HW9 - Algebra - Solution.

1) Assume the contrary, write

$$\sqrt{6} = a + b\sqrt{2} + c\sqrt{3},$$ and raise it to the second power

$$6 = a^2 + 2b^2 + 3c^2 + 2ab\sqrt{2} + 2ac\sqrt{3}$$
$$+ 2bc\sqrt{6}$$

$$0 = \left(a^2 + 2b^2 + 3c^2 - 6\right) + 2ab\sqrt{2} + 2ac\sqrt{3}$$
$$+ 2bc\left(a + b\sqrt{2} + c\sqrt{3}\right),$$

thus,

$$(\underbrace{a^2 + 2b^2 + 3c^2 - 6 + 2abc}_{A}) + (\underbrace{2ab + 2b^2c}_{B})\sqrt{2} + (\underbrace{2ac + 2bc^2}_{C})\sqrt{3} = 0$$

A, B, C are in $\mathbb{Q}$.

$$A + B\sqrt{2} + C\sqrt{3} = 0; \quad A + B\sqrt{2} = -C\sqrt{3}.$$

So $$(A + B\sqrt{2})^2 = C^2 3$$

$$= A^2 + 2B^2 + 2AB\sqrt{2} = 3C^2 \qquad A, B, C \text{ rational}$$

So $\sqrt{2}$ can be written as a rational number ...

2) See Textbook, Thm. 17.22

See also the proof of a similar statement for $\mathbb{Z}$ in the lecture notes "Rings"

3) It is nothing else than the definition of a ring morphism.

We get "for free" the following property:

if $x \neq 0$; $\varphi(x)^{-1} = \varphi(x^{-1})$. ("respects" the inverse)

4) Let $\varphi : \mathbb{Q} \longrightarrow \mathbb{Q}$ be a field morphism.

We must have $\varphi(0) = 0$

$\varphi(1) = 1$

thus $\varphi(p) = p \quad \forall p \in \mathbb{Z}$

thus $\varphi\left(\frac{1}{p}\right) = \frac{1}{p} \quad \forall p \in \mathbb{Z}^*$

thus $\varphi\left(\frac{p}{q}\right) = \frac{\varphi(p)}{\varphi(q)} = \frac{p}{q} \quad \forall p, q \in \mathbb{Z} \times \mathbb{Z}^*$

So φ is the identity map.

5) The map $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is a non trivial isomorphism.