Algebra - Midterm 1 - Fall 2018 - NYU All answers must be justified.

NAME:

Exercise 1 We recall that $GL_2(\mathbb{R})$ denotes the group of invertible 2×2 matrices with real coefficients. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix, we recall that the transpose of A, denoted by A^T is the matrix $A^T := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and that $(AB)^T = B^T A^T$. The identity matrix is denoted by I_2 . We let $O_2(\mathbb{R})$ be the subset of $GL_2(\mathbb{R})$ defined by

$$O_2(\mathbb{R}) := \left\{ A \in GL_2(\mathbb{R}), A^T A = I_2 \right\}.$$

Question 1. Show that $O_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$.

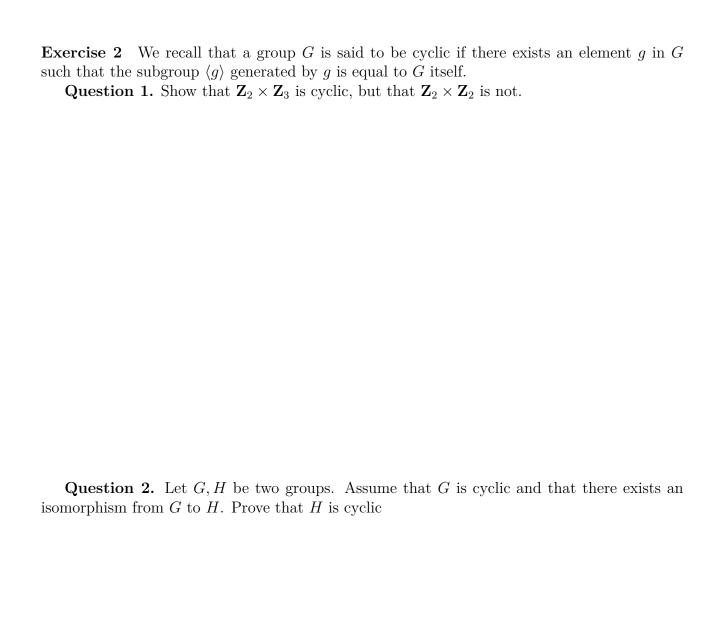
For any A in $O_2(\mathbb{R})$, we define the *commutator of* A as the subset

$$Comm(A) := \{ B \in O_2(\mathbb{R}), AB = BA \}.$$

In plain words, Comm(A) is the set of all matrices in $O_2(\mathbb{R})$ that commute with A. Question 2. Compute $Comm(I_2)$.

Question 3. For all A in $O_2(\mathbb{R})$, show that $\operatorname{Comm}(A)$ is a subgroup of $O_2(\mathbb{R})$.

Question 4. Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Show that A is in $O_2(\mathbb{R})$, compute its order and describe its commutator.





The signature morphism Let $n \geq 2$. We recall that the "signature" or "parity" of a permutation σ in S_n is defined as

$$\varepsilon(\sigma) := \prod_{1 \le i < j \le n} \operatorname{Sign} \left(\sigma(j) - \sigma(i) \right),$$

where Sign denotes the sign (+1 or -1).

We have proven the following facts: $\varepsilon(\mathrm{Id}) = 1$, and if τ is a transposition, $\varepsilon(\tau) = -1$. The goal of this exercise is to prove that ε is a morphism, namely that for all permutations σ_1, σ_2 , we have

$$\varepsilon(\sigma_1 \circ \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2).$$

Question 1. Show that

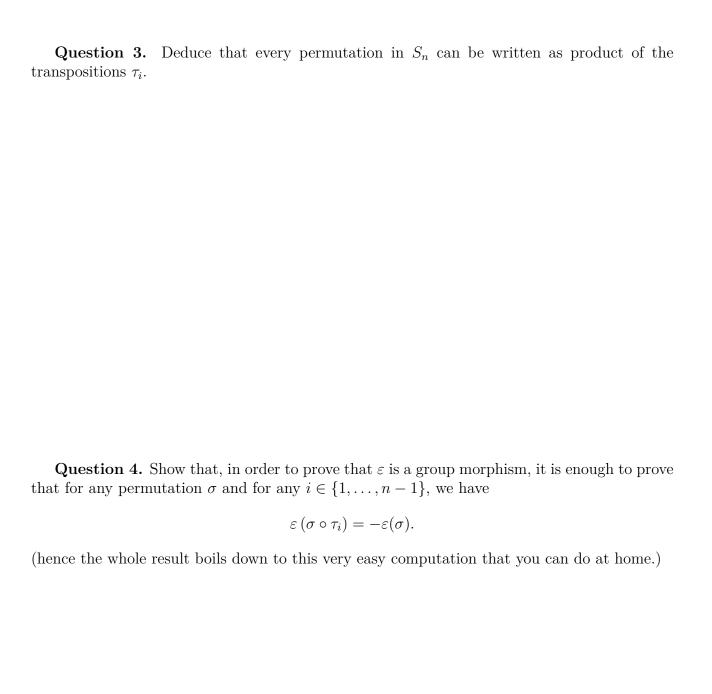
$$(13) = (23)(12)(23).$$

Question 2. For i = 1, ..., n-1, we let τ_i be the transposition

$$\tau_i := (i(i+1)),$$

that switches two "neighbors" in $\{1, \ldots, n\}$. Prove that any transposition can be written as a product of these transpositions τ_i .

You may argue by induction on the "distance" between the elements of the transposition, and let yourself be inspired by Question 2.



 ${\bf Question~5.}~{\bf Compute~(with~minimal~justification)~the~parity~of~the~following~permutations:}$

- 1. (123)(456)(123456)
- 2. $(12)(234)^{-1}(12345)(34)^{-1}$
- 3. (123456789).

Bonus question: compute the center of S_n for $n \geq 3$.