2,7,27,31

Ex. 2 chap. 3

Checking associativity is cumbersome. In example a), we could check that $(a \circ a) \circ b = a \circ b = c$
$a \circ (a \circ b) = a \circ c = d$ so it is not associative

- let us check existence of a neutral element (a line a column $= \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (abcd)$)

a) No ~~c~~ c) Yes it is a

b) Yes it is a d) yes it is a

- Inverses? a) Not even relevant

b) Yes they are their own inverses

c) Yes $a^2 = a$
$c^2 = a$
$bd = db = a$

d) No inverse for d.

Ex. 7 chap. 3.

- Check that $*$ is associative let a, b, c be in S,

we have

$$(a*b)*c = (a+b+ab)*c$$
$$= a+b+ab+c+(a+b+ab)c$$
$$= a+b+c+ab+ac+bc+abc.$$

and also

$$a*(b*c) = a*(b+c+bc)$$
$$= a+b+c+bc+a(b+c+bc)$$
$$= a+b+c+bc+ab+ac+abc.$$

So indeed $(a*b)*c = a*(b*c)$ ✓

- Commutative is clear.
- Neutral element? looking for $e \in \mathbb{R} \setminus \{-1\}$ such that

$$\forall a \in \mathbb{R} - \{-1\}, \quad a + e + ae = a$$

$e = 0$ works well.

HW 1 – Solution

1. Binary operations

1) Let r, r' be such that $\varphi(r) = \varphi(r')$.
By definition, we have $s \bullet r = s \bullet r'$.
Let us apply the left-inverse of s. We get

$$t \bullet (s \bullet r) = (t \bullet s) \bullet r \quad \text{(by associativity)}$$
$$= e \bullet r$$
$$= r$$

and similarly $t \bullet (s \bullet r') = r'$, but since $s \bullet r = s \bullet r'$,
we see that $t \bullet (s \bullet r) = t \bullet (s \bullet r')$, hence $r = r'$,
which shows that φ is one-to-one.

2) A one-to-one map from S to itself (if S is finite) is also onto.

3) Since by question 2) we know that φ is onto, in particular
there exists u in S such that $\varphi(u) = e$, which means

$$s \bullet u = e.$$

4) We have

$$\begin{array}{l} (t \bullet s) \bullet u \underset{\uparrow \text{associativity}}{=} t \bullet (s \bullet u) = t \bullet e = t \\ \parallel \\ e \bullet u \\ \parallel \\ u \end{array}$$

hence $t = u$.

• Inverse? Given $a \in \mathbb{R} - \{-1\}$, look for $b \in \mathbb{R} - \{-1\}$ such that
$$a + b + ab = 0$$
$$\boxed{b = \frac{-a}{1+a}}$$
works well, $a \neq -1$ so it makes sense and $\frac{-a}{1+a} \neq -1$ because $-1 \neq 0$.

Ex. 27 chap. 3

Induction on n.

Ex. 31 chap. 3

Let a, b be in G. Compute $(ab)(ba) = a(bb)a = aea = aa = e$

and $(ba)(ab) = e$ similarly, so ab and ba are inverses of each other.

But on the other hand $(ab)^2 = e$ by assumption, so ab is its own inverse. By uniqueness of inverses, we have $ab = ba$, hence G is abelian.

Ex. 41 chap. 3

- Clearly $1 = 1 + 0 \cdot \sqrt{2}$ is in G
- If $a, b; c, d$ are in $\mathbb{Q}$, $(a+b\sqrt{2})(c+d\sqrt{2}) =$
$$\text{so } \underbrace{\underbrace{(ac + 2bd)}_{\in \mathbb{Q}} + \underbrace{(ad+bc)}_{\in \mathbb{Q}}\sqrt{2}}_{\text{in } G}.$$
- If a, b are in $\mathbb{Q}$ and $\neq 0$,

we have $\frac{1}{a+b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \underbrace{\frac{a}{a^2-2b^2}}_{\in \mathbb{Q}} - \underbrace{\frac{b}{a^2-2b^2}}_{\in \mathbb{Q}}\sqrt{2}$ so $(a+b\sqrt{2})^{-1}$ is in G.

Ex.45 chap 3 let H, H' be ⊢

- By def. ~~the~~ the identity must be in both subgroups, hence in the intersection.
- If $h_1 \in H \cap H'$, $h_2 \in H \cap H'$ then

$h_1 \circ h_2 \in H$ <u>and</u> in H' so $h_1 \circ h_2 \in H \cap H'$

$h_1^{-1} \in H$ <u>and</u> in H' so $h_1^{-1} \in H \cap H'$.