HW7 - Algebra

1)

a) let g, g' be in $\pi^{-1}(H')$

We have $\pi(g(g')^{-1}) = \pi(g)\,\pi(g')^{-1}$ because π is a morphism

$\pi(g)$ in H' and $\pi(g')$ in H' (by definition) and H' is a subgroup,

So $\pi(g)\,\pi(g')^{-1}$ in H' and $g(g')^{-1}$ in $\pi^{-1}(H')$.

So $\pi^{-1}(H')$ is a subgroup.

b) let a, b in $\pi(G')$, write $a = \pi(g)$; $b = \pi(g')$ for g, g' in G'.

Then $ab^{-1} = \pi(g)\,\pi(g')^{-1} = \pi(g(g')^{-1})$ and G' is a subgroup so $g(g')^{-1}$ is in G'. Thus $ab^{-1} \in \pi(G')$ and $\pi(G')$ is a subgroup.

2)

~~a) We already know it is a subgroup by 1) a)~~

~~With the notation of the proof of 1) a), we have~~

$$\cancel{\pi(g \times g') = \pi(g) \times \pi(g') \text{ and } \pi(g) \times \pi(g')}$$

a) We already know that $\pi^{-1}(S')$ is a subgroup of $(R, +)$ by 1) a). Now, let r, r' be in $\pi^{-1}(S')$, we have

$\pi(r \times r') = \pi(r) \times \pi(r')$ with $\pi(r), \pi(r')$ in S',

thus $\pi(r \times r') \in S'$ and $r \times r' \in \pi^{-1}(S')$

So $\pi^{-1}(S')$ is a subring

b) We already know $\pi(R')$ is a subgroup of $(S,+)$ by 1) b).
Now, let r, r' be in R', we have $r \times r'$ in R',
so $\pi(r \times r') = \pi(r) \times \pi(r')$ is in $\pi(R')$, and thus
$\pi(R')$ is a subring of S.

3) a) We already it is a subgroup of $(R,+)$.
if $a \in R$ with $\pi(a) \in J$ and $b \in R$, then
$\pi(a \times b) = \pi(a) \times \pi(b) \in J$ because J is an ideal,
so $a \times b \in \pi^{-1}(J)$ and $\pi^{-1}(J)$ is an ideal of R.

b) Take the inclusion $\mathbb{Z} \to \mathbb{Q}$, $x \mapsto x$, $R = \mathbb{Z}$; $S = \mathbb{Q}$
$I = \mathbb{Z}$.
However $\mathbb{Z}$ is not an ideal of $\mathbb{Q}$.

c) We already know $(\pi(I),+)$ is a subgroup of $(S,+)$ by 1) b).
Take ~~a~~ $a \in \pi(I)$ and $b \in S$. Since π is onto, we write
$b = \pi(c)$ for some $c \in R$. We also write $a = \pi(i)$ for some i in I.
Then $a \times b = \pi(i) \times \pi(c) = \pi(i \times c)$
and $i \times c \in I$ because I is an ideal.
So $a \times b \in \pi(I)$ and $\pi(I)$ is an ideal.

4] It is a classical exercise - many solutions available online.

See 5] next page.

5) a) It follows by induction on $N \geq 2$, using 4) and the fact that the only ideals of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$.

b) Take $(0, 1, 1, 0, 0, \dots)$ where we put as k-th coefficient

$$\begin{cases} \text{a } 0 \text{ if } I_k = \{0\} \\ \text{a } 1 \text{ if } I_k = \mathbb{R} \end{cases}$$

c) They are all prime. The only maximal ideals are those for which $I_k = \{0\}$ for only one k; or $\mathbb{R} \times \cdots \times \mathbb{R}$.