Algebra - Midterm 1 - Fall 2018 - NYU

All answers must be justified.

NAME:

Exercise 1 We recall that $GL_2(\mathbb{R})$ denotes the group of invertible 2×2 matrices with real coefficients. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix, we recall that the transpose of A, denoted by A^T is the matrix $A^T := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and that $(AB)^T = B^T A^T$. The identity matrix is denoted by I_2.

We let $O_2(\mathbb{R})$ be the subset of $GL_2(\mathbb{R})$ defined by

$$O_2(\mathbb{R}) := \left\{ A \in GL_2(\mathbb{R}), A^T A = I_2 \right\}.$$

Question 1. Show that $O_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$.

For any A in $O_2(\mathbb{R})$, we define the *commutator of* A as the subset

$$\mathrm{Comm}(A) := \{ B \in O_2(\mathbb{R}),\ AB = BA \}.$$

In plain words, $\mathrm{Comm}(A)$ is the set of all matrices in $O_2(\mathbb{R})$ that commute with A.

Question 2. Compute $\mathrm{Comm}(I_2)$.

Question 3. For all A in $O_2(\mathbb{R})$, show that $\mathrm{Comm}(A)$ is a subgroup of $O_2(\mathbb{R})$.

Question 4. Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Show that A is in $O_2(\mathbb{R})$, compute its order and describe its commutator.

Exercise 2 We recall that a group G is said to be cyclic if there exists an element g in G such that the subgroup $\langle g \rangle$ generated by g is equal to G itself.

Question 1. Show that $\mathbf{Z}_2 \times \mathbf{Z}_3$ is cyclic, but that $\mathbf{Z}_2 \times \mathbf{Z}_2$ is not.

Question 2. Let G, H be two groups. Assume that G is cyclic and that there exists an isomorphism from G to H. Prove that H is cyclic

Question 3. Let G be a group, and let g be in G. Someone tells you that there is **at most one** element of G not included in $\langle g \rangle$. Show that G is cyclic.

The signature morphism Let $n \geq 2$. We recall that the "signature" or "parity" of a permutation σ in S_n is defined as

$$\varepsilon(\sigma) := \prod_{1 \leq i < j \leq n} \operatorname{Sign}\left(\sigma(j) - \sigma(i)\right),$$

where Sign denotes the sign ($+1$ or -1).

We have proven the following facts: $\varepsilon(\mathrm{Id}) = 1$, and if τ is a transposition, $\varepsilon(\tau) = -1$. The goal of this exercise is to prove that ε is a morphism, namely that for all permutations σ_1, σ_2, we have

$$\varepsilon(\sigma_1 \circ \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2).$$

Question 1. Show that

$$(13) = (23)(12)(23).$$

Question 2. For $i = 1, \ldots n-1$, we let τ_i be the transposition

$$\tau_i := (i(i+1)),$$

that switches two "neighbors" in $\{1, \ldots, n\}$. Prove that any transposition can be written as a product of these transpositions τ_i.

You may argue by induction on the "distance" between the elements of the transposition, and let yourself be inspired by Question 2.

Question 3. Deduce that every permutation in S_n can be written as product of the transpositions τ_i.

Question 4. Show that, in order to prove that ε is a group morphism, it is enough to prove that for any permutation σ and for any $i \in \{1, \ldots, n-1\}$, we have

$$\varepsilon\left(\sigma \circ \tau_i\right) = -\varepsilon(\sigma).$$

(hence the whole result boils down to this very easy computation that you can do at home.)

Question 5. Compute (with minimal justification) the parity of the following permutations:

1. $(123)(456)(123456)$
2. $(12)(234)^{-1}(12345)(34)^{-1}$
3. (123456789).

Bonus question: compute the center of S_n for $n \geq 3$.