Questions with a $(*)$ are more difficult.

1. (a) What are the group morphisms from $(\mathbb{Z}, +)$ to itself?
   (b) What are the ring morphisms from $(\mathbb{Z}, +, \times)$ to itself? Show that the map $x \mapsto x^3$ is not a ring morphism.
   (c) What are the group morphisms from $(\mathbb{Z}_3, +)$ to itself?
   (d) What are the ring morphisms from $(\mathbb{Z}_3, +, \times)$ to itself? Show that the map $x \mapsto x^3$ is a ring morphism.
   (e) For any fixed polynomial $Q \in \mathbb{Z}[X]$, show that the map

   $$P \in \mathbb{Z}[X] \mapsto Q \times P$$

   is a group morphism from $(\mathbb{Z}[X], +)$ to itself.
   (f) (*) Are there group morphisms from $(\mathbb{Z}[X], +)$ to itself which are not of this type?
   (g) Show that the map $P \mapsto P^3$ is **not** a ring morphism $(\mathbb{Z}[X], +)$ to itself.
   (h) (*) What are the ring morphisms from $(\mathbb{Z}[X], +, \times)$ to itself?
   (i) For any fixed polynomial $Q \in \mathbb{Z}_3[X]$, show that the map

   $$P \in \mathbb{Z}_3[X] \mapsto Q \times P$$

   is a group morphism from $(\mathbb{Z}_3[X], +)$ to itself.
   (j) Show that the map $P \mapsto P^3$ **is** a ring morphism from $(\mathbb{Z}_3[X], +)$ to itself.

2. Let $T := \{z \in \mathbb{C}, |z| = 1\}$. We recall that $(T, \times)$ is a subgroup of $(\mathbb{C}^*, \times)$.
   (a) For any $k \geq 1$, we define T_k as the set

   $$T_k := \left\{ z \in \mathbb{C}, z^k = 1 \right\}.$$

   Show that T_k is a subgroup of T.
   (b) (*) Show that T_k is a cyclic group (you really need to find T_k explicitely for that. Look for "roots of unity").
   (c) If G is a group, we let $Aut(G)$ be the set of group isomorphisms from G to G.

i. Show that the identity map and the conjugation map are in $Aut(T)$.

ii. For z_0 in T, show that the map from T to T defined by

$$z \mapsto z_0 \times z,$$

is a bijection from T to T, but not a morphism, unless $z_0 = 1$

iii. For $k \geq 2$, show that the map from T to T defined by

$$z \mapsto z^k$$

is a group morphism from T to T, but not an isomorphism. What is its kernel?

(d) How many elements are there in T_3?

(e) Show that there is a one-to-one map from $Aut(T_3)$ to S_3 (the permutation group with 4 elements).

(f) (*) Is this map onto?

3. Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. We recall that $(\mathcal{F}, +, \times)$ is a commutative ring.

(a) Let A be the set of all functions f in $\mathcal{F}$ such that $f(x) > 0$ for all x in $\mathbb{R}$. Show that $(A, \times)$ is an Abelian group.

(b) Let B be the set of all functions f in $\mathcal{F}$ such that $f(x) > 2$ for all x in $\mathbb{R}$. Is $(B, \times)$ a group?

(c) Let C the set of all functions f in $\mathcal{F}$ such that $f(x)$ is a non-zero rational number for all x in $\mathbb{R}$. Is $(C, \times)$ a group?

(d) Let D the set of all **continuous** functions f in $\mathcal{F}$ such that $f(x)$ is a non-zero rational number for all x in $\mathbb{R}$. Is $(D, \times)$ a group?

(e) (*) (This needs an analytic argument) Show that D is isomorphic to the group $(\mathbb{Q}^*, \times)$.

(f) Recall the definition of the direct product of two groups.

(g) (*) Let S be a set and G be a group. Define an interesting group structure on the set of all maps from S to G. Show that for $S = \{0, 1\}$, the structure you obtain is isomorphic to the direct product $G \times G$.