HW7 - Algebra - Fall 2018

Due date: Friday, Nov 9nd.

1. If G, H are two groups, and G' is a subgroup of G and H' is a sub-group of H, and $\pi : G \to H$ is a group morphism, show that:
   (a) The pre-image $\pi^{-1}(H') := \{g \in G, \pi(g) \in H'\}$ is a subgroup of G.
   (b) The direct image $\pi(G') := \{\pi(g), g \in G'\}$ a subgroup of H.
2. If R, S are two rings, R' is a subring of R and S' is a subring of S and $\pi : R \to S$ is a ring morphism, show that:
   (a) The pre-image $\pi^{-1}(S') := \{r \in R, \pi(r) \in S'\}$ is a subring of R.
   (b) The direct image $\pi(R') := \{\pi(r), r \in R'\}$ a subring of S.
3. Let R, S be two rings, I be an ideal of R, J be an ideal of S and $\pi : R \to S$ be a ring morphism.
   (a) Show that the pre-image $\pi^{-1}(J) := \{a \in R, \pi(a) \in J\}$ is an ideal of R.
   (b) Find an example where the direct image $\pi(I) := \{\pi(a), a \in I\}$ is **not** an ideal of S?
   (c) Show that if π is onto (surjective), then $\pi(I)$ **is** an ideal of S.
4. If R, S are two commutative rings, show that all the ideals of $R \times S$ are of the form $I \times J$, where I is an ideal of I and R is an ideal of S (you may follow the proof given in the lecture notes for $R = S = \mathbb{R}$)
5. Let $N \geq 2$.
   (a) Show that the ideals of $\mathbb{R}^N$ are of the form $I_1 \times \cdots \times I_N$ where I_k is either $\{0\}$ or $\mathbb{R}$.
   (b) Show that all of them are principal, by giving a generator in each case.
   (c) Which are prime? Maximal?

0.1 Extra-credit

1. Show that $\mathbb{R} \times \{0\}$ is isomorphic to $\mathbb{R}$.
2. What is the quotient ring $\mathbb{R}/\mathbb{R}$? What is the quotient ring $\mathbb{R}/\{0\}$?
3. Show that the quotient ring $(\mathbb{R} \times \mathbb{R})/(\mathbb{R} \times \{0\})$ is isomorphic to $\mathbb{R}$.