HW 8 - Algebra - Solution

Chap 17 Ex. 26

Let F be a field. We claim that the polynomial X doesn't have a multiplicative inverse, hence $F[X]$ is not a field.

Assume that there is $P \in F[X]$ such that $P \cdot X = 1$.

Write $P(X) = \sum_{k=0}^{n} a_k X^k$. Then $P(X) \cdot X = \sum_{k=0}^{n} a_k X^{k+1}$, which is not 1 ... so $P \cdot X \neq 1$, contradiction.

Chap 18 1, 4, 6, 9, 10, 11 (abc)

1) Let $z = a + b\sqrt{3}\, i$; a, b in $\mathbb{Z}$.

Let us look for an inverse for z, of the form $z' = a' + b'\sqrt{3}\, i$.

We have $z \cdot z' = (aa' - 3bb') + (ab' + a'b)\sqrt{3}\, i$

* First observation: assume $a^2 + 3b^2 = 1$,

take $a' = a$; $b' = -b$, then

$$z \cdot z' = (a^2 + 3b^2) + (-ab + ab)\sqrt{3}\, i = 1,$$

so z is a unit (i.e. has an inverse).

* Second observation: if z is a unit, i.e. has an inverse, we must be able to find a', b' such that

$$\begin{cases} aa' - 3bb' = 1 \\ ab' + a'b = 0 \end{cases}$$

We must have $a \neq 0$, otherwise $3bb' = 1$, impossible.

So we may write $b' = \frac{-a'b}{a}$, and thus

$aa' + \frac{3b^2a'}{a} = 1$, so $a'(a^2 + 3b^2) = 1$, $b = 0$

and hence $a' = 1$; $a^2 + 3b^2 = 1$, which implies $a = \pm 1$.

4] True. Assume $xy = 0$, and $x \neq 0$, $y \neq 0$, then by applying $\underset{(\text{in } F)}{x^{-1} \cdot y^{-1}}$ we obtain $\underset{(\text{in } F)}{1 = 0}$, absurd.

5] let 1 be the unity of F.

The map $\mathbb{Q} \longrightarrow F$

$$\frac{p}{q} \mapsto (\underbrace{1 + 1 + \cdots + 1}_{p \text{ times}}) \cdot (\underbrace{1 + 1 + \cdots + 1}_{q \text{ times}})^{-1}$$

is a one-to-one ring morphism.

The point is that sine F has characteristic zero, $\underbrace{1 + \cdots + 1}_{q \text{ times}}$ is never 0, and can thus always be inverted.

9] We can observe that $\mathbb{Q}(i)$ is indeed a field, and contains $\mathbb{Z}[i]$. Moreover, any field containing $\mathbb{Z}[i]$ must contain $\mathbb{Q}$, and $\{qi \,;\, q \in \mathbb{Q}\}$, so must contain $\mathbb{Q}(i)$.

10] a) Take 1 to be the unity in F, and let E be the smallest subfield containing 1. It is a prime subfield, and the unique one because every subfield of F contains E.

b) The map $\mathbb{Q} \longrightarrow F$

$$\frac{p}{q} \mapsto (\underbrace{1+1+\cdots+1}_{p \text{ times}}) (\underbrace{1+\cdots+1}_{q \text{ times}})^{-1}$$

is a one-to-one map, and its image is the prime subfield.

c) The map $\mathbb{Z}_p \longrightarrow F$

$$[x] \mapsto (\underbrace{1+\cdots+1}_{x \text{ times}})$$ is a one-to-one map its image is the prime subfield.

11) a) ~~$(a+b\sqrt{2})(a'+b'\sqrt{2})=0$~~

~~$\Rightarrow aa'+2bb'=0$ and $(ab'+ba')$~~ $=0$

The simplest way is to observe that $\mathbb{Q}(\sqrt{2})$ is a field containing $\mathbb{Z}[\sqrt{2}]$, hence, by question 4), $\mathbb{Z}[\sqrt{2}]$ must be an integral domain.

b) Let a, b in $\mathbb{Z}$; assume there exists a', b' in $\mathbb{Z}$ such that

$$(a+\sqrt{2}b)(a'+\sqrt{2}b')=1.$$

Then $aa'+2bb'=1$; $ab'+ba'=0$.

We must have $a\neq 0$ otherwise $2bb'=1$. Then $b'=\frac{-ba'}{a}$,

and $aa'-\frac{2b^2a'}{a}$, so $a'(a^2-2b^2)=1$.

Thus $a^2-2b^2=\pm 1$ and $a'=\frac{1}{a^2-2b^2}$.

Units are exactly the ones with a^2 - 2b^2 = 1 or -1

~~$\Downarrow$ so $a^2=\pm 1$ and $b=0$.~~

~~Units $=\{1, -1\}$~~.

c) it is $\mathbb{Q}(\sqrt{2})$, same reason as Q.9.