The field of complex numbers

1. Define the ideal generated by $X^2 + 1$ in $\mathbb{R}[X]$. We will denote it by $\langle X^2 + 1 \rangle$.

2. Explain why, for every polynomial P in $\mathbb{R}[X]$, there exists a polynomial R of degree (at most) 1 such that

$$P - R \in \langle X^2 + 1 \rangle.$$

3. Let Q be the quotient ring $\mathbb{R}[X]/\langle X^2 + 1 \rangle$, and let π be the usual projection map from $\mathbb{R}[X]$ to Q, that sends an element on its equivalence class. Recall the three main properties of π.

4. Show that $\pi(X^2) = \pi(-1)$.

5. Let φ be the map from $\mathbb{R} \times \mathbb{R}$ to Q defined as

$$\varphi(a, b) := \pi(a + bX).$$

   Prove that φ is one-to-one (injective).

6. Deduce from question 2. that, for every polynomial P in $\mathbb{R}[X]$, there exists two real numbers a, b such that

$$\pi(P) = \pi(a + bX),$$

   In particular, the map φ defined in question 5. is onto (surjective).

7. We define a ring structure on $\mathbb{R} \times \mathbb{R}$ as follows:

   (a) Addition: $(a, b) + (a', b') := (a + a', b + b')$

   (b) Product: $(a, b) \star (a', b') := (aa' - bb', ab' + a'b)$.

   (No need to check that this defines indeed a ring.) Show that $(0, 1) * (0, 1) = (-1, 0)$.

8. Show that φ is a ring isomorphism from $(\mathbb{R} \times \mathbb{R}, +, \star)$ to Q.