\|\psi\|_{C^2(U)}\Big)\|\psi\|_{C^0(U)} .$$

□

B.5. **Proof of Lemma 3.3.**

Proof. First, we solve the equation $\Xi_V[\psi] = \frac{1}{2}\xi + c_\xi$ in $\mathring{\Sigma}_V$, where Ξ_V is operator defined in (1.12). For x in $\mathring{\Sigma}_V$, we have the following Schwinger-Dyson equation

$$\frac{V'(x)}{2} = P.V.\int \frac{1}{x-y}d\mu_V(y). \tag{B.22}$$

In particular, for x in $\mathring{\Sigma}_V$, it implies

$$\Xi_V[\psi](x) := P.V.\int_{\Sigma_V}\frac{\psi(y)}{y-x}\mu_V(y)dy, \tag{B.23}$$

and we might thus try to solve

$$P.V.\int_{\Sigma_V}\frac{\psi(y)}{y-x}\mu_V(y)dy = \frac{1}{2}\xi + c_\xi. \tag{B.24}$$

Equation (B.24) is a singular integral equation, we refer to [Mus92, Chap. 10-11-12] for a detailed treatment. In particular, it is known that if the conditions (1.14) are satisfied, then there exists a solution ψ_0 to

$$P.V.\int_{\Sigma_V}\frac{\psi_0(y)}{y-x}dy = \frac{1}{2}\xi + c_\xi \text{ on } \mathring{\Sigma}_V, \tag{B.25}$$

which is explicitly given by the formula

$$\psi_0(x) = -\frac{\sigma(x)}{2\pi^2}P.V.\int_{\Sigma_V}\frac{\xi(y)}{\sigma(y)(y-x)}dy. \tag{B.26}$$

Since we have, for x in $\mathring{\Sigma}_V$

$$P.V.\int_{\Sigma_V}\frac{1}{\sigma(y)(y-x)}dy = 0,$$

we may re-write (B.26) as

$$\psi_0(x) = -\frac{\sigma(x)}{2\pi^2}\int_{\Sigma_V}\frac{\xi(y)-\xi(x)}{\sigma(y)(y-x)}dy \text{ on } \mathring{\Sigma}_V, \tag{B.27}$$

where the integral is now a definite Riemann integral. From (B.27) we deduce that the map $\frac{\psi_0}{\sigma}$ is of class $C^{\mathsf{r}-1}$ in $\mathring{\Sigma}_V$ and extends readily to a $C^{\mathsf{r}-1}$ function on Σ_V.

For $d = 0, \dots, \mathsf{r}-1$ and for $x \in \Sigma_V$, we compute that

$$\left(\frac{\psi_0}{\sigma}\right)^{(d)}(x) = -\frac{d!}{2\pi^2}\int_{\Sigma_V}\frac{\xi(y)-R_{s_i,d+1}\xi(y)}{\sigma(y)(y-s_i)^{d+1}}dy.$$

In particular, if conditions (1.15) hold, in view of Lemma 3.1 the map

$$\psi(x) := \frac{\psi_0(x)}{S(x)\sigma(x)}$$

extends to a function of class $(\mathsf{p}-3-2\mathsf{k})\wedge(\mathsf{r}-1-\mathsf{k})$, hence C^2 on Σ_V, and in view of (B.25) it satisfies $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$ on Σ_V.

Now, we define ψ outside Σ_V. By definition, for x outside Σ_V, the equation

$$\Xi_V[\psi](x) = \frac{1}{2}\xi(x) + c_\xi$$

can be written as

$$\psi(x)\int\frac{1}{x-y}\,d\mu_V(y) - \int\frac{\psi(y)}{x-y}\,d\mu_V(y) - \frac{1}{2}\psi(x)V'(x) = \frac{1}{2}\xi(x) + c_\xi,$$

and thus the choice (3.5) ensures that $\Xi_V[\psi] = \frac{1}{2}\xi + c_\xi$. Moreover, ψ is clearly of class $C^{\mathsf{r}\wedge(\mathsf{p}-1)}$ on $\mathbb{R}\setminus\Sigma_V$. It remains to check that ψ has the desired regularity at the endpoints of Σ_V. For a given endpoint α we consider $\tilde{\psi}$ the Taylor development of order $l := (\mathsf{p}-3-2\mathsf{k})\wedge(\mathsf{r}-1-\mathsf{k})$ at α of ψ. We can write (3.5) as

$$\begin{aligned}\frac{\int\frac{\psi(y)}{x-y}d\mu_V(y) + \frac{\xi(x)}{2} + c_\xi}{\int\frac{1}{x-y}d\mu_V(y) - \frac{1}{2}V'(x)} &= \frac{-\int\frac{\tilde{\psi}(x)-\tilde{\psi}(y)}{x-y}d\mu_V(y) + \tilde{\psi}(x)\int\frac{1}{x-y}d\mu_V(y) + \frac{\xi(x)}{2} + c_\xi}{\int\frac{1}{x-y}d\mu_V(y) - \frac{1}{2}V'(x)}\\ &= \tilde{\psi}(x) + \frac{\frac{\xi(x)}{2} + c_\xi - \Xi_V[\tilde{\psi}](x)}{\int\frac{1}{x-y}d\mu_V(y) - \frac{1}{2}V'(x)}\ .\end{aligned}$$

As $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$ on Σ_V, the numerator on the right hand side of the last equation and its first l derivatives vanish at α. From Lemma (3.1) we conclude that ψ is of class $l-\mathsf{k} = (\mathsf{p}-3-3\mathsf{k})\wedge(\mathsf{r}-1-2\mathsf{k})$ at α, hence C^2 from (1.13).

□

B.6. **Proof of Lemma 4.7.** Using definition (1.6) we can write $\mathcal{I}_{V_t}(\mu_t)$ in the following form

$$\mathcal{I}_{V_t}(\mu_t) = \int h^{\mu_t}\,d\mu_t + \int V_t\,d\mu_t.$$

To prove Lemma 4.7, we introduce the auxiliary quantity

$$\mathcal{I}(\tilde{\mu}_t) := \int h^{\tilde{\mu}_t}\,d\tilde{\mu}_t + \int V_t\,d\tilde{\mu}_t,$$

and we first prove that $\mathcal{I}(\tilde{\mu}_t)$ is close to $\mathcal{I}_{V_t}(\mu_t)$.

Claim 1. *We have*

$$\mathcal{I}_{V_t}(\mu_t) = \mathcal{I}(\tilde{\mu}_t) + O\left(t^4 \|\psi\|^4_{C^2(U)}\right). \tag{B.28}$$

Proof. Let us write

$$\begin{aligned} \mathcal{I}_{V_t}(\mu_t) &= \int h^{\mu_t}\, d\mu_t + \int V_t\, d\mu_t \\ &= \int h^{\tilde{\mu}_t}\, d\tilde{\mu}_t + \int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) + \int V_t\, d\mu_t. \end{aligned} \tag{B.29}$$

We have used the fact that, integrating by parts twice,

$$\int h^{\mu_t} d\tilde{\mu}_t = \int h^{\tilde{\mu}_t} d\mu_t.$$

We have, using the definition of $\zeta_t, \tilde{\zeta}_t$ and (3.10)

$$\int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) = \int \left(\zeta_t - \frac{1}{2}V_t - c_t + \tilde{\zeta}_t - \frac{1}{2}V_t - \tilde{c}_t + O(t^2\|\psi\|^2_{C^2(U)})\right) d(\mu_t - \tilde{\mu}_t).$$

In view of (4.1), (4.2), we thus get

$$\int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) + \int V_t\, d\mu_t = O(t^4\|\psi\|^4_{C^2(U)}) + \int V_t d\tilde{\mu}_t. \tag{B.30}$$

Combining (B.29) and (B.30) yields the result. □

We may now compare $\mathcal{I}(\tilde{\mu}_t)$ and $\mathcal{I}_V(\mu_V)$ using the transport map.

Claim 2. *We have*

$$\begin{aligned} \mathcal{I}(\tilde{\mu}_t) &= \mathcal{I}_V(\mu_V) + t \int \xi d\mu_V \\ &\quad + \frac{t^2}{2}\left(\iint \left(\frac{\psi(x) - \psi(y)}{x - y}\right)^2 d\mu_V(x) d\mu_V(y) + \int V''\psi^2 d\mu_V + 2\int \xi'\psi d\mu_V\right) \\ &\quad + O(t^3\|\xi\|_{C^2(U)}). \end{aligned} \tag{B.31}$$

Proof. We may write

$$\begin{aligned} \mathcal{I}(\tilde{\mu}_t) &= -\int \log|\phi_t(x) - \phi_t(y)| d\mu_0(x) d\mu_0(y) + \int V \circ \phi_t\, d\mu_0 + t\int \xi \circ \phi_t\, d\mu_0 \\ &= \int h^{\mu_0}\, d\mu_0 - \iint \log\left|1 + t\frac{\psi(x) - \psi(y)}{x - y}\right| d\mu_0(x) d\mu_0(y) + \int V \circ \phi_t\, d\mu_0 + t\int \xi \circ \phi_t\, d\mu_0. \end{aligned}$$

By a Taylor expansion, we obtain

$$\begin{aligned} \mathcal{I}(\tilde{\mu}_t) &= \mathcal{I}_V(\mu_0) - t\iint \frac{\psi(x) - \psi(y)}{x - y} d\mu_0(x) d\mu_0(y) + \frac{t^2}{2}\iint \left(\frac{\psi(x) - \psi(y)}{x - y}\right)^2 d\mu_0(x) d\mu_0(y) \\ &\quad + t\int V'\psi\, d\mu_V + \frac{t^2}{2}\int V''\psi^2 d\mu_V + t\int \xi d\mu_V + t^2\int \xi'\psi d\mu_0 + O(t^3\|\xi\|_{C^2(\mathbb{R})}). \end{aligned}$$

Let us recall that by definition $\mu_0 = \mu_V$. By (B.14) we have

$$\iint \frac{\psi(x) - \psi(y)}{x - y} d\mu_V(x) d\mu_V(y) = \int V'\psi\, d\mu_V,$$

hence we obtain (B.31). □

To conclude the proof of Lemma 4.7 it remains to prove the following identity.

Claim 3.

$$\int \xi' \psi d\mu_V = -\iint \left(\frac{\psi(x)-\psi(y)}{x-y}\right)^2 d\mu_V(x)d\mu_V(y) - \int V''\psi^2 d\mu_V. \tag{B.32}$$

Proof. By definition of ψ we have

$$\frac{1}{2}(\xi + c_\xi) = \int \frac{\psi(x)-\psi(y)}{x-y} d\mu_V(y) - \frac{1}{2}\psi V' \ ,$$

and thus

$$\xi' = 2\int \frac{\psi(y)-\psi(x)-\psi'(x)(y-x)}{(x-y)^2} d\mu_V(y) - \psi' V' - \psi V'' \ .$$

Integrating both sides against $\psi \mu_V$ yields

$$\int \xi' \psi d\mu_V = 2\iint \frac{(\psi(y)-\psi(x)-\psi'(x)(y-x))\psi(x)}{(x-y)^2} d\mu_V(y)d\mu_V(x) - \int \psi\psi' V' d\mu_V - \int V''\psi^2 d\mu_V.$$

Using (B.14) for the second term we obtain

$$\int \xi' \psi d\mu_V = 2\iint \frac{(\psi(y)-\psi(x)-\psi'(x)(y-x))\psi(x)}{(y-x)^2} d\mu_V(y)d\mu_V(x) - \iint \frac{\psi\psi'(y)-\psi\psi'(x)}{y-x} d\mu_V(x)d\mu_V(y) - \int V''\psi^2 d\mu_V.$$

We may then combine the first two terms in the right-hand side to obtain (B.32). ☐

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