Algebra - Midterm 2 - Fall 2018 - NYU

All answers must be justified.

Exercise 1 Let R be a commutative ring, and let I, J be two ideals of R. We define the subset $I+J$ as follows

$$I+J := \{i+j, \text{ for } i \in I \text{ and } j \in J\}.$$

1. Show that $I+J$ is an ideal of R.
2. Why is it true that $I \subset I+J$?
3. Recall the definition of a maximal ideal.
4. If I is maximal, and if J is not included in I, show that $I+J=R$.

Exercise 2 The following questions are independent.

1. A Boolean ring R is a ring where $a^2 = a$ for any a in R. Prove that if R is a Boolean ring, then R is anti-commutative, i.e. $a \times b = -b \times a$ for all a, b in R.
2. Let $(R, +, \times)$ be a commutative ring, and a be a fixed element of R. Show that the set

$$J := \{r \in R, \text{ such that } r \times a = 0\}$$

is an ideal of R.

Exercise 3 Let $T_2(\mathbb{R})$ be the ring of 2×2 upper-triangular matrices with real coefficients

$$T_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \quad a, b, c \in \mathbb{R}. \right\}$$

1. Let I be a left-ideal of $T_2(\mathbb{R})$
   (a) If I contains a matrix $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ with $a \neq 0$ and $b \neq 0$, show that I is trivial.
   (b) Deduce that if I contains two matrices $\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & c' \\ 0 & b \end{pmatrix}$ with $a \neq 0$ and $b \neq 0$, then I is trivial.
   (c) (**Bonus question**) Find all the left-ideals of $T_2(\mathbb{R})$, all the right-ideals of $T_2(\mathbb{R})$ and all the two-sided ideals of $T_2(\mathbb{R})$.

Exercise 4 Let us define the sets $\mathbf{Z}[\sqrt{2}], \mathbf{Q}[\sqrt{2}]$ as

$$\mathbf{Z}[\sqrt{2}] := \left\{a + b\sqrt{2}, \text{ for } a, b \text{ in } \mathbf{Z}\right\}, \quad \mathbf{Q}[\sqrt{2}] := \left\{a + b\sqrt{2}, \text{ for } a, b \text{ in } \mathbf{Q}\right\}$$

1. Show that $\mathbf{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$. Is it an ideal of $\mathbb{R}$?

2. Is $\mathbf{Z}[\sqrt{2}]$ an ideal of $\mathbf{Q}[\sqrt{2}]$?

3. Write down a non-trivial ideal of $\mathbf{Z}[\sqrt{2}]$.

4. Let A be the subset of $\mathbf{Z}[\sqrt{2}]$ defined by $A := \left\{2n + 2m\sqrt{2},\ \text{for } n, m \text{ in } \mathbf{Z}\right\}$. It is easy to check that A is an ideal of $\mathbf{Z}[\sqrt{2}]$, and we admit it.

   (a) Show that A is a **principal** ideal.

   (b) Show that A is not a **maximal** ideal (hint: consider the set of all $2n + m\sqrt{2}$, for n, m in $\mathbf{Z}$)