Chapter 9 1, 2, 7, 8, 9, 34, 36, 41, 42 HW4 – Algebra

1] The map $x \mapsto nx$ from $\mathbb{Z}$ to $n\mathbb{Z}$ is easily seen to be an isomorphism.

2] Let A be the subgroup mentioned. We consider the map

$$\varphi : \begin{array}{ccc} \mathbb{C}^* & \longrightarrow & A \\ z & \mapsto & \begin{pmatrix} Re(z) & Im(z) \\ -Im(z) & Re(z) \end{pmatrix} \end{array}$$

It is clearly a bijection. Let us check that it is a morphism

If $z = x+iy$ $z' = x'+iy'$, we have: $zz' = xx' - yy' + i(xy' + x'y)$

so $Re(zz') = Re(z)Re(z') - Im(z)Im(z')$

and $Im(zz') = Re(z)Im(z') + Re(z')Im(z)$

$$\varphi(zz') = \begin{pmatrix} Re(zz') & Im(zz') \\ -Im(zz') & Re(zz') \end{pmatrix}$$

On the other hand

$$\varphi(z)\varphi(z') = \begin{pmatrix} Re(z) & Im(z) \\ -Im(z) & Re(z) \end{pmatrix} \begin{pmatrix} Re(z') & Im(z') \\ -Im(z') & Re(z') \end{pmatrix}$$

$$= \begin{pmatrix} Re(z)Re(z') - Im(z)Im(z') & Re(z)Im(z') + Re(z')Im(z) \\ -Im(z)Re(z') - Im(z')Re(z) & -Im(z)Im(z') + Re(z)Re(z') \end{pmatrix}$$

so $\varphi(zz') = \varphi(z)\varphi(z')$ and φ is a morphism.

7] Let G be a cyclic group of order n, and g be a generator of G.

The map $\begin{array}{ccc} \mathbb{Z}_n & \longrightarrow & G \\ \underline{k} & \longmapsto & g^k \end{array}$ is an isomorphism from $\mathbb{Z}_n$ to G.

- it is onto (and thus bijective since $|\mathbb{Z}_n| = |G|$)
- it is a morphism by the property $g^k \cdot g^{k'} = g^{k+k'}$.

8) Let $\varphi : \mathbb{Z} \longrightarrow \mathbb{Q}$ be an hypothetical isomorphism.

We must have $\varphi(0) = 0$ (neutral element gets mapped to the neutral element)

Let us denote $\varphi(1)$ by a; for some $a \in \mathbb{Q}$ (with $a \neq 0$).

Since φ is a morphism, $\underset{\varphi(2)}{\underset{\|}{\varphi(1+1)}} = \varphi(1) + \varphi(1) = a + a = 2a$

and by an easy induction, $\varphi(n) = na$ for all $n \in \mathbb{Z}$.

Let us show that this cannot possibly be onto. Since a is a nonzero rational number, we may write it as $a = p/q$ with $p \neq 0$; $q \neq 0$ and $\gcd(p, q) = 1$.

So $\varphi(n) = \dfrac{np}{q}$.

We claim that $\varphi(n)$ is never equal to $\dfrac{1}{q^2}$. Indeed, we would have

$$\frac{np}{q} = \frac{1}{q^2}, \quad \text{so} \quad npq = 1, \quad \text{which implies } n = p = q = \pm 1,$$

so φ would be the ~~identity~~ map $\varphi(n) = n$, which is not onto from $\mathbb{Z}$ to $\mathbb{Q}$!

So $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

9) We already proved that G was a group in a previous HW.

Let $\varphi : \begin{array}{ccc} \mathbb{R}^* & \longrightarrow & G \\ x & \longmapsto & x - 1 \end{array}$ it is clearly a bijection.

We may check that
$$\begin{aligned} \varphi(xy) &= xy - 1 \\ &= (x-1)(y-1) + (x-1) + (y-1) \\ &= \varphi(x) * \varphi(y) \end{aligned}$$

So φ is an isomorphism

34) let us consider $z \mapsto \bar{z}$ from $(\mathbb{C},+)$ to $(\mathbb{C},+)$
it is clearly a bijection, and we have

$$\overline{z+z'} = \bar{z}+\bar{z}' \quad \text{so it is a morphism.}$$

(elementary properties of complex numbers, easy to check by hand)

36) First, we observe that if $A \in SL_2(\mathbb{R})$ and $B \in GL_2(\mathbb{R})$, we have
$\det(B^{-1}AB) = \det(B)^{-1}\det(A)\det(B) = \det(A) = 1$, so
$B^{-1}AB$ is indeed in $SL_2(\mathbb{R})$

The map is clearly one-to-one: if $B^{-1}AB = B^{-1}A'B$, then by applying B on the left and B^{-1} on the right, we get $A = A'$

it is also onto: given A' in $SL_2(\mathbb{R})$, to solve $B^{-1}AB = A'$ it is enough to consider $A = BA'B^{-1}$, which is in $SL_2(\mathbb{R})$.

Finally, it is a morphism, because given A, A' in $SL_2(\mathbb{R})$, we have

$$(B^{-1}AA'B) = B^{-1}ABB^{-1}A'B = (B^{-1}AB)(B^{-1}A'B).$$

41) It is a map from G to G because G is a group. It is clearly one-to-one (same computation as above), and onto (idem).
and it is a morphism (same computation as above).

42) • The identity $Id : G \to G$ (which is the neutral element of $Aut(G)$) is indeed in $Im(G)$, consider the map i_e, where e is the neutral element of G.

• If $g \in G$, the inverse of i_g is $i_{g^{-1}}$ (easily checked).

• If g, g' are in G, we have $i_g \circ i_{g'} = i_{gg'}$ so $Im(G)$ is closed under composition.

Rem. $\left(\begin{array}{l} G \longrightarrow Im(G) \\ g \mapsto i_g \end{array} \text{ is an isomorphism}\right)$