Algebra - Midterm 2 - Fall 2018 - NYU
All answers must be justified.

NAME:

$30 = 10+5+5+10$

$20 = 10+10$

$15 = 10+5$

$35 = 10+5+5+5+10$

Exercise 1 Let R be a commutative ring, and let I, J be two ideals of R. We define the subset $I + J$ as follows

$$I + J := \{i + j, \text{ for } i \in I \text{ and } j \in J\}.$$

1. Show that $I + J$ is an ideal of R.

a) let i, i' be in I and j, j' be in J.

Then $(i+j)-(i'+j') = (i-i')+(j-j')$ and $i-i' \in I$, $j-j' \in J$ because I, J are ideals

Thus $(i+j)-(i'+j') \in I+J$ and $I+J$ is a subgroup of R.

b) ~~with~~ let (i,j) be in $I \times J$ and let a be in R.

$a \times (i+j) = a \times i + a \times j$, and $a \times i \in I$, $a \times j \in J$ because I, J are ideals.

Thus $a \times (i+j) \in I+J$.

So $I+J$ is an ideal of R

2. Why is it true that $I \subset I + J$?

Take $i \in I$. Since $J \ni 0$, we have $i + 0 \in I+J$,

thus $i \in I+J$. So $I \subset I+J$.

3. Recall the definition of a maximal ideal.

A maximal ideal of R is an ideal I such that, for any ideal I', if $I \subset I'$ then $I' = I$ or $I' = R$.

4. If I is maximal, and if ~~$I \neq J$~~ J is not included in I, show that $I + J = R$.

We know by Q.1 that $I+J$ is an ideal, and by Q.2 that $I \subset I+J$. If I is maximal, we must have

$$I+J = I \quad \text{or} \quad I+J = R.$$

Since J is not included in I, there exists $j \in J \setminus I$, but $0+j$ is in $I+J$, so there is an element of $I+J$ that is not in I, and hence $I \neq I+J$. The only possibility is thus $I+J = R$.

Exercise 2 The following questions are independent.

1. A Boolean ring R is a ring where $a^2 = a$ for any a in R. Prove that if R is a Boolean ring, then R is ~~commutative.~~ anticommutative .

Let a, b be in R. Compute

$$(a+b)^2 = (a+b) \times (a+b) = a^2 + a \times b + b \times a + b^2$$

Since R is Boolean, we have $(a+b)^2 = a+b$

$$a^2 = a \text{ and } b^2 = b.$$

Thus $a + b = a + b + ab + ba$ and

$ab = -ba$, so R is anti-commutative.

2. Let $(R, +, \times)$ be a commutative ring, and a be a fixed element of R. Show that the set

$$J := \{r \in R, \text{ such that } r \times a = 0\}$$

is an ideal of R.

- Let r, r' be in J. We have

  $$(r - r') \times a = r \times a - r' \times a = 0 - 0 = 0$$

  So $r - r'$ is in J, and J is a subgroup of R

- Let r be in J and $b \in R$, we have

  $$(r \times b) \times a = b \times (r \times a) = b \times 0 = 0,$$

  So $r \times b \in J$.

So J is an ideal of R.

Exercise 2 Let $T_2(\mathbb{R})$ be the ring of 2×2 upper-triangular matrices with real coefficients

$$T_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \quad a, b, c \in \mathbb{R}. \right\}$$

1. Let I be a left-ideal of $T_2(\mathbb{R})$

   (a) If I contains a matrix $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ with $a \neq 0$ and $b \neq 0$, show that I is trivial.

   If $a \neq 0$, $b \neq 0$, the matrix $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ is invertible because $\det \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = ab \neq 0$. ~~By~~ Multiplying $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ by its inverse, we obtain that Id_2 (the identity matrix) is in I, so I is trivial.

   (b) Deduce that if I contains two matrices $\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & c' \\ 0 & b \end{pmatrix}$ with $a \neq 0$ and $b \neq 0$, then I is trivial.

   Their sum would be a matrix of the type treated in question 1a).

(c) (**Bonus question**) Find all the left-ideals of $T_2(\mathbb{R})$, all the right-ideals of $T_2(\mathbb{R})$ and all the two-sided ideals of $T_2(\mathbb{R})$.

A non-trivial ideal contains only matrices of the type

$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ or of the type $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$ [by 1a)]

and cannot contain both types [by 1c)].

Let us compute

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\alpha & \alpha b \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta + b\gamma \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & \alpha b + \beta c \\ 0 & \gamma c \end{pmatrix}$$

$$\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & b\gamma \\ 0 & c\gamma \end{pmatrix}$$

<u>Therefore.</u> Take a left-ideal I

- if $I \neq \{0\}$, it contains $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ $a \neq 0$ or $b \neq 0$

  or $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$ $b \neq 0$ or $c \neq 0$

<u>Case 1</u> It contains

Exercise 3 Let us define the sets $\mathbf{Z}[\sqrt{2}], \mathbf{Q}[\sqrt{2}]$ as

$$\mathbf{Z}[\sqrt{2}] := \left\{a + b\sqrt{2},\ \text{for } a, b \text{ in } \mathbf{Z}\right\}, \quad \mathbf{Q}[\sqrt{2}] := \left\{a + b\sqrt{2},\ \text{for } a, b \text{ in } \mathbf{Q}\right\}$$

1. Show that $\mathbf{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$. Is it an ideal of $\mathbb{R}$?

Let a, b, a', b' be in $\mathbb{Q}$. We have

$$(a+b\sqrt{2}) - (a' + b'\sqrt{2}) = (a-a') + (b-b')\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$$(a+b\sqrt{2})(a'+b'\sqrt{2}) = (aa' + 2bb') + (ab' + ba')\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

So $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

The ideals of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$, and $\mathbb{Q}[\sqrt{2}]$ is not one of them.

2. Is $\mathbf{Z}[\sqrt{2}]$ an ideal of $\mathbf{Q}[\sqrt{2}]$?

No. $\mathbb{Z}[\sqrt{2}]$ contains 1! If $\mathbb{Z}[\sqrt{2}]$ were an ideal of $\mathbb{Q}[\sqrt{2}]$, it would be $\mathbb{Q}[\sqrt{2}]$ itself, which is clearly not the case.

3. Write down a non-trivial ideal of ~~$\mathbf{Q}[\sqrt{2}]$.~~ $\mathbb{Z}[\sqrt{2}]$ (Typo!!)

A principal ideal: $\sqrt{2}\,\mathbb{Z}[\sqrt{2}]$ is a non-trivial ideal of $\mathbb{Z}[\sqrt{2}]$, (it does not contain 1 for example).

($\mathbb{Q}[\sqrt{2}]$ has no non-trivial ideals) – the question, as stated, was incorrect).

4. Let A be the subset of $\mathbf{Z}[\sqrt{2}]$ defined by $A := \{2n + 2m\sqrt{2},\ \text{for } n, m \text{ in } \mathbf{Z}\}$. It is easy to check that A is an ideal of $\mathbf{Z}[\sqrt{2}]$, and we admit it.

(a) Show that A is a principal ideal.

It is the ideal generated by 2!

(b) Show that A is not a **maximal** ideal (hint: consider the set of all $2n + m\sqrt{2}$, for n, m in $\mathbf{Z}$)

Let $B = \{2n + m\sqrt{2},\ \text{for } n, m \text{ in } \mathbb{Z}\}$.

Clearly $A \subset B$, and $B \subsetneq \mathbb{Z}[\sqrt{2}]$, so if we prove that B is an ideal, it will imply that A is not maximal.

Take n, m, n', m' in $\mathbb{Z}$.

$$(2n + m\sqrt{2}) - (2n' + m'\sqrt{2}) = 2(n-n') + (m-m')\sqrt{2}$$

is in B

So B is a subgroup.

Take n, m in $\mathbb{Z}$, n', m' in $\mathbb{Z}$

$$(2n + m\sqrt{2})(n' + m'\sqrt{2}) = 2nn' + 2mm' + (2nm' + mn')\sqrt{2}$$
$$= 2(nn' + mm') + (2nm' + mn')\sqrt{2}$$

is in B.

So B is an ideal, which concludes the proof.

Left ideals of $T_2(\mathbb{R})$. Let I be a non-trivial left-ideal of $T_2(\mathbb{R})$

Case 1 I contains a matrix of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ $b \neq 0$.

Then I contains all of $A_1 := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} ; b \in \mathbb{R} \right\}$

(multiplying ~~by~~ $\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & cb \\ 0 & 0 \end{pmatrix}$ and b is $\neq 0$ so we obtain all of A_1).

Rem. A_1 is a left-ideal, and also a right-ideal

Case 1.1 I contains no other matrix, then $I = A_1$, a 2-sided ideal.

Case 1.2 I contains another matrix

Case 1.2.1 I contains $\begin{pmatrix} a & \alpha \\ 0 & 0 \end{pmatrix}$ with $a \neq 0$.

Then by using elements in A_1, we see that I contains $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and multiplying by $\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ we see that I contains all $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, for x in $\mathbb{R}$. Adding this to an element of A_1, we see that I contains $A_2 := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} ; a, b \text{ in } \mathbb{R} \right\}$.

Rem. A_2 is a left-ideal, and also a right-ideal.

By Q. 1b, I cannot contain another matrix, so $I = A_2$, a 2-sided ideal.

Case 1.2.2 I contains $\begin{pmatrix} 0 & \alpha \\ 0 & c \end{pmatrix}$ with $c \neq 0$.

Then it contains $\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ (using an element of A_1). Then it contains all the $\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$, $y \in \mathbb{R}$. Adding this to A_1, we see that I contains $A_3 = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} ; b, c \text{ in } \mathbb{R} \right\}$

Rem. A_3 is a two-sided ideal.

and $I = A_3$ (it cannot contain anything else)

Case 2 I does not contain $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, $b \neq 0$.

Case 2.1 It contains $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ $a \neq 0$,

then by multiplying by $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ on the left, we obtain

all the $\begin{pmatrix} \alpha a, & \alpha b \\ 0 & 0 \end{pmatrix}$ a, b fixed $\alpha \in \mathbb{R}$

so it contains $\tilde{A}_{\frac{b}{a}} : \left\{ \begin{pmatrix} x & \frac{b}{a} x \\ 0 & 0 \end{pmatrix}, x \in \mathbb{R} \right\}$.

Rem. $\tilde{A}_s := \left\{ \begin{pmatrix} x & sx \\ 0 & 0 \end{pmatrix}, x \in \mathbb{R} \right\}$ is an left-ideal, but not a right-ideal.

I cannot contain something else, because we would obtain either an invertible matrix or an element of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ $b \neq 0$.

so $I = \tilde{A}_s$.

Case 2.2. It contains $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$ $c \neq 0$.

But multiplying it by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we obtain $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \in I$,

which contradicts the assumption of case 2.

We have, in conclusion:

Two-sided ideals: A_1, A_2, A_3

left-ideals : $\left\{ \begin{pmatrix} x & sx \\ 0 & 0 \end{pmatrix} \in \mathbb{R} \right\}$; Right-ideals $\left\{ \begin{pmatrix} 0 & x \\ 0 & sx \end{pmatrix}, x \in \mathbb{R} \right\}$.