HW5 - Solution

Chapter 9

47 let φ be an isomorphism from G to $\overline{G}$
ψ " " H to $\overline{H}$.

The map
$$\begin{array}{ccc} G \times H & \longrightarrow & \overline{G} \times \overline{H} \\ (x, y) & \longmapsto & (\varphi(x), \psi(y)) \end{array}$$
is an isomorphism. (Easy to check).

48 The map
$$\begin{array}{ccc} G \times H & \longrightarrow & H \times G \\ (x, y) & \longmapsto & (y, x) \end{array}$$
is an isomorphism.

50 If A, B are Abelian then clearly so is $A \times B$. Now if $A \times B$ is Abelian, for any $a \in A$, a' in A we have $(a, 0) \times (a', 0) = (a', 0) \times (a, 0)$
so $(aa', 0) = (a'a, 0)$ hence $aa' = a'a$.
So A is Abelian, and B also for similar reasons.

Chapter 11 (2, 8, 10, 11)

2 a) is, Kernel $= \{1\}$

b) let us compute
$$\phi(a+b) = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix} \quad \text{and} \quad \phi(a)\,\phi(b) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix} = \phi(a+b)$$

so ϕ is a morphism.
Kernel $= \{0\}$

c) Take $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in $GL_2(\mathbb{R})$

$$\phi(MN) = \phi\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = 4$$

$\phi(M) = 3$; $\phi(N) = 3$ $\quad \phi(M) + \phi(N) = 6$

So $\phi(MN) \neq \phi(M)\phi(N)$ and ϕ is not a morphism

d) Yes, it is the determinant. Kernel $= SL_2(\mathbb{R})$ (determinant $= 1$)

e) Yes, kernel $= \{0\}$.

8) $\phi(g \cdot g') = (gg')^n = g^n g'^n$ (because G Abelian)

$= \phi(g)\phi(g')$

10) If g is a generator of G, it is easy to check that $\phi(g)$ is a generator of $\phi(G)$. Indeed any element of $\phi(G)$ is equal to $\phi(g')$ for some g' in G, but $g' = g^k$ for some k because G is cyclic, and thus $\phi(g') = \phi(g^k) = \phi(g)^k$.

11) if we know a generator g and its image $\phi(g)$, we know $\phi(g^k)$ for any k because $\phi(g^k) = \phi(g)^k$, and thus we know ϕ on the ~~~~group $\langle g \rangle$.