HW 6 — Algebra

7, 8, 18 (abc), 27, 33, 34 chapter 16.

7] let $\varphi : \mathbb{C} \to \mathbb{R}$ be an isomorphism, then

$$\varphi(i^2) = \varphi(i)^2 = \varphi(-1) = -1\varphi(1) = -1$$

So $\varphi(i)^2 = -1$ which is impossible in $\mathbb{R}$.

8] let $\varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ be an isomorphism.

Then $\varphi((\sqrt{2})^2) = \varphi(\sqrt{2})^2 = \varphi(2) = \varphi(1) + \varphi(1) = 2$

So $\varphi(\sqrt{2})^2 = 2$ which is impossible in $\mathbb{Q}(\sqrt{3})$.

18] a) if x, y are in $\phi(R)$ we can find a, b in R such that $\phi(a) = x$; $\phi(b) = y$, but then

$$\phi(a \times b) = x \times y = \phi(b \times a) = y \times x,$$

so $xy = yx$ and $\phi(R)$ is commutative.

b) $\cancel{\phi(a+0) = \phi(a) + \phi(0) = \phi(a) \text{ so } \phi(0) = 0}$

$$\underbrace{\phi(a - a)}_{= \phi(0)} = \phi(a) - \phi(a) = 0$$

c) for any s in S, since ϕ is onto we have x in R such that $\phi(x) = s$, but then $\phi(1_R \times x) = \phi(x) = s$

$$\phi(1_R \times x) = \phi(1_R) \times \phi(x) = \phi(1_R) \cdot s$$

So $s = \phi(1_R) \cdot s$ for any s in S and thus $\phi(1_R) = 1_S$.

27) i) it is a subgroup: take a, b nilpotents,
so $\exists\, n, m$ such that $a^n = 0$; $b^m = 0$.
Show $(a-b)$ is nilpotent.

Newton's Binomial Thm for commutative rings: for $s \geq 1$

$$(a-b)^s = \sum_{k=0}^{s} \binom{s}{k} a^k b^{s-k}$$

Taking $s = m+n$, we see that for any $k \in [\![0, m+n]\!]$,
we have $a^k b^{m+n-k} = 0$ because
either $k \leq m$, and then $m+n-k \geq m$ so $b^{m+n-k} = 0$
or $k \geq n$ and then $a^k = 0$.

So $(a-b)$ is nilpotent.

ii) for a nilpotent and b in R, we have
$(ab)^k = a^k b^k$ because R is commutative,
so taking k large enough we have $a^k = 0$ and thus $(ab)^k = 0$.

33) ~~we already~~ $\triangle$ it is not the center in the group theory sense

- subgroup? a, b in $Z(R)$, r in R

$$(a-b)r = ar - br = ra - rb = r(a-b)$$

So $a-b \in Z(R)$

2. moreover if a, b in $Z(R)$ then $(abr) = (arb) = (rab)$ for all r so $ab \in Z(R)$

Commutativity is clear.

34 | Note: the question is a bit strange. They mean "is a ring with the usual operations on fractions", or "is a subring of $\mathbb{Q}$".

$$\frac{a}{b} - \frac{c}{d} \qquad \gcd(b, p) = 1 \; ; \; \gcd(d, p) = 1$$

$$\Downarrow$$

$$= \frac{ad - bc}{bd} \qquad \gcd(bd, p) = 1$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \Leftarrow \text{idem.}$$