Algebra - Final exam - Fall 2018 - NYU

NAME:

There are 12 independent questions, and an exercise. The 12 questions are all taken from homework, midterms, or ask for a definition/proof seen in class.

Preliminary grade scheme: 7 points per question, so 84 in total, and 16 points for the exercise. This might be adjusted later.

All answers must be justified (except when you are asked to give a definition, of course).

Use scratch/scrap paper at the end, and postone your answers there if you need more space to write them.

Good luck!

Series of questions

1. Let $(G, \cdot)$ be a group, and let H, H' be two subgroups of G. Show that $H \cap H'$ is a subgroup of G.

2. What is the order of 8 in $\mathbf{Z}_{14}$? Both the “theoretical” and the “computational” answers are accepted.

3. Define the following notions: "subgroup generated by an element", and "cyclic group". Give two different examples of (non-trivial) cyclic groups.

4. Define the group S_n. Show that for $n \geq 3$, the group S_n is not Abelian.

5. What is the parity/sign of the following permutations?

   (a) (123456789)

   (b) $(1234)(56789)$

   (c) $(123)(123)(123)(123)(123)\dots(123)$, where (123) is repeated 137 times.

6. Prove that the groups $(\mathbf{Z}, +)$ and $(\mathbf{Q}, +)$ are not isomorphic.

7. Show that $\mathbf{Z}_2 \times \mathbf{Z}_3$ is cyclic, but that $\mathbf{Z}_2 \times \mathbf{Z}_2$ is not.

8. Let R be a commutative ring, and let I be an ideal of R. Assume that I contains an element of R that has a multiplicative inverse (formally speaking, assume there exists $r \in I$ and $r' \in R$ such that $r' \times r = 1$). Prove that $I = R$.

9. Prove **one** of the following statements:

   - Every subgroup of a cyclic group is cyclic.
   - Every ideal of $\mathbf{Z}$ is principal. *(Might be the easiest.)*
   - Every ideal of $\mathbb{R}[X]$ is principal.

10. Give the definition of a prime ideal **and** of a maximal ideal (no not need to define what an *ideal* is). Give an example of a ring R and a non-trivial, maximal ideal I in R.

11. Let R, S be two commutative rings, and let $\varphi : R \to S$ be a ring morphism. Define the kernel of φ and prove that it is an ideal of R.

12. Let i be the usual complex number. Show that the set

$$\mathbf{Q}(i) := \{x + iy, \text{ for } x, y \text{ in } \mathbf{Q}\},$$

is a subfield of the field $\mathbf{C}$ of complex numbers.

GL_2 and SL_2 We recall that $\mathsf{GL}_2(\mathbb{R})$ is the group of 2×2 invertible matrices with real coefficients, and that $\mathsf{SL}_2(\mathbb{R})$ is the subset of $\mathsf{GL}_2(\mathbb{R})$ formed by matrices whose determinant is equal to 1.

1. Prove that $\mathsf{SL}_2(\mathbb{R})$ is a subgroup of $\mathsf{GL}_2(\mathbb{R})$.

2. For any M in $\mathsf{GL}_2(\mathbb{R})$, and any S in $\mathsf{SL}_2(\mathbb{R})$, show that the matrix MSM^{-1} is in $\mathsf{SL}_2(\mathbb{R})$.

3. What is the name of the property of $\mathsf{SL}_2(\mathbb{R})$ that you have just checked?

4. For any M in $\mathsf{GL}_2(\mathbb{R})$, show that the map from $\mathsf{SL}_2(\mathbb{R})$ to $\mathsf{SL}_2(\mathbb{R})$ defined by

$$\varphi_M(S) := MSM^{-1}$$

is a group isomorphism.

5. (Bonus question) The quotient group $\mathsf{GL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{R})$ is isomorphic to a simple thing. What is your guess?